Strong convergence of term rewriting using strong dependency pairs

(Extended abstract)

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Introduction. Let $\mathcal{R}$ be a finite term rewrite system over a finite signature [3]. Recall the theorem of Arts and Giesl [1] that $\mathcal{R}$ is terminating if and only if $\mathcal{R}$ satisfies the dependency pair criterion. Inspection of the proof learns that if we decrease the set of dependency pairs, then the set of infinite reduction sequences that can be rejected by the dependency pair criterion may decrease. An interesting extension of the set of finite reduction sequences in $\mathcal{R}$ is the set of strongly converging reductions in $\mathcal{R}$ of Kennaway e.a. [2]. In general not all infinite reductions in $\mathcal{R}$ will be strongly converging.

In this note we will reduce the set of dependency pairs for $\mathcal{R}$ to a subset of *strong dependency pairs*. We will prove that $\mathcal{R}$ satisfies a similar strong dependency pair criterion if and only if all reductions in $\mathcal{R}$ are strongly convergent.

Strongly converging reductions. Recall that a reduction is *strongly converging* [2], if either it is a finite reduction or it is an infinite reduction $t_0 \to t_1 \to \ldots$ satisfying $\lim_{n\to\infty} d_n = \infty$, where d_n is the depth of the redex contracted in the reduction step $t_n \to t_{n+1}$. Among the possibilities [**?**] the standard one is to measure depth by the number of nodes on the path from the root to the redex. Like termination, strong convergence of term rewriting systems is undecidable in general. Observe that the terms of an infinite strongly convergent reduction converge to a (possibly infinite) limit. Strong convergence captures the idea of progressing approximation: the limit of an infinite strongly convergent reduction can alternatively be described as the limit $\lim_{n\to\infty} s_n$ of approximating prefixes $s_n \sqsubseteq t_n$, where the s_n remain reduction free in the rest of the reduction. Here we will not consider transfinite reductions and infinite terms.

Strong Dependency Pairs. We will now define strong dependency pairs as dependency pairs with an extra depth dependent condition. As in [**?**] it is notationally convenient to extend the $\mathcal{R}$'s signature with a fresh, capitalised symbol F for each defined function symbol f of $\mathcal{R}$. If $f(s_1,\ldots,s_n) \to C[g(t_1,\ldots,t_m)]$ is a rule of $\mathcal{R}$, g is a defined symbol of $\mathcal{R}$ and the hole $[\ ]$ occurs at depth 0 in $C[\ ]$, then the pair $\langle F(s_1,\ldots,s_n), G(t_1,\ldots,t_n)\rangle$ is called a *strong dependency pair*.

As in [?] we define that a (possibly infinite) sequence $\langle s_1, t_1\rangle$, $\langle s_2, t_2\rangle$, ... of pairs of terms in $\mathcal{R}$ is a *chain* of if there is a substitution σ such that $t_j\sigma \to^* s_{j+1}\sigma$ in $\mathcal{R}$ for each $j \geq 1$. The hard work now goes into proving:

Theorem 1. *There is a reduction sequence in $\mathcal{R}$ in which infinitely many reduction steps take place at depth 0 if and only if there is an infinite chain of strong dependency pairs.*

Corollary 1. *A term rewrite system $\mathcal{R}$ is strongly converging if and only if there is no infinite chain of strong dependency pairs.*

The strong dependency pairs form a subset of the dependency pairs. A similar proof as in [?] involving contexts with holes at depth 0 shows:

Theorem 2. *A term rewrite system $\mathcal{R}$ is strongly converging if and only if there exists a well-founded weakly monotonic quasi-order $\geq$, such that*

- *both $\geq$ and $>$ are closed under substitution,*
- *$l \geq r$ for all rules $l \to r$ in $\mathcal{R}$ and*
- *$s > t$ for all strong dependency pairs $\langle s, t\rangle$ of $\mathcal{R}$.*

Conclusion and example. As in the case of termination the benefit of the last theorem is that the proof of strong convergence of a term rewrite system is now reduced to the search of a suitable quasi-order. As example one may prove that the following non-terminating term rewriting system is strongly converging. Finally, one may observe that the concept of depth is actually a parameter of this note as in [?]. Similar theorems hold for suitable variations: the original theorem of Arts and Giesl can be recognized as an instance.

$filter(x : y, 0, m)$	$\to$	$0 : filter(y, m, m)$
$filter(x : y, s(n), m)$	$\to$	$x : filter(y, n, m)$
$sieve(0 : y)$	$\to$	$sieve(y)$
$sieve(s(n) : y)$	$\to$	$s(n) : sieve(filter(y, n, n))$
$odds(n)$	$\to$	$n : odds(s(s(n)))$
$primes$	$\to$	$s(s(0)) : sieve(odds(s(s(s(0)))))$
$take(0, x : y)$	$\to$	x
$take(s(n), x : y)$	$\to$	$take(n, y)$
$prime(n)$	$\to$	$take(n, primes)$

References

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4. Jan Willem Klop. Term rewriting systems. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 2, chapter 1, pages 1–117. Oxford University Press, Oxford, 1992.

1 Erratum (as corrected at the workshop)

Theorem 1 is not correct. It is not true for TRSs containing collapse rules. The following TRS is not strongly converging, yet it has no infinite chain of strong dependency pairs.

$$\begin{array}{lcr} f(x) & \to & x \\ a & \to & f(a) \end{array}$$

However the theorem is true for arbitrary TRSs when the set of strong dependency pairs gets extended with dependency pairs of the form $\langle F(s_1, \ldots, s_n) \to C[x] \rangle$ for each collapse rule of the form $l \equiv f(s_1, \ldots, s_n) \to x$.