Centrum voor Wiskunde en Informatica

Centre for Mathematics and Computer Science

J.R. Kennaway, J.W. Klop, M.R. Sleep, F.J. de Vries

Transfinite reductions in orthogonal term rewriting systems

Computer Science/Department of Software Technology Report CS-R9041 December

1990

Centrum voor Wiskunde en Informatica
Centre for Mathematics and Computer Science

J.R. Kennaway, J.W. Klop, M.R. Sleep, F.J. de Vries

Transfinite reductions in orthogonal term rewriting systems

Computer Science/Department of Software Technology   Report CS-R9041 December

The Centre for Mathematics and Computer Science is a research institute of the Stichting Mathematisch Centrum, which was founded on February 11, 1946, as a nonprofit institution aiming at the promotion of mathematics, computer science, and their applications. It is sponsored by the Dutch Government through the Netherlands Organization for the Advancement of Research (N.W.O.).

Transfinite Reductions in Orthogonal Term Rewriting Systems

J.R. KENNAWAY [1], J.W. KLOP [2], M.R. SLEEP [3] & F.J. DE VRIES [4]

(1) jrk@sys.uea.ac.uk, (2) jwk@cwi.nl, (3) mrs@sys.uea.ac.uk, (4) ferjan@cwi.nl

(1,3) *School of Information Systems, University of East Anglia, Norwich NR4 7TJ, U.K.*

(2,4) *CWI, P.O. Box 4079, 1009 AB Amsterdam, The Netherlands*

Abstract. First we establish some fundamental facts in the theory of infinitary orthogonal term rewriting systems (OTRSs): for strongly convergent reductions we prove the Infinitary Parallel Moves Lemma and the Compression Lemma. Strongness is necessary as shown by counterexamples. Normal forms (finite or infinite) are unique, in contrast to ω-normal forms. Strongly converging, fair reductions result in normal forms.

Secondly we address the infinite Church-Rosser property, which in general OTRSs fails both for strongly converging reductions and for converging reductions. For OTRSs with no collapsing rules other than one rule of the form $I(x) \to x$ the infinite Church-Rosser Property holds for strongly converging reductions. Non-unifiable OTRSs form a special class of them: here any converging reduction is strongly converging. The top-terminating OTRSs of Dershowitz c.s. are examples of non-unifiable OTRSs. We generalize head normal form, Böhm reduction and Böhm tree from Lambda-Calculus to Term rewriting. For OTRSs any term has a unique Böhm tree, and Böhm reduction satisfies the infinite Church-Rosser property.

Thirdly, results concerning needed redexes from finitary orthogonal rewriting carry over to the infinite setting by adding fairness considerations: needed-fair reductions are normalizing, parallel-outermost reduction is transfinitely hypernormalizing and depth-increasing reduction is hypernormalizing.

Finally the relation between graph rewriting and infinitary term rewriting is considered. The link with infinitary rewriting allows us to treat cyclic graphs as well. Sekar and Ramakrishnan's notion of necessary set is useful to handle needed redexes in a graph: Needed redexes in a graph correspond to necessary sets of redexes in the unraveling of the graph. It follows that for strongly sequential orthogonal term graph rewrite systems an effective normalizing strategy exists. Graph rewrite systems are tree-reducible and OTRSs are graph-reducible.

1985 Mathematics Subject Classification: 68Q50
1987 CR Categories: F4.1, F4.2

Keywords and Phrases: orthogonal term rewriting systems, infinitary rewriting, infinite Church-Rosser Properties, normal forms, Böhm Trees, head normal forms, non-unifiable term rewriting systems, needed redexes, fair reductions, normalising strategies, graph rewriting systems.

All authors were partially sponsored by SEMAGRAPH, ESPRIT Basic Research Action 3074. The first author was also partially surported by a SERC Advanced Fellowship, and by SERC grant no. GR/F 91582.

Report CS-R9041
Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

CONTENTS

1. INTRODUCTION

The theory of Orthogonal Term Rewrite Systems (OTRS) is now well established within theoretical computer science. Comprehensive surveys have appeared recently in [Der90a, Klo91]. In this paper we consider extensions of the established theory to cover infinite terms and infinite rewriting reductions.

1.1. Motivation.

At first sight, the motivation for such extensions might appear of theoretical interest only, with little practical relevance. However, it turns out that both infinite terms and infinite rewriting reductions do have practical relevance.

A practical motivation for studying infinite terms and term rewriting arises in the context of lazy functional languages such as Miranda [Tur85] and Haskell [Hud88]. In such languages it is possible to work with infinite terms, such as the list of all Fibonacci numbers or the list of all primes. This style of programming has been advocated by Turner [Tur85], Peyton-Jones [Pey87] and others. Of course the outcome of a particular computation must be finite, but it is pleasant to define such results as finite portions of an infinite term. It would be even more pleasant to know that nice properties (for example the Church-Rosser Property) held for infinite as well as finite rewriting, but the standard theory does not tell us this. As we show in Section 5, the Church-Rosser Property is one of several standard results which does *not* hold for infinite rewriting in general, although it does hold for fairly natural classes of orthogonal TRSs.

A second practical motivation for considering infinite reductions arises from the common graph-rewrite based implementations of functional languages. The correspondence between graph rewriting and term rewriting was studied in [Bar87] for acyclic graphs. When cyclic graphs are considered, the correspondence with term rewriting immediately requires consideration of infinite terms and infinite reductions. For example, the cyclic graph r:B(r) 'unravels' to the infinite term B(B(B(......))). Reduction of a single redex of a cyclic graph — for example, by applying the rule $B(x) \to C(x)$ to the above graph — can correspond to reduction of infinitely many redexes of such an infinite term. Furthermore, a finite reduction of graph reductions may correspond to an *infinitary* term reduction — a composition of reductions which may themselves be infinite. For example, applying $B(x) \to C(x)$, and then $C(x) \to D$ to the example graph corresponds to an infinite term reduction $B(B(B(\ldots))) \to^{\omega} C(C(C(\ldots))) \to D$. The correspondence with graphs is the motivation for [Far89], which presents a treatment of infinite reduction similar to the one below.

1.2. Overview.

The paper consists of seven sections. You are reading Section 1, the introduction.

In Section 2 the preliminary definitions for finitary and infinitary term rewriting systems are given. Infinite reductions result in infinite sequences of terms. In a natural fashion terms over some given signature can be thought of as elements of in a metric space (cf. [Arn80]). So it seems just as natural to consider converging reductions as the basic reduction notion for infinite term rewriting. This is the line Dershowitz, Kaplan and Plaisted have followed (cf. [Der89a,b and Der90b]). We take a different approach:

In Section 3 we define strongly converging reductions of transfinite length, and present some elementary results. Strongly converging reduction sequences of length at most ω seem to have been considered for the first time by Farmer and Watro (cf.[Far89])

Section 4 contains fundamental properties for strongly converging reductions. First, for OTRSs we prove the Infinitary Parallel Moves Lemma, the infinitary generalization of the finite Parallel Moves Lemma. Then we show for left-linear TRSs that any strongly converging reduction of length greater then ω can be compressed into a strongly converging reduction with same final term but of length lesser than or equal to ω. This puts the Compression Lemma of Dershowitz, Kaplan and Plaisted in a more general perspective. Next, reductions of length at most ω to normal form are characterized as stable reductions. Fourth, we prove that normal forms ((in-)finite terms not containing any redex) are unique in OTRSs. Finally we show that in OTRSs the limit of any fair, strongly converging sequence is a normal form.

The following table summarizes the results for strongly converging reductions in contrast with related results for converging reductions:

<u>*Basic facts for infinitary orthogonal term rewrite systems*</u>		
	converging reductions	*strongly converging reductions*
Transf. Parallel Moves Lemma	NO (4.1.3)	YES (4.1.2)
Inf. Church-Rosser Property	NO (5.1.1)	NO (5.1.1)
Unique ω–normal forms	NO (5.1.1)	NO (5.1.1)
Unique normal forms	YES (4.3.10)	YES (4.3.10)
Compression Lemma	NO [Far89], (4.3.1)	YES (4.3.5) partial result in [Far89]
Fair reductions result in	ω-normal forms [Der90b], (4.3.12.i)	normal forms (4.3.12.ii)

(Table 1.1)

Section 5 is devoted to the infinite Church-Rosser property. First a counterexample is presented showing that the infinite Church-Rosser property does not hold for either the strongly converging reductions we are interested in nor for converging reductions of Dershowitz c.s.. We provide several ways out.

- Consider a more restricted class of reductions. The infinite Church-Rosser property holds rather trivially for stable reductions because of the unique normal form property. This is not a very informative solution.
- Investigate for which classes of OTRSs for which the infinite Church-Rosser property does hold for strongly converging reductions. First we prove the infinite Church-Rosser property for depth preseving OTRS, then we extend this result to OTRSs that contain no collaps rules other than one rule of the form $I(x) \rightarrow x$. This result is optimal in the sense that addition of more collapsing rules generates the counterexample. It explains and generalizes the infinite Church-Rosser property for top-terminating OTRSs, as occurring more or less implicit in the work of Dershowitz c.s..
- Consider Böhm reduction, a more liberal notion of reduction borrowed from Lambda Calculus

in which one is allowed to replace subterms that have no head normal form for the reduction relation of the given OTRS by a special symbol $\perp$. Böhm-reduction is normalizing and satisfies the infinite Church-Rosser property both for strongly converging reductions and converging reductions.

Section 6 is devoted to the extension of some theorems by Huet and Lévy on needed redexes to the context of infinitary rewriting. This generalization turns out to be unproblematic: needed-fair reduction is normalizing; parallel-outermost reduction is transfinitely hypernormalizing in contrast to depth-increasing reduction which is hypernormalizing. That parallel-outermost reduction is not just hypernormalizing is due to the possibility that an infinite term may contain infinitely many outermost redexes.

In Section 7 we give some applications to graph rewriting. We concentrate on term graph rewriting, which seems to be emerging as central object of study in the Semagraph project. Our study of infinite term rewriting allows us to extend results by [Bar87] on the relationship between (term-)graph rewriting and term rewriting to a context including cyclic graphs and transfinite reductions. We obtain that every GRS (term graph rewrite system) is tree reducible and every orthogonal TRS is graph reducible in its standard lifting. Sekar and Ramakrishnan's notion of necessary set is useful to handle needed redexes in a graph: Needed redexes in a graph correspond to necessary sets of redexes in the unraveling of the graph. It follows that for strongly sequential orthogonal term graph rewrite systems an effective normalizing strategy exists.

Finally we discuss relations of our present work with works of others.

Acknowledgments

We acknowledge useful and pleasant discussions with Aart Middeldorp and Jeroen Warmerdam.

2. Preliminaries on term rewriting systems

We briefly recall the definition of a finitary term rewriting system, before we define infinitary orthogonal term rewriting systems involving both finite and infinite terms. For more details the reader is referred to [Der90a] and [Klo91]

2.1. Finitary term rewriting systems

A *finitary term rewriting system* over a signature Σ is a pair $(\mathrm{Ter}(\Sigma),R)$ consisting of the set $\mathrm{Ter}(\Sigma)$ of finite terms over the signature Σ and a set of rewrite rules $R \subseteq \mathrm{Ter}(\Sigma)\times\mathrm{Ter}(\Sigma)$.

The *signature* Σ consists of a countably infinite set Var_Σ of variables (x,y,z,...) and a non-empty set of function symbols (A,B,C,...,F,G,...) of various finite arities ≥ 0. Constants are function symbols with arity 0. The set $\mathrm{Ter}(\Sigma)$ of *finite* terms (t,s,...) over Σ can be defined as usual: the smallest set containing the variables and closed under function application.

The set $O(t)$ of *occurrences* in t is defined by induction on the structure of t as follows: $O(t) = \{< >\}$ if t is a variable and $O(t) = \{< >\} \cup \{<i,u> \mid 1\leq i\leq n \text{ and } <u>\in O(t_i)\}$ if t is of the form $F(t_1,\ldots,t_n)$. If $u\in O(t)$ then the subterm t/u at occurrence u is defined as follows: $t/< > = t$ and $F(t_1,\ldots,t_n)/<i,u> = t_i/u$. The *depth* of a subterm of t at occurrence u is the length of u.

Contexts are terms in $\mathrm{Ter}(\Sigma\cup\{\square\})$, in which the special constant $\square$, denoting an empty place, occurs exactly once. Contexts are denoted by C[ ] and the result of substituting a term t in place of □ is C[t]∈Ter(Σ). A *proper* context is a context not equal to □.

Substitutions are maps $\sigma:Var_\Sigma \rightarrow Ter(\Sigma)$ satisfying $\sigma(F(t_1,...,t_n)) = F(\sigma(t_1),...,\sigma(t_n))$.

The set R of *rewrite rules* contains pairs (l,r) of terms in Ter(Σ), written as $l \rightarrow r$, such that the left-hand side l is not a variable and the variables of the right-hand side r are contained in l. The result l^σ of the application of the substitution of σ to the term l is called an instance of l. A *redex* (reducible expression) is an instance of a left-hand side of a rewrite rule. A *reduction step* $t \rightarrow s$ is a pair of terms of the form $C[l^\sigma] \rightarrow C[r^\sigma]$, where $l \rightarrow r$ is a rewrite rule in R. Concatenating reduction steps we get either a *finite reduction* $t_0 \rightarrow t_1 \rightarrow ... \rightarrow t_n$, which we also denote by $t_0 \rightarrow_n t_n$, or an infinite reduction $t_0 \rightarrow t_1 \rightarrow ...$.

Finally we can now give the definition of an orthogonal TRS.

2.1.1. DEFINITION. Let R be a finitary TRS.

(i) R is *left-linear* if no variable occurs more than once in a left-hand side of R's rewrite rules.

(ii) (informally) R is *non-overlapping* (or non-ambiguous) if non-variable parts of different rewrite rules don't overlap and non-variable parts of the same rewrite rule overlap only entirely:

(ii') (formally) R is *non-overlapping* if for any two left-hand sides s and t, any occurrence u in t, and any substitutions σ and $\tau:Var_\Sigma \rightarrow Ter(\Sigma)$ it holds that if $(t/u)^\sigma = s^\tau$ then either t/u is a variable or t and s are left-hand sides of the same rewrite rule and u is the empty occurrence < >, the position of the root.

(iii) R is *orthogonal* if R is both left-linear and non-overlapping.

It is well-known (cf. [Ros73], [Klo91]) that finitary orthogonal TRSs satisfy the finitary Church-Rosser property, i.e., ${}_{<\omega}\!\leftarrow \circ \rightarrow_{<\omega} \subseteq \rightarrow_{<\omega} \circ {}_{<\omega}\!\leftarrow$, where $\rightarrow_{<\omega}$ is our notation for the transitive, reflexive closure of the relation $\rightarrow$.

2.2. Infinitary term rewriting systems

An *infinitary term rewriting system* over a signature Σ is a pair $(Ter^\infty(\Sigma),R)$ consisting of the set $Ter^\infty(\Sigma)$ of finite and infinite terms over the signature Σ and a set of rewrite rules $R \subseteq Ter(\Sigma) \times Ter^\infty(\Sigma)$. We don't consider rewrite rules with infinite left-hand sides. But we allow right-hand sides to be infinite in order to be able to interpret various liberal forms of graph rewriting in infinitary term rewriting. In [Der90b] only finite right-hand sides are considered.

It takes some elaboration to define the set $Ter^\infty(\Sigma)$ of *finite and infinite terms* precisely. Finite terms may be represented as finite trees, well-labelled with variables and function symbols. Well-labelled means that a node with n ≥ 1 successors is labelled with a function symbol of arity n and that a node with no successors is labelled either with a constant or a variable. Now *infinite terms* are infinite well-labelled trees with nodes at finite distance to the root. Substitutions, contexts and reduction steps generalize trivially to the set of infinitary terms $Ter^\infty(\Sigma)$.

To introduce the *prefix ordering* ≤ on terms we extend the signature Σ with a fresh symbol Ω. The prefix ordering ≤ on $Ter^\infty(\Sigma \cup \{\Omega\})$ is defined inductively: $x \le x$ for any variable x, $\Omega \le t$ for any term t and if $t_1 \le s_1, ..., t_n \le s_n$ then $F(t_1,...,t_n) \le F(s_1,...,s_n)$. For terms t in $Ter^\infty(\Sigma \cup \{\Omega\})$ we denote by $|t|$ the minimal distance of an occurrence of Ω in t to the root, if there is any, otherwise $|t| = \infty$.

If all function symbols of Σ occur in R we will write just R for $(Ter^\infty(\Sigma),R)$.

The definition of orthogonality for finitary TRSs extends verbatim to infinitary TRSs:

2.2.1. DEFINITION. Let R be an infinitary TRS.

(i) R is *left-linear* if no variable occurs more than once in a left-hand side of R's rewrite rules,

(ii) R is *non-overlapping* if for any two left-hand sides s and t, any occurrence u in t, and any substitutions σ and τ:$Var_\Sigma \to Ter(\Sigma)$ it holds that if $(t/u)^\sigma = s^\tau$ then either t/u is a variable or t and s are left-hand sides of the same rewrite rule and u is the empty occurrence < >, the position of the root.

(iii) R is *orthogonal* if R is both left-linear and non-overlapping.

2.3. Metric spaces of terms

In this paper we will consider limit behavior of infinite reduction sequences. As reduction sequences are just a special kind of sequences in which the terms of the sequence have a particular relation with each other, we can borrow terminology regarding converging sequences from Topology (any handbook on Topology will do; we refer to [Kel55]). To discuss convergence of sequences it is convenient to recognize the set $Ter^\infty(\Sigma)$ as a complete metric space. We will briefly recall this well-known fact, see for instance [Arn80].

The set $Ter(\Sigma)$ of finite terms for a signature Σ can be provided with an ultra-metric d: $Ter(\Sigma)\times Ter(\Sigma) \to [0,1]$. The distance d(t,s) of two terms t and s is 0 if t and s are equal, and otherwise 2^{-k}, where $k \in N$ is the largest number such that the labels of all nodes of s and t at depth less than or equal to k are equally labelled. For example: $d(F(F(x,y),z), F(F(x,F(F(C,C),C)),z)) = 2^{-2}$, as the depth of the node of y, up to which depth both terms are equal, is 3.

The metric completion of $Ter(\Sigma)$ is isomorphic to the set of infinitary terms $Ter^\infty(\Sigma)$ (cf. [Arn80]). As a result in $Ter^\infty(\Sigma)$ all Cauchy sequences of ordinal length ω have a limit in $Ter^\infty(\Sigma)$.

We will consider sequences of infinitary length:

2.3.1. DEFINITION. A *sequence* is a set of elements indexed by some ordinal, denoted by $(t_\beta)_{\beta<\alpha}$. Instead of $(t_\beta)_{\beta<\alpha+1}$ we often write $(t_\beta)_{\beta\leq\alpha}$.

The notion of Cauchy sequence generalizes to finite and transfinite sequences. In Topology convergence of sequences has been generalized to convergence of nets, sets of elements indexed by a directed set; ordinals are directed sets. This comes down to:

2.3.2. DEFINITION. A sequence $(t_\beta)_{\beta\leq\alpha}$ is *converging* (or the sequence $(t_\beta)_{\beta<\alpha}$ converges to t_α), i.e. $\lim_{\beta<\alpha} t_\beta = t_\alpha$, iff for any neighborhood V of t_α there is an ordinal $\beta<\alpha$ such that for all $\beta<\gamma\leq\alpha$ the terms t_γ are in the neighborhood V, that is $\forall\varepsilon>0\, \exists\beta<\alpha\, \forall\gamma\, (\beta\leq\gamma<\alpha \to d(t_\gamma,t_\alpha) < \varepsilon)$.

For the purpose of term rewriting this notion of convergence is not enough. Needed is a notion of what could be called *everywhere converging sequence*: a sequence of which all its initial sequences are converging to the next element, or more formally:

2.3.3. DEFINITION. A sequence $(t_\beta)_{\beta\leq\alpha}$ is *everywhere converging* if the initial sequences $(t_\beta)_{\beta\leq\gamma}$ converge to t_γ for all $1\leq\gamma\leq\alpha$.

We do not know whether everywhere converging sequences have been studied in Topology.

3. STRONGLY CONVERGING REDUCTIONS

In Section 3.1 we will introduce the basic notion of a strongly converging reduction of arbitrary length. Then in section 3.2 we will prove an elementary but important fact: the number of steps in a strongly converging reduction that contribute to a finite prefix of the final term of the reduction is finite. The proof of the Compression Lemma will be based on this fact.

We start with some examples.

(i) $A \to B \to A \to B \to \ldots$, in a TRS with rules $A \to B$ and $B \to A$,

(ii) $C \to S(C) \to S(S(C)) \to \ldots$, in a TRS with rule $C \to S(C)$;

(iii) $D(E) \to D(S(E)) \to D(S(S(E))) \to \ldots$, in a TRS with rule $D(x) \to D(S(x))$.

The first example illustrates a reduction sequence that does not converge to any limit. In the second example it is tempting to say that the limit of C will be S^ω, an infinite reduction of S (plus all the necessary brackets), and similar D(E) should have as limit $D(S^\omega)$. Cauchy convergence is the natural formalism in which to express all this. The difference between the second and the third example is that in the third example the contracted redex is at depth 0 in the successive steps, whereas in the second example the depth of the reduced redexes tends to infinity. The third example is an example of a *converging* reduction, the second example is an example of *strongly converging* reduction.

As the limits are themselves terms which can be reduced it is natural to study reduction sequences of length possibly greater than ω: transfinite reduction sequences.

3.1. Strongly convergence

We will now introduce reduction sequences as special sequences.

3.1.1. DEFINITION. (i) A *reduction* $(t_\beta)_{\beta<\alpha}$ is a sequence such that $t_\beta \to t_{\beta+1}$ for all $\beta+1 < \alpha$. The redex contracted $t_\beta \to t_{\beta+1}$ will be denoted by R_β, its depth as subterm of t_β by d_β.

(ii) A reduction $(t_\beta)_{\beta<\alpha}$ is *closed* if α is a successor ordinal; a reduction $(t_\beta)_{\beta<\alpha}$ is *open* if α is a limit ordinal.

(iii) The length of a closed reduction $(t_\beta)_{\beta<\alpha+1}$ is α. The length of a open reduction $(t_\beta)_{\beta<\lambda}$ is λ.

For example, the length of the *open* reduction $t_0 \to t_1 \to \ldots \to t_n \to t_{n+1} \to \ldots$ is ω, just as the length of the *closed* reduction $t_0 \to t_1 \to \ldots \to t_n \to t_{n+1} \to \ldots t_\omega$.

This notion of reduction on its own is a bit peculiar from the point of view of computing. In, for example, the TRS with the rules $A \to B$ and $B \to A$ the following closed reduction of length ω is allowed: $A \to B \to A \to \ldots C$. What is missing in this example is any relation between the initial terms A and B and the final term C. The notion of everywhere converging sequence will remedy this defect.

3.1.2. DEFINITION. A *converging* reduction is a reduction whose underlying sequence is everywhere converging.

Converging reduction is the notion of reduction as introduced by Dershowitz, Kaplan (cf.

[Der89a], [Der90b]). Despite its naturality, we need a stronger form of converging reduction in order to state and prove the fundamental facts for infinitary term rewriting. More precisely, we will define when a reduction is strong and then consider strongly converging reductions. Strong reductions are reductions in which the depth of the reduced redexes tend to infinity. We present the definition for reductions of arbitrary infinitary length. Strongly converging reductions are introduced by Farmer and Watro [Far89].

3.1.3. DEFINITION. By induction on the ordinal α we define when a reduction $(t_\beta)_{\beta<\alpha}$ is a *strong* reduction:

(zero) $(t_\beta)_{\beta<0}$ is a strong reduction,

(successor) $(t_\gamma)_{\gamma<\beta+1}$ is a strong reduction if $(t_\gamma)_{\gamma<\beta}$ is a strong reduction,

(limit) $(t_\gamma)_{\gamma<\lambda}$ is a strong reduction if for all $\beta<\lambda$ the reduction $(t_\gamma)_{\gamma<\beta}$ is strong and $\lim_{\beta<\alpha} d_\beta = \infty$, that is $\forall d>0\ \exists\beta<\lambda\ \forall\gamma\ (\beta\leq\gamma<\lambda \rightarrow d_\gamma>d)$.

3.1.4. DEFINITION. A *strongly converging* reduction is a strong reduction that is a everywhere converging sequence.

We will use the following notation exhibiting our preference for strongly converging reductions.

3.1.5. NOTATION. (i) We will denote a strongly converging reduction $(t_\beta)_{\beta\leq\alpha}$ by $t_0 \rightarrow_\alpha t_\alpha$,

(ii) We will denote a converging reduction $(t_\beta)_{\beta\leq\alpha}$ by $t_0 \overset{c}{\rightarrow}_\alpha t_\alpha$,

(iii) By $t \rightarrow_{\leq\alpha} s$ we denote the existence of a strong reduction of length less or equal to α converging towards limit s,

(iii) By $t \overset{c}{\rightarrow}_{\leq\alpha} s$ we denote the existence of a reduction of length less or equal to α converging towards limit s.

3.2. Counting steps in strongly convergent reductions

Convergent reductions exist of any length. Consider for example the TRS with the single rule $A \rightarrow A$. Reductions of the form $A \overset{c}{\rightarrow}_\alpha A$ are converging for any ordinal α. However these sequences are not strongly convergent. The example $A \overset{c}{\rightarrow}_\alpha A$ shows also that in a converging reduction any number of reduction steps may be performed below some depth. For strongly converging reductions this is different:

3.2.1. THEOREM. *If* $t_0 \rightarrow_\lambda t_\lambda$ *is strongly convergent, then the number of steps in* $t_0 \rightarrow_\lambda t_\lambda$ *reducing a redex at depth* $\leq$ n *is finite.*

PROOF. Assume $t_0 \rightarrow_\lambda t_\lambda$ is strongly convergent. As this reduction is strong there is a last step $t_\alpha \rightarrow t_{\alpha+1}$ at which a redex is contracted at depth $\leq$ n. Consider the initial segment $t_0 \rightarrow_\alpha t_\alpha$, and repeat the argument. By the well-ordering of the ordinals (no infinite descending chains of ordinals) this process stops in finitely many steps. □

We have the following informative corollary:

3.2.2. COROLLARY. *A strongly converging infinitary reduction has countable length.*

PROOF. By the previous Theorem 3.2.1 a strongly convergent infinitary reduction can only perform finitely many reductions at any given depth $d \in \mathbb{N}$. □

For any countable ordinal α it is possible to construct a strongly converging reduction of length α. Exercise: construct such reductions in the Binary Tree TRS: $C \to B(C,C)$.

In the setting of left-linear TRSs we have a similar theorem for the number of reduction steps that somehow have been relevant or have contributed to a particular occurrence in the final term of a reduction sequence. To this end we generalize Huet's and Lévy's notion [Hue79] of preservation of occurrences of a term to infinite strong converging reductions in left-linear TRSs. First we need a lemma. Here, our assumption that left-hand sides of rules have to be finite plays a crucial role.

3.2.3. LEMMA. *For some limit ordinal λ, let $t_0 \to_\lambda t_\lambda$ be a strongly convergent reduction in a left linear TRS. For any redex* R *at occurrence* v *present in t_λ it holds that there is a last step $t_\beta \to t_{\beta+1}$ where a redex R_β is reduced at occurrence $v_\beta \leq v$ and at* v *in $t_{\beta+1}$ occurs a redex.*

PROOF. In a left-linear TRS a redex is determined by a finite prefix. In a strong converging reduction $t_0 \to_\lambda t_\lambda$ this finite prefix is already present at some earlier moment $\beta < \lambda$. □

Left linearity is necessary as the following example taken from [Der90b] shows. Take the non-left-linear TRS with rules $F(x,x) \to C$ and $G(x) \to H(G(x))$, then $F(G(A),H(G(B)) \to_\omega F(G^\omega,G^\omega) \to C$. There is no redex for rule $F(x,x) \to A$ present in any predecessor of $F(G^\omega,G^\omega)$. Note that the term $F(G(A),H(G(A))$ has a unique normal form C, which can be not be reached in finitely many steps.

The following definition is based on the Lemma 3.2.3.

3.2.4. DEFINITION. Let $t_0 \to_\alpha t_\alpha$ be a strongly convergent reduction in a left-linear TRS.

(i) A strongly converging reduction $t_0 \to_\alpha t_\alpha$ *preserves* an occurrence u in t_0 if no reduction step of the reduction is performed at an occurrence which is a proper prefix of u.

(ii) Let u be an occurrence of t_α. The set $C(u,t_0 \to_\alpha t_\alpha)$ of steps of $t_0 \to_\alpha t_\alpha$ which *contribute to* u is defined thus:

$$C(u,t_0 \to_\alpha t_\alpha) = \begin{cases} \bigcup \{C(v_\beta w,t_0 \to_\beta t_\beta) \mid w \in O(l_\beta)\} & \text{if there is a last } \beta \text{ with } v_\beta \leq u \\ \varnothing & \text{otherwise} \end{cases}$$

where v_β is the occurrence of redex R_β contracted in step $t_\beta \to t_{\beta+1}$ and $O(l_\beta)$ is the set of occurrences of function symbols in the corresponding LHS of R_β.

(iii) The set of steps of $t_0 \to_\alpha t_\alpha$ which contribute to a set of occurrences U of t_α is the set of steps which contribute to any member of U.

In words: if no step of $t_0 \to_\alpha t_\alpha$ is performed at an occurrence which is a prefix of u, then no step of $t_0 \to_\alpha t_\alpha$ contributes to u. Otherwise, since $t_0 \to_\alpha t_\alpha$ is strongly convergent, there must be a last step $t_\beta \to t_{\beta+1}$ reducing R_β at an occurrence v that is a prefix of u. Then $t_\beta \to t_{\beta+1}$ contributes to u, and every step of $t_0 \to_\beta t_\beta$ which contributes to v or to any node of t_β pattern-matched by R_β (the variable free part of the left-hand side of the rule for which R_β is a redex) contributes to u. (Cf. also the link with needed redexes explained after corollary 3.2.6.)

3.2.5. THEOREM. *Let $t_0 \to_\alpha t_\alpha$ be a strongly convergent reduction in a left-linear TRS. For every* *finite prefix of* t_α, *there are only finitely many steps in* $t_0 \to_\alpha t_\alpha$ *contributing to all occurrences of the prefix.*

PROOF. A variation on the proof of Theorem 3.2.1 works. Let s be a finite prefix of t_α. The crucial step in repeating the proof of 3.2.1 is the insight that there is a last step $t_\beta \to t_{\beta+1}$ contributing to the prefix s. We prove this step as follows.

Suppose there is no such last step. Then consider the supremum λ of the ordinals β of these contributing steps. This supremum itself does not correspond to a contributing step, otherwise it would have been the last step itself. Clearly $\lambda \leq \alpha$. Observe that after each ordinal less than λ there is a step contributing to the prefix and reducing at height less than the height of the prefix $s \leq t_\alpha$. Otherwise from a certain point on they no longer contribute to the prefix s. Hence the depth of the contracted redexes of the contributing steps also does not go to infinity.

On the other hand the shorter sequence $t_0 \to_\lambda t_\lambda$ is also strongly convergent. Thus the depth of the contracted redex of the contributing steps also goes to infinity.

Contradiction. □

In the present infinitary context it is natural to define that a term *is a normal form* if it contains no redexes, just like in the finitary context.

3.2.6. COROLLARY. *For orthogonal TRSs, if a term (even an infinite one) strongly converges to a finite normal form, then it can be reduced to normal form in finitely many steps.*

PROOF. Instead of using Huet and Lévy's theory of needed redexes form which implies this theorem (cf. [Hue79]), we use the previous counting theorem. In Theorem 4.3.7 we will show that any reduction to normal form in an orthogonal TRS is strongly converging. Then the corollary follows from the previous Theorem 3.2.4 and the observation that the finitely many contributing steps given by Theorem 3.2.4 form a reduction on their own. □

Another way of thinking of those steps in a reduction to finite normal form that actually contribute to that normal form can be given with help of Huet and Lévy's notion of needed redex. Recall that a redex in a term is needed if any reduction from that term to normal form contracts a descendant of the redex. The steps in an infinite reduction to finite normal form that contribute to the normal form are exactly the steps in which a needed redex gets contracted. We owe this remark to Yoshihito Toyama.

4. THE FUNDAMENTALS OF INFINITARY TERM REWRITING

In this chapter we prove some fundamental facts of infinitary term rewriting.

In 4.1 we will show that the Parallel Moves Lemma generalizes to infinitary strongly converging reductions in infinitary orthogonal TRSs. We also present a counterexample against the construction embodied in Infinitary Parallel Moves Lemma for converging reductions.

In 4.2 we prove a sligthly stronger version of the Compression Lemma (Theorem 1 in [Der90b]): in a left-linear TRS any strongly convergent reduction $t_0 \to_\alpha t_\alpha$ for $\alpha > \omega$ can be compressed to a shorter reduction $t_0 \to_{\leq\omega} t_\alpha$.

Recall that a term is a *normal form* if it contains no redexes, and that a term is an ω-normal form when, if this term can reduce, it reduces in one step to itself. In 4.4 we will show for infinitary orthogonal TRSs that each term has at most one normal form. The limit of a fair, strongly converging reduction will be proved to be a normal form in 4.5. The unique ω-normal form property does not hold in general. Yet, there is a parallel between normal forms and ω-normal forms: the limit of a fair converging reduction will be an ω-normal form.

To obtain the results in 4.4 and 4.5 we introduce the notion of a stable reduction. Informally, an infinite reduction will be called *stable* if the sequence of stable prefixes of its terms converges to its limit: a stable prefix of a term t is a prefix of t such that no occurrence of that prefix can become an occurrence of a redex in any reduction sequence starting from t. In 4.3 we introduce stable reductions and prove the important theorem that in a orthogonal TRS any converging reduction to normal form is strongly converging and stable.

4.1. The Infinitary Parallel Moves Lemma

In this section we will prove a generalisation of the Parallel Moves Lemma—well-known in the setting of finitary orthogonal term rewriting—to infinitary orthogonal term rewriting with infinite reductions and rules in which the right-hand sides may be infinite. It may come as a surprise that this Infinitary Parallel Moves Lemma will only be provable for strongly converging reductions: we present a counterexample for arbitary converging reductions. For the statement of the lemma and for its proof the notion of descendant has to be extended to transfinite reductions.

In $t \to s$ let s be obtained by contraction of the redex S in t. Recall the notation u\S of the set descendants of a redex occurrence u of t in the contraction of S (cf. [Hue79]). We extend descendance to transfinite reductions:

4.1.1. DEFINITION. Let $t_0 \to_\alpha t_\alpha$ be a strongly converging reduction such that for all $\beta<\alpha$ t_β reduces to $t_{\beta+1}$ by contraction of the redex R_β. By induction on the ordinal α we define the set of *descendants* $u\backslash\alpha$ in t_α that descend from the redex occurrence u in t_0:

(i) $u\backslash 0 = \{u\}$

(ii) $u\backslash(\beta+1) = \bigcup\{v\backslash R_\beta \mid v \in u\backslash\beta\}$

(iii) $u\backslash\lambda = \{v \mid \exists\beta<\lambda\ \forall\gamma\ (\beta\leq\gamma<\lambda \to v\in u\backslash\gamma)\}$

4.1.2. INFINITARY PARALLEL MOVES LEMMA. *Let* $t_0 \to_\alpha t_\alpha$ *be a strongly converging reduction sequence of* t_0 *with limit* t_α *and let* $t_0 \to s_0$ *be a reduction of a redex* S *of* t_0. *Then for each* $\beta \leq \alpha$ *a term* s_β *can be constructed by outermost reduction of all descendants of* S *in* t_β *such that* $s_\beta \to_{\leq\omega} s_{\beta+1}$ *via outermost reduction of all descendants of* R_β *in* s_β *for each* $\beta \leq \alpha$ *and all these reductions together form a strongly converging reduction from* s_0 to s_α. *Moreover, if* t_α *is a normal form, then* s_α *and* t_α *are equal.*

$$\begin{tikzcd}
t_0 \arrow[r, "R_0"] \arrow[d, "S"', "\leq\omega"] & t_1 \arrow[r] \arrow[d, "\leq\omega"] & \cdots\cdots\ t_\beta \arrow[r, "R_\beta"] \arrow[d, "\leq\omega"] & t_{\beta+1}\ \cdots\cdots \arrow[d, "\leq\omega"] & t_\alpha \arrow[d, "\leq\omega"] \\
s_0 \arrow[r, "\leq\omega"'] & s_1 \arrow[r, "\leq\omega"'] & \cdots\cdots\ s_\beta \arrow[r, "\leq\omega"'] & s_{\beta+1}\ \cdots\cdots & s_\alpha
\end{tikzcd}$$

(Figure 4.1)

PROOF. First note that outermost reduction of a finite or an infinite number of disjoint redexes in some term gives a strongly converging reduction. hence all vertical reductions in Figure 4.1 are strongly converging.

We prove the lemma by induction on the ordinal α. The case with zero is easy.

Next, let α be of the form $\beta+1$. Assume as induction hypothesis that we have the Infinitary Parallel Moves Lemma for $\gamma \leq \beta$. It suffices to show that in the following Figure (4.2) the rightmost square can be constructed and indeed is commutative. As in the traditional proof of the finitary Parallel Moves Lemma this can be proved by an easy analysis of the position of R_β with the positions of the disjoint (this is where orthogonality comes in) descendants of S in t_β. If the reduction from s_0 to s_β is strong converging then composition with the strong convergent reduction from s_β to $s_{\beta+1}$ gives a strong converging reduction from s_0 to $s_{\beta+1}$.

$$\begin{tikzcd}
t_0 \arrow[r, "R_0"] \arrow[d, "S"', "\leq\omega"] & t_1 \arrow[r] \arrow[d, "\leq\omega"] & \cdots\cdots\ t_\beta \arrow[r, "R_\beta"] \arrow[d, "\leq\omega"] & t_{\beta+1} \arrow[d, "\leq\omega"] \\
s_0 \arrow[r, "\leq\omega"'] & s_1 \arrow[r, "\leq\omega"'] & \cdots\cdots\ s_\beta \arrow[r, "\leq\omega"'] & s_{\beta+1}
\end{tikzcd}$$

(Figure 4.2)

Finally, let α be a limit ordinal λ. Assume as induction hypothesis that we have the Infinitary Parallel Moves Lemma for $\beta < \lambda$. There are two possibilities: there exists a $\beta < \lambda$ such that the actual length of the reduction sequence $t_\beta \rightarrow_{\leq\omega} s_\beta$ is zero, that is there are no descendants of S in t_β, or there is no such β. The first possibility is easy: we find that $t_\gamma = s_\gamma$ for all γ with $\beta \leq \gamma < \lambda$. It follows that s_0 strongly converges to s_λ.

So let us pursue the second possibility and suppose there is no such β.

Let $(v_\beta)_{\beta\leq\mu}$ be the reduction of the bottom line of Figure 4.1 obtained by refining the sequence $(s_\beta)_{\beta\leq\lambda}$ with reductions $s_\beta \rightarrow_{\leq\omega} s_{\beta+1}$ for each $\beta < \alpha$. That such a μ exists follows by an exercise on well-orderings: refining a well-ordering with well-orderings gives again a well-ordering. In order to conclude $s_0 = v_0 \rightarrow_\mu v_\mu = s_\lambda$ we have to show: (i) the reduction $(v_\beta)_{\beta\leq\mu}$ is strong, (ii) the reduction $(v_\beta)_{\beta\leq\mu}$ is converging.

PROOF OF (i): By induction clause in the definition of strong sequence we only have to show $\forall d>0$ $\exists\beta<\mu\ \forall\gamma\ (\beta<\gamma<\mu \rightarrow d v_\gamma > d)$ to conclude that $(v_\beta)_{\beta\leq\mu}$ is strong.

Observe that the depth of the redexes contracted in $s_\beta \rightarrow_{\leq\omega} s_{\beta+1}$ (the descendants of redex R_β

under $t_\beta \to_{\leq\omega} s_\beta$) is at least $dt_\beta - h$, where dt_β is the depth of R_β in t_β and h is the maximal distance in the left-hand side of the rule applied to R from its root to any variable. As the depth of the redexes R_β tends to infinity with β tending to μ we get $\forall d>0\ \exists\beta<\mu\ \forall\gamma\ (\beta<\gamma<\mu \to dv_\gamma>d)$.

PROOF OF (ii): By induction hypothesis it suffices to show that $\forall\varepsilon>0\ \exists\beta<\mu\ \forall\gamma\ (\beta<\gamma<\mu \to d(v_\gamma,s_\lambda) < \varepsilon)$. So, let $\varepsilon > 0$. Let $2^{-k} < \varepsilon$ for some natural number k.

Let $t_\lambda = r_0 \to r_1 \to \ldots \to_{\leq\omega} s_\mu$ be a (possible finite) reduction obtained by outermost contraction of the descendants of R in t_λ. Consider the rule $l \to r$ of which R is a redex. Let h be the maximum of the differences of the depth of a variable in r and the depth of the same variable in l.

For some N large enough we have $d(r_n,s_\lambda) \leq 2^{-k}$ for $n \geq N$. For some ξ large enough all the descendants of S in t_λ contracted in the reduction up to r_{N+1} are present in all t_γ for $\gamma \geq \xi$. For some ζ large enough the redexes reduced in t_γ for $\gamma \geq \zeta$ are at depth larger than k. Hence for $\gamma \geq \max(\zeta,\xi)$ the initial part of t_γ and t_λ up to level k+1 are equal.

If we now contract the (disjoint!) descendants of R in t_γ and in t_λ, and compare the result s_γ and s_λ, then we see that up to level (k+1)-h the terms s_γ and s_λ are equal. By (ii) we find that for η large enough the depth of the redexes contracted in $v_\gamma \to v_{\gamma+1}$ for $\gamma \geq \eta$ is at least k. So finally if we take $\beta = \max(\zeta,\xi,\eta)$ then up to level (k+1)-h the terms v_γ and s_λ are identical for $\gamma \geq \beta$.

Hence for any $\varepsilon > 0$ there is a β such that for $\beta \leq \gamma < \mu$ the distance of v_γ and s_λ is smaller than ε.

END PROOF OF (ii) □

It seems natural to ask whether an infinitary parallel moves lemma exists for the larger class of converging reductions. The following example shows that the *construction* embodied in the Infinitary Parallel Moves Lemma for strongly converging reductions does not generalize.

4.1.3. COUNTEREXAMPLE.

Rules: $A(x,y) \to A(y,x)$, $C \to D$

Sequences:

$$
\begin{array}{ccccccccc}
A(C,C) & \to & A(C,C) & \to & A(C,C) & \to & A(C,C) & \to \ \ldots \to^{c}_{\omega} & A(C,C) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \not\downarrow \\
A(C,D) & \to & A(D,C) & \to & A(C,D) & \to & A(D,C) & \to \ \ldots & \text{NO LIMIT}
\end{array}
$$

The bottom infinite reduction obtained by standard projection over the one step reduction $C \to D$ does not converge to any limit. □

Note that this example is a counterexample not to the Parallel Moves Lemma, but to a method of proving it. It might be possible that by altering the construction, perhaps by considering a more liberal notion of descendant, the parallel moves lemma holds for transfinite converging reductions. After all, every term occurring in the counterexample can reduce to A(D,D).

For the sake of the next section we now state a special instance of the infinitary parallel moves lemma which is valid for left-linear TRSs.

4.1.2. SPECIAL INFINITARY PARALLEL MOVES LEMMA. *For left-linear TRSs, let* $t_0 \to_\alpha t_\alpha$ *be a strongly converging reduction sequence of* t_0 *with limit* t_α *and let* $t_0 \to s_0$ *be a reduction of a redex* S *of* t_0. *If the set of descendants of* S *in* t_β *contains at most one element then the construction of the Infinitary Parallel Moves Lemma can be performed.* □

4.2. The Compression Lemma

In this section we will prove the Compression Lemma for infinitary left-linear TRSs: if $t \rightarrow_{\alpha} s$ is strongly converging, then $t \rightarrow_{\leq\omega} s$. That is: any strongly converging reduction from t into s of length α can be compressed in a closed reduction of length less or equal than ω. Dershowitz, Kaplan and Plaisted were the first to conjecture and partially prove compressing lemmas. In their final paper [Der90b] they prove using an elegant topological argument that in a left-linear topterminating rewrite system the Compression Lemma holds. It is not difficult to see that their result and argument remain true under the weaker assumption that converging reductions are strong. (That this is indeed a weaker assumption we will explain in section 5.3.) However, it is more apt and general to state and prove the Compression Lemma for strongly converging reductions. We will present a different proof for this lemma, explicitly based on the strong convergence properties of the reduction.

The following Table (4.1) collects counterexamples against some other conditions for the Compression Lemma:

Validity of Compression Lemma under various conditions:

	converging	strongly converging
left-linear	NO overlapping: [Far89], (4.2.1.i) non-overlapping: (4.2.1.i)	YES (4.2.5)
non-left-linear	NO [Der89a], (4.2.1.i)	NO [Der89a]
	[Der89a] is presented in (4.2.1.ii)	

(Table 4.1)

4.2.1. COUNTEREXAMPLES.

(i) Example against a compressing lemma for converging reductions in orthogonal TRSs.

Rules: $A(x) \rightarrow A(B(x))$, $B(x) \rightarrow E(x)$.

Sequence: $A(C) \rightarrow_{\omega} A(B(B^{\omega})) \rightarrow A(E(B^{\omega}))$.

Note: $A(C)$ cannot reduce to $A(E(B^{\omega}))$ in $\leq \omega$ steps. The reduction is converging but not strong.

(ii) Example of [Der89a] against a compressing lemma for strongly converging reductions in non-left-linear TRSs.

Rules: $A \rightarrow S(A)$, $B \rightarrow S(B)$, $H(x,x) \rightarrow C$.

Sequence: $H(A,B) \rightarrow^{*} H(S(A),S(B)) \rightarrow^{*} H(S(S(A)),S(S(B))) \rightarrow_{\omega} H(S^{\omega},S^{\omega}) \rightarrow C$.

Note: The term $H(A,B)$ of Dershowitz and Kaplan (cf. [Der89a]) can reduce via the limit $H(S^{\omega},S^{\omega})$ to C. But not $H(A,B) \rightarrow_{\leq\omega} C$. The sequence is strongly converging.

The proof of the Compression Lemma goes in two steps. First we compress the closed reduction up to the last limit ordinal to a closed reduction of length $\leq \omega$. Then, if necessary, we apply the Compression Lemma for $\omega+1$. The Compression Lemma for $\omega+1$ is simple to prove:

4.2.2. COMPRESSING LEMMA for ω+1. *If* $t \to_{\omega+1} s$ *is strongly converging, then* $t \to_{\leq\omega} s$.

PROOF. Suppose $t_0 \to_\omega t_\omega$ is strongly converging and $t_\omega \to s$. Let the redex R_s contracted in $t_\omega \to s$ have depth S. By strongness there exists an N such that for n≥N the depth of the redex R_n contracted in $t_n \to t_{n+1}$ is larger than S+h, where h is the height of the non-variable part of the redex R_s. The set of descendants in t_m of the copy of R_s in t_N is a singleton for all m>N. We will now construct a strongly converging reduction $t_0 \to_{\leq\omega} s$. For the first N steps we take $t_0 \to t_1 \to \ldots \to t_N$. Then we reduce $t_N \to s_N$ by contracting R_s in t_N. By the Special Infinitary Parallel Moves Lemma 4.1.2 applied to $t_N \to s_N$ and $t_N \to_\omega t_\omega$ we obtain a strongly converging reduction $t \to_{\leq\omega} s$. □

The proof of the Compression Lemma for limit ordinals is more involved and needs some preliminary theory.

4.2.3. LEMMA. *Let* $t_0 \to_\alpha t_\alpha$ *be a strongly convergent reduction. Let* s *be a finite prefix of* t_α. *Then the reduction* $t_0 \to_\alpha t_\alpha$ *can be factorized in a strongly convergent reduction* $t_0 \to^* t_1 \to_\gamma t_\alpha$ *such that all steps in* $t_0 \to^* t_1$ *contribute to the prefix* s *and there are no steps contributing to* s *in* $t_1 \to_\gamma t_\alpha$.

PROOF. By Theorem 3.2.5 there are finitely many steps that contribute to the prefix s. We will handle them one by one. Let R_0 be the contracted redex of the first of these finitely many steps, say in step $t_\beta \to t_{\beta+1}$. If R_0 is not a redex in t_0, then somewhere in the reduction R_0 has been constructed. But then the reduction step using R_0 was not the first reduction step contributing to the finite prefix s. Hence R_0 is a redex of t_0. In $t_0 \to_{\beta+1} t_{\beta+1}$ there are no terms containing multiple copies of R_0 in t_0: otherwise $t_\beta \to t_{\beta+1}$ would not have been the first step contributing to the finite s of t_ω. Also no terms contain no copy of R_0, for the same reason. So we can apply the Special Infinitary parallel Moves Lemma to get a strongly converging reduction $r_0 \to^* r_1 \to^* r_2 \to \ldots r_\beta$, where each r_α is obtained from t_α ($0 \leq \alpha \leq \beta$) by reduction of the unique occurrence of the descendant of the redex R_0. By construction r_β equals $t_{\beta+1}$. Hence we have factorized $t_0 \to_\lambda t_\lambda$ in $t_0 \to r_0 \to_\delta t_\beta \to_\gamma t_\lambda$. Clearly the remaining n-1 steps contributing to the prefix s are performed beyond t_β, so that sufficient repetition of the construction yields the desired factorization. □

4.2.4. COMPRESSING LEMMA for limit ordinals. *If* $t_0 \to_\lambda t_\lambda$ *is strongly convergent, then there exists a strongly convergent reduction* $t_0 \to_{\leq\omega} t_\lambda$.

PROOF. Choose some depth n. Apply Theorem 3.2.5 to find the finitely many steps of $t_0 \to_\lambda t_\lambda$ contributing to occurrences of t_λ at depth ≤ n. With an appeal to Lemma 4.2.3 perform the finitely many contributing steps first to find a strongly converging reduction $t_0 \to^* t_1 \to_\gamma t_\lambda$ where all steps in $t_0 \to^* t_1$ contribute to occurrences of t_λ at depth ≤ n and no steps contribute to occurrences of t_λ at depth ≤ n in $t_1 \to_\gamma t_\lambda$.

Now choose a bigger n and repeat the argument for $t_1 \to_\alpha t_\lambda$, getting a sequence $t_1 \to^* t_2 \to_\beta t_\lambda$ for some $\beta \leq \alpha$. Repeat ad infinitum: we obtain the sequence $t_0 \to^* t_1 \to^* t_2 \to^* \ldots$ which by construction is a strongly converging reduction to t_λ. □

4.2.5. COMPRESSING LEMMA. *For any ordinal* α, *if* $t \rightarrow_\alpha t_\alpha$ *is strongly convergent, then there exists a strongly convergent reduction* $t \rightarrow_{\leq\omega} t_\lambda$.

PROOF. Together 4.2.2 and 4.2.4 establish the Compression Lemma. Every infinite ordinal has the form λ+n, for a limit ordinal λ and a finite n. For any strongly convergent sequence $t \rightarrow_{\lambda+n} t_\alpha$, we apply Theorem 4.2.4 to the first λ steps, to obtain a sequence $t \rightarrow_{\leq \omega+n} t_\alpha$, then apply 4.2.2 n times to obtain $t \rightarrow_{\leq\omega} t_\alpha$. □

4.3. Stable reductions and normal forms

In this section we will show that in orthogonal TRSs reductions to normal form are stable reductions. Our proof will depend on the Infinitary Parallel Moves Lemma.

Recall that a term is a normal form if it contains no redex. We will say that a term t *has a normal form* s if there is a strongly converging reduction starting at t to some normal form s. This will only be an apparent restriction: any reduction from t to a normal form s has to be strongly converging (see 4.3.6).

Dershowitz, Kaplan and Plaisted [Der90b] consider a weaker notion of normal form: the ω-*normal forms*. An ω-normal form is a term such that if this term can reduce, then it reduces in one step to itself. In orthogonal TRS's we will show that the unique normal form property holds (Theorem 4.4.10), and contrastingly that one term can have different ω-normal forms: cf. 5.1.1.

Note that already restricted to finite terms normal forms and ω-normal forms are different concepts. For example, take the TRS consisting of one rule $A \rightarrow A$. The term A is an ω-normal form but not a normal form.

Informally, an infinite reduction will be called *stable* if the sequence of stable prefixes of its terms converges to its limit: a stable prefix of a term t is a prefix of t such that no occurrence of that prefix can become an occurrence of a redex in any strongly converging reduction sequence starting from t. Stable reductions will be strongly converging, but not conversely. In fact stability can be defined more general for a reduction relation extending $\rightarrow$.

The formal definition of stability requires some preliminaries.

4.3.1. DEFINITION. (i) A prefix $s \leq t$ is called *stable* with respect to a strongly converging reduction starting from t if no proper occurrence of s becomes an occurrence of a redex during that reduction.

(ii) A prefix $s \leq t$ is called *stable* if s is stable for all strongly converging reductions starting from t.

4.3.2. PROPOSITION. *In an orthogonal TRS: If a prefix* t *of* t_0 *is stable with respect to a strong reduction from* t_0 *which converges to normal form, then it is stable.*

PROOF. Without loss of generality consider the prefix $F(\Omega,...,\Omega)$ consisting only of the top symbol of $t_0 = F(t_1,...,t_n)$. Assume $F(\Omega,...,\Omega)$ is stable with respect to a strong, closed reduction $\mathcal{B}$, which converges to normal form, say s, and not stable for some other strongly converging $\mathcal{B}'$. By the Compression Lemma we may assume that the length of $\mathcal{B}$ is at most ω. Then at some finite position in $\mathcal{B}$ the symbol F is reducible for the first time. Let $\mathcal{B}^*$ be the finite reduction up to this point. By applying the Infinitary Parallel Moves Lemma repeatedly to $\mathcal{B}$ and $\mathcal{B}^*$ we obtain a strongly convergent reduction of t_0 to the same normal form s, which does not reduce F, and in which the terms after $\mathcal{B}^*$

all have the prefix $F(\Omega,\ldots,\Omega)$. By orthogonality, the redex at the root of t_0 cannot be destroyed, the redex at F is still present in the normal form s of t_0. Contradiction. Hence such a $\mathcal{B}$ does not exist.

□

4.3.3. DEFINITION. Let $\Sigma(t)$ denote the maximal stable prefix of t. A converging reduction $t_0 \twoheadrightarrow_{\leq\omega} t_\omega$ is *stable* if $\forall d \exists N \forall k \geq N \; |\Sigma(t_k)| > d$.

Stability is a very strong condition on reductions. The limit of a stable reduction sequence of length ω is already in normal form, as there is no redex in any finite prefix of the limit of the stable sequence, there is no redex in the limit at all. Hence, stable reductions don't exist of length greater than ω. We also conclude that stable reductions have the infinitary Church-Rosser property.

The proof of the following lemma is routine and therefore omitted.

4.3.4. LEMMA. (i) *If* $t \twoheadrightarrow s$ *then* $\Sigma(t) \leq \Sigma(s)$.

(ii) *For reductions: stable* $\Rightarrow$ *strongly convergent* $\Rightarrow$ *convergent. But not conversely.*

(iii) *The limit of a stable reduction sequence is a normal form.* □

4.3.5. COROLLARY. *The infinite Church-Rosser property holds for stable reductions.* □

4.3.6. THEOREM. *The following are equivalent:*

(i) $t \twoheadrightarrow_{\leq\omega} s$ *is a converging reduction to normal form,*

(ii) $t \twoheadrightarrow_{\leq\omega} s$ *is a strongly converging reduction to normal form,*

(iii) $t \twoheadrightarrow_{\leq\omega} s$ *is a stable reduction*

PROOF. It is trivial to see that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i).
(i) $\Rightarrow$ (ii): Let $t \twoheadrightarrow_{\leq\omega} s$ be converging to normal form. Suppose it is not strongly converging to normal form. Then there must be some depth d such that from some t_i onwards, every term has a redex at depth d. Since arities are finite, this implies that at some occurrence u, infinitely many reductions are performed. But then convergence implies that u is also an occurrence of a redex in the limit, contrary to hypothesis. (In fact, the implication is still true when operators of infinite arity are allowed.)
(ii) $\Rightarrow$ (iii): Let $t \twoheadrightarrow_{\leq\omega} s$ be a strongly converging reduction normal form. Let p_i be the largest prefix of the i'th term t_i which is stable with respect to the remainder of the sequence. Then by Proposition 4.3.2, p_i is equal to the largest stable prefix $\Sigma(t_i)$ of t_i. Since the sequence is strongly convergent, the depths of the prefixes p_i grow without bound, hence the sequence $t \twoheadrightarrow_{\leq\omega} s$ is stable. □

The equivalence of (i) and (ii) of the previous theorem extends to reductions whose length can be measured by a limit ordinal.

We end this section with a more general definition of stable reduction. Some of its instances will be stronger than the just defined notion. This part of the section may be skipped, as it is not used in the sequel.

4.3.9. DEFINITION. Let R denote some reduction relation on a set of terms. $\Sigma_R(t)$ denotes the maximal stable prefix of t with respect to R-reductions.

4.3.10. DEFINITION. A reduction (t_i) w.r.t. a reduction relation R is called R-*stable* if $\forall d \exists N \forall k \geq N$

$|\Sigma_R(t_i)| > d$.

The proof of the following lemma is routine and therefore omitted.

4.3.11. LEMMA. *Let* R *be a reduction relation on a set of terms.*

(i) $t \rightarrow_R s \Rightarrow \Sigma_R(t) \leq \Sigma_R(s)$.

Let S *be a reduction relation on the same set of terms.*

(ii) *if* $R \supseteq S$ *then* $\Sigma_R(t) \leq \Sigma_S(t)$ *and if also* (t_i) *is* R-*stable then* (t_i) *is* S-*stable.* □

4.3.12. DEFINITION. (i) The reduction relation $\rightarrow_?$ (arbitrary reduction) is defined as $C[s] \rightarrow_? C[t]$ for every context C[ ], redex s and arbitrary term t.

(ii) The reduction relation $\rightarrow_!$ (less arbitrary reduction) is defined by $C[s] \rightarrow_? C[r(t_1,...,t_n)]$ for every context C[ ], redex s and arbitrary terms $t_1, ..., t_n$ substituted for variables in the right-hand side $r(x_1,...,x_n)$ of the rule for which s is a redex.

The proof of the following lemma is routine and therefore omitted.

4.3.13. LEMMA.

(i) $\rightarrow \subseteq \rightarrow_! \subseteq \rightarrow_?$;

(ii) $\rightarrow_?$-stable $\Rightarrow$ $\rightarrow_!$-stable $\Rightarrow$ $\rightarrow$-stable $\Rightarrow$ strongly convergent $\Rightarrow$ convergent. □

We have called $\rightarrow$-stability just stability. The stronger notions based on $\rightarrow_!$- and $\rightarrow_?$-reduction arise naturally from the studies of sequentiality by Huet and Lévy [Hue79], for $\rightarrow_!$, and Oyamaguchi [Oya87], for $\rightarrow_?$. We introduce them only to show that they are in fact strictly stronger than stability, as is demonstrated by the following examples. One might initially expect that for strongly sequential systems, the three would coincide, and that for sufficiently sequential systems (defined by [Oya87]), stablility and $\rightarrow_?$-stability would be the same. However, the following examples contradict that expectation: each uses an orthogonal, strongly sequential TRS.

4.3.14. EXAMPLES.

(i) convergence $\not\Rightarrow$ strong convergence

Rule: $A \rightarrow A$

Sequence: $A \rightarrow A \rightarrow A \rightarrow \ldots$

Note: The reduction is convergent, but the depth of each reduction step is 0.

(ii) strong convergence $\not\Rightarrow$ stability

Rules: $A \rightarrow B(A)$, $C(x) \rightarrow D$

Sequence: $C(A) \rightarrow C(B(A)) \rightarrow C(B(B(A))) \rightarrow C(B(B(B(A)))) \rightarrow \ldots$

Note: This is strongly convergent, but the stable prefix of each term in the reduction is just Ω (since each term is a redex).

(iii) Stability $\not\Rightarrow$ $\rightarrow_?$-stability

Rules: $A(D(E),x) \rightarrow x$, $C(x) \rightarrow A(x,C(x))$

Sequence: $C(D(F)) \rightarrow A(D(F),C(D(F))) \rightarrow A(D(F),A(D(F),C(D(F)))) \rightarrow \ldots$

Note: The stable prefixes of the terms in the reduction are Ω, $A(D(F),\Omega)$, $A(D(F),A(D(F),\Omega))$, ..., but the $\rightarrow_?$-stable prefixes of the terms are all Ω.

(iv) $\rightarrow_?$-stability $\not\Rightarrow$ $\rightarrow_!$-stability

Rules: $A(B) \to D$, $C \to A(C)$

Sequence: $C \to A(C) \to A(A(C)) \to A(A(A(C))) \to \ldots$

Note: The $\to_?$-stable prefixes of the terms in the reduction are Ω, $A(\Omega)$, $A(A(\Omega))$, ..., but the $\to_!$-stable prefixes of the terms are all Ω.

4.4. Unique normal form property

We are now ready to prove the *unique normal form property* for infinitary orthogonal TRSs. By way of contrast one should observe that in the orthogonal TRS given by the rules $A(x) \to x$, $B(x) \to x$ and $C \to A(B(C))$ the term C has two *different* ω*-normal forms*: A^ω and B^ω (see for more details 5.1.1).

4.4.1. UNIQUE NORMAL FORM PROPERTY. *Normal forms are unique in infinitary orthogonal TRSs.*

PROOF. Suppose a term t admits two converging reductions $t \to s_1 \to s_2 \to \ldots \to^{c}_{\leq\omega} s$ and $t \to r_1 \to r_2 \to \ldots \to^{c}_{\leq\omega} r$ to normal form. By Theorem 4.3.6 these reductions are stable. By the finite Church-Rosser property, for each n there exists u_n such that $s_n \to^* u_n$ and $r_n \to^* u_n$. Hence we get a reduction $t \to^* u_1 \to^* u_2 \to^* \ldots$. Using Lemma 4.3.4 (i) the newly constructed reduction $(u_n)_{n\in\mathbb{N}}$ inherits its stability from the stable reductions $(s_n)_{n\in\mathbb{N}}$ and $(r_n)_{n\in\mathbb{N}}$. Thus we see by Theorem 4.3.6 that the limit u of (u_n) is a normal form. Once more applying Lemma 4.3.5 (i) we see that $\Sigma(s_n) \leq \Sigma(u_n)$ and $\Sigma(r_n) \leq \Sigma(u_n)$. Hence $s \equiv \lim_{n\to\infty} \Sigma(s_n) \leq \lim_{n\to\infty} \Sigma(u_n) \equiv u \geq \lim_{n\to\infty} \Sigma(r_n) \equiv r$. Since normal forms are maximal in the prefix ordering (in contrast to ω-normal forms) s and r are equal. □

We obtain the following useful theorem as a corollary:

4.4.2. THEOREM. *Any strongly convergent reduction starting from a term having normal form can be extended to a strongly convergent reduction ending in that normal form.*

PROOF. Let $t \to_{\leq\omega} s$ be a stable reduction of t to normal form. By the compressing Lemma it suffices to consider a strongly convergent reduction $t \to_{\leq\omega} r$ of length lesser than or equal to ω. Apply the Parallel Moves Lemma to $t \to_{\leq\omega} r$ and each step of $t \to_{\leq\omega} s$ in order to construct an infinitary reduction $r \to_{\leq\omega^2} u$. This reduction must be strongly converging, because $t \to_{\leq\omega} s$ is stable. Let its limit be u. This u has to be a normal form. Apply the Compression Lemma to $r \to_{\leq\omega^2} u$ to obtain a strong convergent reduction $r \to_{\leq\omega} u$. Now $t \to_{\leq\omega} r \to_{\leq\omega} u$ is also strongly convergent. By the unique normal form property we see that s and u must be equal. □

4.5. Fair reductions

Theorem 4.3.6 implies that for orthogonal TRSs stable converging infinite reductions result in normal forms. If we add a fairness condition to strongly converging reductions, then their limits will also be normal forms. Similarly, the same fairness condition added to converging reductions results in converging reductions to ω-normal form [Der89b]. Fairness of a reduction will express that, whenever a redex occurs in a term during this reduction, the redex itself or a term containing the redex will be reduced within a finite number of steps.

4.5.1. DEFINITION. (i) Let r be a redex of t at occurrence u. A reduction $t \rightarrow_{\leq\omega} t'$ *preserves* r if no step of this performs a contraction at an occurrence $\leq u$.

(ii) A reduction $t \rightarrow_{\leq\omega} t'$ is *fair* if for every term t" in the reduction, and every redex r of t" some finite part of the reduction starting at t" does not preserve r.

Note that a finite sequence is fair if and only if it ends in a normal form, and fair reductions don't need to be converging (for example, think of the reduction $A \rightarrow B \rightarrow A \rightarrow B \rightarrow \ldots$). Note also that orthogonality guarantees that if the reduction $t \rightarrow_{\leq\omega} t'$ preserves a redex in t of a certain rule, then t' contains a redex of the same rule.

4.5.2. THEOREM. (i) [Der89b] *The limit of a fair, converging reduction is an ω-normal form.*

(ii) *The limit of a fair, strongly converging reduction is a normal form.*

PROOF. By the previous remark we only have to consider sequences of length ω.

(i) Consider the limit of a fair, converging reduction. If it contains no redexes then the limit is a normal form and *a fortiori* an ω-normal form. So let us suppose the limit contains a redex. Assume that contraction of the redex results in a term that differs at depth n with the limit. By convergence there is a point in the reduction such that all later terms in the sequence have the same initial part up to depth n+1. By fairness, it follows that there will be a later point in the reduction where the redex is contracted. At that point k we see that the initial part of the k-th term up to level n+1 is equal to the similar initial parts of further terms. Hence in the limit there can be no difference at depth n. Contradiction. Therefore contraction of the redex in the limit results in the limit itself.

(ii) Use (i): strong convergence and fairness rule out that the limit reduces to itself. □

4.5.3. COROLLARY. (i) *If a reduction sequence is fair and convergent then it is ω–stable.*

(ii) *A reduction sequence is fair and strongly convergent if and only if it is stable.*

PROOF. The proof of (i) is similar to "only if" part in the proof of (ii):

(ii) "If": by Theorem 4.3.6, stable reductions end in normal form; hence by an easy *reductio ad absurdum* stable reductions are fair. "Only if": by Theorem 4.5.2, any fair and strongly convergent reduction ends in normal form; hence by Theorem 4.3.6 the reduction is stable. □

The converse of 4.5.3.i does not hold: consider for example the TRS with two rules: $A(x) \rightarrow A(x)$ and $B \rightarrow B$. Then $A(B) \rightarrow^{c}_{\leq\omega} A(B)$ via just B reductions. This reduction is ω-stable but not fair.

5. THE INFINITE CHURCH-ROSSER PROPERTY

The finite Church-Rosser property holds for infinitary orthogonal TRSs as it holds for finitary orthogonal TRSs. One might check that the usual proofs go through verbatim. Or one might realize that we have proved the Parallel Moves Lemma for strongly converging reductions of any ordinal length, in particular of finite length. Finite reductions are strongly converging. Repeated application of the Parallel Moves Lemma then gives the finite Church-Rosser property for infinitary orthogonal TRSs.

Perhaps the reader would have expected a treatment of the infinite Church-Rosser Property,

$$\leftarrow_\omega \circ {}_\omega\!\!\rightarrow \subseteq {}_{\leq\omega}\!\!\rightarrow \circ \leftarrow_{\leq\omega},$$

in the previous section on fundamental facts of infinitary rewriting. The reason for this omission is that in arbitrary orthogonal TRSs the infinite Church-Rosser property fails for strongly converging reductions as well as for converging reductions. Only for stable reductions the infinite Church-Rosser property holds because of the unique normal form property.

In the present section we investigate to what extent the infinite Church-Rosser property is valid for strongly converging reductions. For strongly converging reductions we will prove the infinite Church-Rosser property for *depth-preserving* OTRSs directly. Then using a technique based on Park's notion of hiaton and requiring König's Lemma we generalize this result to those OTRSs *with no collapsing rules other than possibly* $I(x) \rightarrow x$. With respect to strongly converging reductions this result is optimal.

In non-unifiable OTRSs any converging reduction is strongly converging. With help of this observation we will show the infinite Church-Rosser property for converging reductions in *non-unifiable* OTRSs. This improves and explains the infinite Church-Rosser property implicit in [Der90b].

The infinitary Church-Rosser property for converging reductions in OTRSs with no collapsing rules other than possibly $I(x) \rightarrow x$ is a wide open problem, even already in the situation of one ω long converging reduction versus a one step reduction. However this problem is in our view not in the main stream of what we perceive as the "canonical development" of infinitary term rewriting.

We will finish the section with an analysis of stable forms, Böhm trees and Böhm reduction. We will prove for arbitrary OTRSs that any term has a convergent reduction to stable form, and that the infinite Church-Rosser property holds for (in-)finite Böhm reductions. This can be interpreted as another solution in the quest for a general Church-Rosser property. The possibility of identifying subterms that cause "bad" behavior, ie. subterms without head normal form, during a reduction by replacing them by a new symbol $\perp$ circumvents the counterexamples to the Church-Rosser property.

5.1. Failure of the infinite Church-Rosser Property for orthogonal TRSs

The following counterexamples show that the infinite Church Rosser property does not hold even for strongly converging reductions of length ω.

5.1.1. COUNTEREXAMPLES.

(i) Rules:

$A(x) \rightarrow x,$

$B(x) \rightarrow x,$

$C \rightarrow A(B(x))$

Sequences:

$C \rightarrow A(B(C)) \rightarrow A(C) \rightarrow A(A(B(C))) \rightarrow A(A(C)) \rightarrow_\omega A^\omega$

$C \rightarrow A(B(C)) \rightarrow B(C) \rightarrow B(A(B(C))) \rightarrow B(B(C)) \rightarrow_\omega B^\omega$

Hence $C \rightarrow_{\leq\omega} A^\omega$ as well as $C \rightarrow_{\leq\omega} B^\omega$.

But there is no term t such that $A^\omega \rightarrow_{\leq\omega} t \leftarrow_{\leq\omega} B^\omega$ be it converging or strongly converging.

(ii) Rules: $D(x,y) \rightarrow x$, $C \rightarrow D(A,D(B,C))$

Sequences:

$$C \to D(A,D(B,C)) \to D(A,C) \to^* D(A,D(A,C)) \to^* D(A,D(A,D(A,C))) \to ...$$
$$C \to D(A,D(B,C)) \to D(B,C) \to^* D(B,D(B,C)) \to^* D(B,D(B,D(B,C))) \to ...$$

It is not possible to join the limits of these two sequences.

5.2. Depth preserving orthogonal term rewriting systems

What to do about this failure of the infinite Church-Rosser property? We can think of some *prima vista* solutions. For instance, using the unique normal form property, it is not difficult to see that the infinite Church-Rosser property holds for (strongly) converging reductions starting with a term that has a normal form. Or another rather weak way out is the restriction to stable reductions: the infinite Church-Rosser property clearly holds for stable reductions by, again, courtesy of the unique normal form property. These solutions are however rather restricted.

In the present section and the next we will consider two natural classes of orthogonal TRSs for which the infinite Church-Rosser property holds for strongly convergent sequences without extra conditions.

5.2.1. DEFINITION. A *depth preserving* TRS is a left linear TRS such that for all rules the depth of any variable in a right-hand side is greater than or equal to the depth of the same variable in the corresponding left-hand side.

For example, the rules $A(x) \to x$ and $B(A(z),C(y,A(x))) \to D(A(x),x)$ are not depth preserving.

5.2.2. LEMMA. *Depth preserving TRSs are distance preserving in the following sense: Let* $l \to r$ *be a depth-preserving rule. Then for all contexts* $C[\ ]$, *all* $t_1,...,t_n$ *and* $s_1,...,s_n$ *it holds that*

$$d(C[l(t_1,...,t_n)],\ C[l(s_1,...,s_n)]) \geq d(C[r(t_1,...,t_n)],\ C[r(s_1,...,s_n)]). \qquad \square$$

We recall a useful lemma by Farmer and Watro (cf. [Far89]):

5.2.3. LEMMA OF FARMER AND WATRO [Far89]. *Let* $t_{n,0} \to_{\leq\omega} t_{n,\omega} = t_{n+1,0}$ *be strongly converging for all* $n \in \mathbb{N}$. *Let* $d_{n,k}$ *denote the depth of the contracted redex* $R_{n,k}$ *in* $t_{n,k} \to t_{n,k+1}$. *If for all* n *there is a* d_n *such that for all* k *it holds that* $d_{n,k} > d_n$, *and* $\lim d_k = \infty$, *then there exists a term* $t_{\omega,\omega}$ *such that* $t_{0,0} \to_{\omega\omega} t_{\omega,\omega}$ *via the strongly converging reduction* $t_{0,0} \to_{\leq\omega} t_{0,\omega} = t_{1,0} \to_{\leq\omega} t_{1,\omega} = t_{2,0} \to_{\leq\omega} ... \to_{\omega\omega} t_{\omega,\omega}$.

$\square$

5.2.4. THEOREM. *Any depth preserving orthogonal TRS has the infinite Church Rosser Property for strongly converging sequences.*

PROOF. Let $t_{0,0} \to t_{0,1} \to ... \to_{\leq\omega} t_{0,\omega}$ and $t_{0,0} \to t_{1,0} \to ... \to_{\leq\omega} t_{\omega,0}$ be strongly convergent.

(i) Using the infinite Parallel Moves Lemma for strongly convergent reductions we construct the horizontal strongly converging sequences $t_{n,0} \to^* t_{n,1} \to^* ... \to_{\leq\omega} t_{n,\omega}$ as depicted in Figure 5.1. The vertical reductions are constructed similarly.

$$\begin{tikzcd}
t_{0,0} \arrow[r] \arrow[d] & t_{0,1} \arrow[r] \arrow[d, "*"] & \ldots \arrow[r] & t_{0,m} \arrow[r] \arrow[d, "*"] & \ldots \arrow[r, "\leq\omega"'] & t_{0,\omega} \arrow[d, "\leq\omega"] \\
t_{1,0} \arrow[r, "*"] \arrow[d] & t_{1,1} \arrow[r, "*"] \arrow[d, "*"] & \ldots \arrow[r, "*"] & t_{1,m} \arrow[r, "*"] \arrow[d, "*"] & \ldots \arrow[r, "\leq\omega"'] & t_{1,\omega} \arrow[d, "\leq\omega"] \\
\vdots \arrow[d] & \vdots \arrow[d, "*"] & & \vdots \arrow[d, "*"] & & \vdots \arrow[d, "\leq\omega"] \\
t_{n,0} \arrow[r, "*"] \arrow[d] & t_{n,1} \arrow[r, "*"] \arrow[d, "*"] & \ldots \arrow[r, "*"] & t_{n,m} \arrow[r, "*"] \arrow[d, "*"] & \ldots \arrow[r, "\leq\omega"'] & t_{n,\omega} \arrow[d, "\leq\omega"] \\
\vdots \arrow[d, "\leq\omega"] & \vdots \arrow[d, "\leq\omega"] & & \vdots \arrow[d, "\leq\omega"] & & \vdots \arrow[d, "\leq\omega"] \\
t_{\omega,0} \arrow[r, "\leq\omega"'] & t_{\omega,1} \arrow[r, "\leq\omega"'] & \ldots \arrow[r, "\leq\omega"'] & t_{\omega,m} \arrow[r, "\leq\omega"'] & \ldots \arrow[r, "\leq\omega"'] & t_{\omega,\omega}
\end{tikzcd}$$

(figure 5.1)

(ii) The construction of the infinite Parallel Moves Lemma also implies that the reduction $t_{n,\omega} \rightarrow_{\leq\omega} t_{n+1,\omega}$ is strongly converging.

(iii) By the depth preserving property it holds for all $m,n \in \mathbb{N}\cup\{\omega\}$ the depth of the reduced redexes in $t_{n,m} \rightarrow^* t_{n,m+1}$, which are all descendants of the redex $R_{0,m}$ in $t_{0,m} \rightarrow t_{0,m+1}$, is at least the depth of $R_{0,m}$ itself. Because $t_{0,0} \rightarrow t_{0,1} \rightarrow \ldots \rightarrow_{\leq\omega} t_{0,\omega}$ is strongly convergent we find by Farmer's and Watro's Lemma 5.2.3 that $t_{\omega,0} \rightarrow_{\leq\omega} t_{\omega,1} \rightarrow_{\leq\omega} t_{\omega,2} \ldots$ is strongly converging. Let us call its limit $t_{\omega,\omega}$.

(iv) In the same way the terms $t_{n,\omega}$ are part of a strongly converging sequence. The limit of this sequence is also equal to $t_{\omega,\omega}$, as can be seen with the following argument.

Let $\varepsilon > 0$. There is N_1 such that for all $m \geq N_1$ we have $d(t_{\omega,m},t_{\omega,\omega}) < \frac{1}{3}\varepsilon$.

Because of the strong convergence of $t_{0,0} \rightarrow t_{1,0} \rightarrow \ldots \rightarrow_{\leq\omega} t_{\omega,0}$ there is an N_2 such that for $n \geq N_2$ we have that $2^{-d_n} < \frac{1}{3}\varepsilon$ where d_n is the depth of the redex R_n reduced at step $t_{n,0} \rightarrow t_{n+1,0}$. Since the depth of the descendants of this redex R_n occur at least at the same depth, and since the TRS is the depth preserving we get $d(t_{\omega,m},t_{n,m}) < \frac{1}{3}\varepsilon$ for all $m \in \mathbb{N}\cup\{\omega\}$ and all $n \geq N_2$.

For similar reasons there is N_3 such that for all $n \in \mathbb{N}\cup\{\omega\}$ and all $m \geq N_3$ we have that $d(t_{n,\omega},t_{n,m}) < \frac{1}{3}\varepsilon$.

Let N be the maximum of N_1, N_2 and N_3. Then for $n \geq N$ we find

$$\begin{aligned}
d(t_{n,\omega},t_{\omega,\omega}) &\leq d(t_{n,\omega},t_{n,N})+d(t_{n,N},t_{\omega,\omega}) \text{ for any } m \geq N \\
&\leq d(t_{n,\omega},t_{n,N})+d(t_{n,N},t_{\omega,N})+d(t_{\omega,N},t_{\omega,\omega}) \\
&\leq \tfrac{1}{3}\varepsilon+\tfrac{1}{3}\varepsilon+\tfrac{1}{3}\varepsilon \\
&\leq \varepsilon.
\end{aligned}$$

□

5.2.5. REMARK. Observe that in this proof there are two places where it is essential that the reductions are strong convergent. The first is the appeal to the infinite Parallel Moves Lemma. The second is in the argument that the sequences $(t_{\omega,n})$ and $(t_{n,\omega})$ have the same limit.

5.3. Non-collapsing orthogonal term rewriting systems

5.3.1. DEFINITION. A TRS R is *non-collapsing* if there is no rewrite rule in R whose right-hand side is a single variable.

We will show that any non-collapsing orthogonal TRS satisfies the infinitary Church-Rosser property with respect to strong convergence. The proofs will use a variant of Parks notion of hiaton. The idea is to replace a depth-reducing rule like $A(x,B(y)) \to B(x)$ by a depth-preserving variant $A(x,B(y)) \to B(\varepsilon(x))$. In order to keep the rewrite rules applicable to terms involving hiatons, we also have to add many more variants: $A(x,\varepsilon^m(B(y))) \to_\varepsilon B(\varepsilon^{m+1}(y))$ for all $m > 0$. We will call the new TRS the ε-*completion* of the old one.

5.3.2. CONSTRUCTION. Let R be a left-linear TRS. The ε-*completion* R_ε is defined as the TRS $(\Sigma\cup\{\varepsilon\},R_\varepsilon)$. The symbol ε is a fresh unary symbol with respect to R. The TRS R_ε consists of all rewrite rules $l_\varepsilon \to r_\varepsilon$, described as follows. The new left-hand side l_ε is obtained from the left-hand side of a rewrite rule $l \to r$ in R by substituting any proper subterm t (i.e., not a variable, nor l itself) in l by $\varepsilon^n(t)$ for some $n\in \mathbb{N}$. The new right-hand side r_ε is obtained from the corresponding right-hand side r by replacing each occurrence of a variable, say x, by $\varepsilon^m(x)$, where m is the minimum of 0 and the depth of x in l_ε minus the depth of this occurrence of x in r.

The proof of the following proposition is straightforward and omitted.

5.3.3. PROPOSITION. *The ε-completion of an orthogonal TRS is orthogonal and depth preserving.*

□

5.3.4. LEMMA. *Let* R *be an orthogonal TRS with no collapsing rules other than possibly* $I(x) \to x$. *Let* $t \to_\omega s$ *be an infinite, strongly converging reduction of length* ω *in* R. *Let* $t \overset{\varepsilon}{\to}_\omega \varepsilon(s)$ *be the corresponding reduction sequence in* R_ε. *Then*

(i) *there are no branches ending in an infinite string of ε's in the tree representation of* $\varepsilon(s)$,

(ii) $t \overset{\varepsilon}{\to}_\omega \varepsilon(s)$ *is strongly converging.*

Let t *be a term without infinite strings of ε's, and let* $t \to_\omega s$ *be a strongly converging reduction in* R_ε. *Let* $t/\varepsilon \to_\omega s/\varepsilon$ *be the reduction obtained from* $t \overset{\varepsilon}{\to}_\omega s$ *by erasing all finite strings of ε's.*

(iii) *If* t *does not contain an infinite string of ε's then neither does s.*

(iv) $t/\varepsilon \to_\omega s/\varepsilon$ *is strongly converging*

(v) *if* $t \to_\omega s$ *is strongly converging in* R, *then there exists a strongly converging reduction* $t \overset{\varepsilon}{\to}_\omega r$ *in* R_ε *such that erasure of all ε's in* $t \overset{\varepsilon}{\to}_\omega r$ *results again in the sequence* $t \to_\omega s$.

PROOF. (i) In the limit term of a strongly converging R_ε-reduction starting with an ε-free term one easily sees that an infinite string of ε's can only be produced by infinite applications of rules containing no function symbols in the right-hand side. Suppose there is such an infinite string: in the original sequence the collapsing contractions necessary to compute this infinite string must have been applied all at the same occurrence, hence, the reduction is not strongly convergent. Contradiction. Note that this argument remains valid if the initial term t contains finite strings of ε's.

(ii) Trivial, by construction.

(iii) See (i).

(iv) Suppose $t \to_\omega s$ is a strongly convergent sequence R_ε. Let $p\in \mathbb{N}$.

Let q be the minimal natural number below which depth at any branch of s a function symbol F can be found for which there are p function symbols not equal to ε on the branch in between F and the root. Such a number q has to exist, since all infinite branches contain infinitely many function symbols unequal to ε. The construction of q actually involves König's Lemma. If we cut all infinite branches at the point where we count the p^{th} function symbol from the root, we end up with a finitely branching tree with finite branches. Then by the contraposition of König's Lemma there is an upperbound on the length of the branches in the truncated tree. For q we take this upperbound.

Because $t \to_{\omega}^{\varepsilon} s$ is strongly converging we can find an $N \in \mathbb{N}$ such that $d_n > q$ for all $n \geq N$, where d_n is the depth of the redex contracted in the n^{th} step. Clearly, after deleting all ε in t_n and s we get as remaining depth $d_{n/\varepsilon} < 2^{-p}$. Hence $t \to_{\omega}^{\varepsilon} s/\varepsilon$ is strongly convergent.

(v) Let $t \to_{\omega} s$ be strongly converging in R. By imitating the steps of R with corresponding steps in R_ε we can construct a strongly converging reduction $t \to_{\omega}^{\varepsilon} r$ in R_ε. If we now erase all finite strings of ε's in $t \to_{\omega}^{\varepsilon} r$, we obtain again the sequence $t \to_{\omega} s$. □

The results in [Der90a] imply that top-terminating OTRSs, that is OTRSs such that there are no derivations of length ω with infinitely many rewrites at topmost position, satisfy the infinite Church-Rosser property for Cauchy converging reductions: combine Theorem 1, Proposition 2, Theorem 10 (which is true under the condition of top-termination) with Theorem 9 in [Der90a]. We will strengthen this in the next Theorem 5.3.5 to: non-collapsing OTRSs satisfy the infinite Church-Rosser property for strongly converging reductions. This is stronger than the result implicit in [Der90a] because (i) under the assumption of top-termination every Cauchy converging reduction is strongly converging, (ii) any top-terminating infinitary TRS is non-collapsing, as one easily sees. Actually it will follow from our construction that the Church-Rosser property holds for OTRSs with no collapsing rules other than possibly $I(x) \to x$, i.e., a collapsing rule that contains only one variable in its left-hand side (cf. the counterexample in 5.1.1).

5.3.5. THEOREM. *Any orthogonal TRS with no collapsing rules other than possibly* $I(x) \to x$ *satisfies the infinite Church-Rosser Property for strongly converging reductions.*

PROOF. Let R be an OTRS. Construct its ε-completion R_ε. By Theorem 5.2.4 the depth-preserving OTRS R_ε satisfies the infinite Church-Rosser property. So if we start with two strongly convergent reductions $t \to_{\leq\omega} s_1$ and $t \to_{\leq\omega} s_2$, then by Lemma 5.3.4 (ii) we can lift these to strongly converging reductions in R_ε, let us say $t \to_{\leq\omega}^{\varepsilon} r_1$ and $t \to_{\leq\omega}^{\varepsilon} r_2$. By Theorem 5.2.4 we find a join u for the two lifted reductions such that $r_1 \to_{\leq\omega}^{\varepsilon} u$ as well as $r_2 \to_{\leq\omega}^{\varepsilon} u$. Both reduction are strongly convergent. Hence, erasing all finite strings of ε's we see by Lemma 5.3.4 (iv) and (v) that in R the term u/ε is the join of the strongly convergent reductions $t \to_{\leq\omega} s_1$ and $t \to_{\leq\omega} s_2$. □

5.4. Non-unifiable orthogonal TRSs

From the work of Dershowitz, Plaisted and Kaplan on convergent reductions it follows that any left-linear, top-terminating and semi-ω-confluent (terminology to be explained next) TRS satisfies the infinite Church-Rosser property:

$$_{\omega}\overset{c}{\leftarrow} \circ \overset{c}{\to}_{\omega} \subseteq \overset{c}{\to}_{\leq\omega} \circ \, _{\leq\omega}\overset{c}{\leftarrow}$$

(cf. [Der90b]: combine Theorem 1, Proposition 2 with Theorem 9). A TRS is *top-terminating* if there are no top-terminating reductions of length ω, that is reductions with infinitely many rewrites at the root of the initial term of the reduction. Semi-ω-confluence, that is

$$_{<\omega}\leftarrow \circ \overset{c}{\rightarrow}_{\omega} \subseteq \overset{c}{\rightarrow}_{\leq\omega} \circ\, _{\leq\omega}\overset{c}{\leftarrow}$$

holds if the Infinitary Parallel Moves Lemma holds for converging reductions. On the assumption that we are in an orthogonal TRS in which all convergent reductions are strong the infinite Church-Rosser Property holds for this TRS. Top-termination implies this assumption.

Hence in top-terminating orthogonal TRSs the infinite Church-Rosser Property holds. We can prove this result by proving a slightly stronger version using the following syntactic equivalent of the previous assumption.

5.4.1. DEFINITION. A TRS is called *unifiable* if the TRS contains a *unifiable* rule, that is a rule $l \rightarrow r$ such that for some substitution σ with finite and infinite terms for variables we have $l^\sigma = r^\sigma$.

Note that unifiability in the space of finite and infinite terms means unifiability "without the occurs check": the terms I(x) and x are unifiable in this setting, and their most general unifier is the infinite term I^ω. Collapsing rules, i.e. rules whose right-hand side is a variable are unifiable.

5.4.2. LEMMA. *The following are equivalent for an orthogonal TRS:*
(i) *the TRS is non-unifiable,*
(ii) *all convergent reductions of the TRS are strong,*
(iii) *all convergent reductions are top-terminating.*

PROOF. (i) $\Rightarrow$ (ii): If a convergent sequence were not strongly convergent, then there would be some redex in its limit which reduces to itself. But condition (i) rules this out.

(ii) $\Rightarrow$ (iii): By easy contraposition.

(iii) $\Rightarrow$ (i): If an orthogonal TRS is non-unifiable, then one can construct the infinite, convergent and not top-terminating reduction $l^\sigma \rightarrow r^\sigma = l^\sigma \rightarrow l^\sigma \rightarrow \dots$. □

5.4.3. THEOREM. *Any non-unifiable orthogonal TRS has the infinite Church-Rosser Property for converging reductions.*

PROOF. Trivial: since in a non-unifiable OTRS any converging reduction is strongly converging, and a non-unifiable OTRS does not contain collapsing rules we can apply Theorem 5.3.5. □

5.4.4. COROLLARY. *A non-unifiable orthogonal TRS has the unique ω-normal form property.*

□

5.4.5. OPEN PROBLEM. Is it possible to weaken the condition non-unifiable to non-collapsing with the usual exception of allowing a single collapsing rule $I(x) \rightarrow x$?

The problem is related to 5.6.13. But as pointed out in the introduction to this chapter the relevance of the problem for the general theory of infinitary term rewriting is not clear.

5.5. Head normal forms, top-termination and stable prefixes

As in Lambda Calculus (cf. [Bar84]) it is possible to introduce a notion of head normal form in term rewriting. We study head normal forms as stepping stone to Böhm reduction.

5.5.1. DEFINITION. (i) A term t is a *head normal form* if it has a stable root.
(ii) A term t has a *head normal form* if there is a finite reduction from t to a head normal form.

The restriction to finite reductions in part (ii) of this definition is technically convenient. In fact it can be weakened to converging reductions:

5.5.2. LEMMA. *If there is a converging reduction from a term* t *to a term* s *in head normal form, then there exist a finite reduction from* t *to some term* s' *with the same root as* s.

PROOF. Suppose $t_0 \xrightarrow{c}_{\alpha} t_\alpha$ for some ordinal $\alpha \geq \omega$. Then $\alpha = \lambda+n$. Suppose t_α is a head normal form. By working backwards from the root prefix $s_{\lambda+n}$ of $t_{\lambda+n}$ we will construct a prefix s_λ of t_λ such that any term with prefix s_λ reduces in less than n+1 steps to head normal form.

Suppose we have constructed $s_{\lambda+n-i}$. Then we construct $s_{\lambda+n-(i+1)}$ as follows. Consider the reduction step $t_{\lambda+n-(i+1)} \rightarrow t_{\lambda+n-i}$. If the rule applied in this step is a collapsing rule, say for example $A(x,y) \rightarrow x$, then we take $s_{\lambda+n-(i+1)} = A(s_{\lambda+n-i},\Omega)$. If the applied rule was not a collapsing rule, then we proceed as follows. If $s_{\lambda+n-i}$ shares symbols with the non-variable part of the right-hand side of the applied rule, then we replace the fragment of the right-hand side with the whole pattern of the left-hand side. Otherwise we take $s_{\lambda+n-(i+1)} = s_{\lambda+n-i}$. It is clear that any term with prefix s_λ can reduce in less than n+1 steps to head normal form.

By convergence the constructed prefix s_λ of t_λ is prefix of some t_β with $\beta < \lambda$. So from t_β it is possible to reach head normal form in less then n steps. Hence we can reach head normal form in less then α steps, say α'. We repeat this argument as long as $\alpha' \geq \omega$. By well-foundedness of the ordinals, we can not repeat this argument ad infinitum. Hence by repeating the argument as long as possible we eventually find a finite reduction from t to head normal form. □

5.5.3. LEMMA. *If a term has a head normal form, then all its head normal forms have the same root symbol.*

PROOF. By the previous Lemma 5.5.2 and the finite Church-Rosser property. □

5.5.4. COROLLARY. *If* t *strongly converges to* t', *then* t *has a head normal form iff* t' *has a head normal form.*

PROOF. "Only if": Suppose t reduces to t', and t reduces in finitely many steps to head normal form t". Then by the infinite Church-Rosser property for strongly convergent reductions, there is a strongly converging reduction from t" to a common reduct of t' and t". Clearly this common reduct is a head normal form. By Lemma 5.5.2 there is also a finite reduction from t" to head normal form.

"If": Suppose t reduces to t' and t' reduces in finitely many steps to head normal form t". The combined reduction from t via t' to t" is strongly convergent. Hence by Lemma 5.5.2 there is a finite reduction to head normal from t. □

The concept "having a head normal form" is equivalent to the concept "being top-terminating" of Dershowitz, Kaplan and Plaisted.

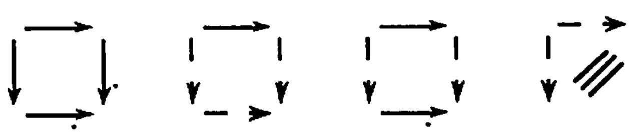
5.5.5. DEFINITION. (i) A converging reduction is *top-terminating* (cf. [Der90a]) if it contains at most finitely many root-reductions (reductions which take place at the root).

(ii) A term is *top-terminating* (cf. [Der90a]) if any reduction $t = t_0 \to t_1 \to t_2 \to \ldots$ not longer than ω reduces only finitely often at root level.

5.5.6. LEMMA. *If* $t \to_{<\omega} t'$, *and* t' *is top-terminating, then* t *is top-terminating.*

PROOF. It suffices to prove: if t reduces in one step to t', and t' is top-terminating, then t is top-terminating.

Consider a non-top-terminating reduction $\mathcal{R}$: $t \to_\omega t''$. We shall construct a non-top-terminating reduction from t'. Apply the Infinitary Parallel Moves Lemma to $\mathcal{R}$ and an one step reduction $\mathcal{S}$: $t \to t'$. There are four elementary one-one step Church Rosser situations, as in the figure:



The dashed arrows are root contractions.

It follows by an easy argument that $\mathcal{R}/\mathcal{S}$ has at least as many root contractions as $\mathcal{R}$. Hence if $\mathcal{R}$ is non-top-terminating so is $\mathcal{R}/\mathcal{S}$. □

5.5.7. THEOREM. *The following are equivalent for any term* t:

(i) t *has finite reduction to head normal form.*

(ii) t *is top-terminating.*

PROOF. (i) ⇒ (ii) (We will need this in the sequel.) Suppose t reduces in finitely many steps to head normal form t'. Being a head normal form t' is top-terminating. By the previous Lemma 5.5.6 we find that t itself is top-terminating.

(ii) ⇒ (i) Trivial, by contradiction. □

Let us recall the definition of stable prefix as presented in 4.3.1. Some of the results for head normal forms generalize to a context with stable prefixes.

5.5.8. DEFINITION. (i) A prefix $s \leq t$ is called *stable for a reduction* starting from t if no proper occurrence of s becomes an occurrence of a redex during that reduction.

(ii) A prefix $s \leq t$ is called *stable* if s is stable for all strongly converging reductions starting from t.

(iii) A prefix $s \leq t$ is *maximally stable* if $r \leq s$ for any stable prefix $r \leq t$.

The next lemma establishes the link with head normal forms. Its proof is obvious.

5.5.9. LEMMA. *A prefix* $s \leq t$ *is stable if and only if all the subterms with root in* s *are in head normal* *form.* □

The restriction of stability to strongly converging reductions is no real restriction:

5.5.10. COROLLARY. *A prefix $s \leq t$ is stable if and only if* s *is stable for any infinite reduction starting from* t.

PROOF. (⇒) Suppose s is unstable for some non-strongly converging reduction. Then there is a lowest occurrence of s that becomes reducible. Hence the subterm of t at this occurrence is not a head normal form. Contradiction, via Lemma 5.5.9. □

5.5.11. LEMMA. *If a term* t *strongly converges to a term with a finite stable prefix, then there exist a finite reduction from* t *to some term with the same stable prefix* s.

PROOF. We combine the Transfinite Parallel Moves Lemma and 5.5.2. Using both we obtain a finite reduction from t to t' such that t' converges strongly to a term with stable prefix s and the root of t' is stable. Now we can repeat the construction with t', creating stable roots in the subterms at depth 1 of t'. Etc. □

5.5.12. LEMMA. *Let* s_i *be a finite stable prefix of* t_i *for* $i = 1, 2$. *Assume* t *converges to both* t_1 *and* t_2. *Then there exist a term* t_3 *with a finite stable prefix* s_3 *such that both* s_1 *and* s_2 *are prefixes of* s_3 *and* t *converges to* t_3.

PROOF. Apply Lemma 5.5.12 and the finite Church-Rosser property. □

5.6. Böhm reduction

From Lambda Calculus (cf. [Bar84]) we will borrow the idea for Böhm reduction $\to_\perp$ which extends the rewite relation $\to$ of a given TRS with an extra possibility: we allow ourselves to replace a subterm t which has no head normal form by a fresh symbol $\perp$, that we have added to the signature of the TRS. As in Lambda Calculus normal forms with respect to $\to_\perp$ will be called *Böhm trees.* We will call the reduction $\to_\perp$-Böhm reduction.

Each term has a unique Böhm tree. This implies that Böhm reduction in a OTRS satisfies the infinitary Church-Rosser property, both for strong converging Böhm reduction and converging Böhm reduction.

Before defining Böhm reduction and strict Böhm reduction we define an auxiliary reduction $\stackrel{\perp}{\to}$ on the terms of the signature of the given TRS extended with the fresh symbol $\perp$. The idea is to replace a subterm by $\perp$ when the subterm has no head normal form.

5.6.1. DEFINITION. (i) Let us denote by $\stackrel{\perp}{\to}$ the rewrite relation $\{\langle C[t],C[\perp]\rangle \mid t$ has no head normal form for $\to$, $C[\ ]$ is a one-place context$\}$.

(ii) Let the rewrite relation underlying *Böhm-reduction* (notation $\to_\perp$) be $\to \cup \stackrel{\perp}{\to}$.

(iii) A term t has a *Böhm tree* if there exists a strongly converging Böhm-reduction from t to $\to_\perp$-normal form.

First we give a argument for the finite Church-Rosser property of Böhm-reduction combining the finite Church-Rosser Property of the given OTRS and $\stackrel{\perp}{\to}$ with help of the Hindley-Rosen Lemma

[Bar84].

5.6.2. LEMMA. *The reduction relation* $\overset{\perp}{\longrightarrow}$ *has the finite Church-Rosser property*

PROOF. We give a direct proof. First we show WCR by a case analysis. Suppose $t_0 \overset{\perp}{\longrightarrow} t_1$ and $t_0 \overset{\perp}{\longrightarrow} t_2$. The reduced redexes are either disjoint, identical or one is inside the other. WCR is trivial for the first two cases. For the third case it suffices to prove that $C[\perp] \overset{\perp}{\longrightarrow} \perp$ assuming that $C[t] \overset{\perp}{\longrightarrow} C[\perp]$ as well as $C[t] \overset{\perp}{\longrightarrow} \perp$. That is, we must prove that $C[\perp]$ has no head normal form from the assumption that both $C[t]$ and t have no head normal form. Well, suppose $C[\perp] \rightarrow_\alpha s$ for some α and s. Now s is a term that may or may not contain $\perp$, say $s = E(\perp)$. Then clearly $C[t] \rightarrow_\alpha E(t)$. As $C[t]$ has no head normal form, we see that $E(t)$ cannot reduce to a redex. Hence $E(\perp)$ cannot reduce to redex (by contraposition).

We have established WCR in the following form:

$$
\begin{tikzcd}
\bullet \arrow[r, "\perp"] \arrow[d, "\perp"'] & \bullet \arrow[d, dashed, "\perp"', "\leq 1"] \\
\bullet \arrow[r, dashed, "\perp"', "\leq 1"] & \bullet
\end{tikzcd}
$$

As the dashed arrows stand for at most one reduction step, the full finite Church-Rosser property follows by an easy diagram chase. □

5.6.3. COROLLARY. $\rightarrow_\perp$ *has the finite Church-Rosser property.*

PROOF. The reduction relation $\rightarrow_\perp$ is generated by the two relations $\rightarrow$ and $\overset{\perp}{\longrightarrow}$. These relations $\rightarrow$ and $\overset{\perp}{\longrightarrow}$ satisfy the following diagrams (to be understood as saying that when the solid arrows exist, so do the shaded arrows):

$$
\begin{tikzcd}
\bullet \arrow[r, "*"] \arrow[d, "*"] & \bullet \arrow[d, dotted, "*"] \\
\bullet \arrow[r, dotted, "*"] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "\perp\ *"] \arrow[d, "*", "\perp"'] & \bullet \arrow[d, dotted, "*"', "\perp"] \\
\bullet \arrow[r, dotted, "*", "\perp"'] & \bullet
\end{tikzcd}
\qquad
\begin{tikzcd}
\bullet \arrow[r, "*"] \arrow[d, "*", "\perp"'] & \bullet \arrow[d, dotted, "*"', "\perp"] \\
\bullet \arrow[r, dotted, "*"] & \bullet
\end{tikzcd}
$$

The first two diagrams state the finite Church-Rosser properties for the separate relations.

The third is true because the only conflict among $\rightarrow$ and $\rightarrow_\perp$ arises when a $\rightarrow$-redex is contained in a $\rightarrow_\perp$-redex. But in that case, the $\rightarrow_\perp$-redex still has a unique residual by the $\rightarrow$-redex, whose contraction gives the same result as contracting the original $\rightarrow_\perp$-redex. The finite Church-Rosser Property follows from these facts by the Hindley-Rosen Lemma [Bar84]. □

5.6.4. LEMMA. *Each finite part of a Böhm tree of can be found in finitely many* $\rightarrow_\perp$*-steps.*

PROOF. Let $t \rightarrow_{\perp\alpha} t'$ be a strongly converging Böhm-reduction of length α from t to a Böhm tree t'. If we delete all the $\overset{\perp}{\longrightarrow}$ steps, then we obtain a strongly converging $\rightarrow$-reduction to say t''. The finite parts of the Böhm tree correspond to finite stable prefixes of the maximal stable prefix of t''. By Lemma 5.5.11 there exist finite reductions from t to the finite stable prefixes of t''. If we now apply $\overset{\perp}{\longrightarrow}$ steps up to sufficient depth, we obtain finite $\rightarrow_\perp$-reductions to finite prefixes of the Böhm tree for t. □

5.6.5. THEOREM. *A term of an orthogonal TRS has at most one Böhm tree.*

PROOF. By a similar argument as in the previous lemma; this time by an appeal to Lemma 5.5.12. □

Next we want to prove the existence of a strongly converging reduction to Böhm tree from any term. First we focus on a strict version of Böhm-reduction which gives priority over usual reduction to replacement by $\perp$ of subterms that have no head normal forms. This strict Böhm-reduction has some pleasant properties.

5.6.6. DEFINITION. *Strict Böhm-reduction* (notation $\rightarrow_{[\perp]}$) is the rewrite relation included in $\rightarrow \cup {}^{\perp}\!\!\rightarrow$ in which a ${}^{\perp}\!\!\rightarrow$-step have priority over ordinary $\rightarrow$-steps: a $\rightarrow$-step is allowed only if no ${}^{\perp}\!\!\rightarrow$-steps are possible.

5.6.7. LEMMA. (i) *Böhm reduction* $\rightarrow_{\perp}$ *and strict Böhm reduction* $\rightarrow_{\perp}$ *have the same normal forms.*

(ii) *Infinite* $\rightarrow_{[\perp]}$*-reductions are strongly convergent and of lenght at most* ω.

(iii) *Every term has a normal form with respect to* $\rightarrow_{[\perp]}$.

PROOF. (i) Trivial: both reductions have the same redexes.

(ii) If an infinite $\rightarrow_{[\perp]}$-reduction is not strongly convergent, there is be an occurrence u which is contracted infinitely often in the reduction. Hence the subterm at such an occurrence is not be top-terminating, and hence by Lemma 5.5.6 has no head normal form. But the only thing which such a subterm can be reduced to by $\rightarrow_{[\perp]}$ is $\perp$, contradiction.

(iii) Any outermost $\rightarrow_{[\perp]}$-reduction is strongly converging to Böhm normal form. □

5.6.8. THEOREM. *Any term of an orthogonal TRS admits a strongly converging reduction to Böhm tree.*

PROOF. Apply the previous lemma on strict Böhm reduction. □

5.6.9. COROLLARY. (i) *Böhm reduction* $\rightarrow_{\perp}$ *satisfies the infinite Church-Rosser Property for strongly converging* $\rightarrow_{\perp}$*-reductions.*

(ii) *Böhm reduction* $\rightarrow_{\perp}$ *satisfies the infinite Church-Rosser Property for converging* $\rightarrow_{\perp}$*-reductions.*

PROOF. (i) Suppose we have two strongly convergent reductions $t \rightarrow_{\perp\alpha} t_1$ and $t \rightarrow_{\perp\beta} t_2$. Using the previous Lemma 5.6.7 we construct strongly converging $\rightarrow_{[\perp]}$-reductions to Böhm tree starting from t_1 and t_2. By the unique Böhm tree property for t these reductions have the same reduct, the unique Böhm tree of t.

(ii) As (i). □

6. NEEDED REDEXES

The concept of a needed redex has been studied by Huet and Lévy [Hue79] for orthogonal TRSs.

In this section we shall recall some facts about needed redexes for finite rewriting first, and then generalise them to the infinitary setting.

6.1. Recollection of some results of Huet and Lévy

6.1.1. DEFINITION [Hue79]. A redex s of a term t is *finitely needed* if in every reduction of t to finite normal form a residual of s is rewritten.

6.1.2. FACTS [Hue79]. *In any orthogonal TRS, neededness has the following properties, where only finite terms, reductions and normal forms are considered.*

(i) If a term contains redexes, then it contains a finitely needed redex.

(ii) If a term has a finite normal form, then repeated rewriting of finitely needed redexes leads to that normal form, even if diluted by finite strings of non-needed reductions.

In general, neededness of redexes is undecidable for orthogonal TRSs: so the needed strategy (i.e. repeated rewriting of finitely needed redexes) is not effective. For this reason Huet and Lévy restrict themselves to strongly sequential orthogonal TRSs.

6.2. Neededness and infinitary reduction

We now extend the above ideas to infinite reductions and infinitary normal forms of orthogonal TRSs. The theorems concerning the existence of needed redexes and the sufficiency of needed reduction for computing normal forms will in this section be extended, with certain modifications, to the case of infinite reductions. Of course, generalization to the infinitary setting has no effect on the undecidability of neededness of redexes.

6.2.1. DEFINITION. A redex s of a term t is *needed* if in every strongly converging reduction of t to normal form a residual of s is rewritten.

6.2.2. THEOREM. *For orthogonal TRSs, in every term having a normal form but not in normal form, there is at least one needed redex.*

PROOF. Huet and Lévy prove this for finite terms. A study of their proof reveals that it applies equally to infinite terms and strongly convergent reductions to normal form. We only note the few points where the infinitary aspects need some care.

Lemma 3.11 of [Hue79], proving that every reduction A has, in Huet and Lévy's terminology, an external redex, is proved by induction on the length of A. To apply the proof to an infinite reduction, we note that we need only consider the initial segment of A which is terminated by the last step of A which reduces for the first time some residual of a member of $\mathcal{R}(A)$.

Lemma 3.16 of [Hue79], proving that every term having a normal form but not in normal form has an external redex, proceeds by induction on the size of the term, applying the inductive hypothesis to the immediate subterms of the given term. For infinite terms, such an induction would not be well-founded. However, it is clear that the induction can be recast as an induction on the stable depth of the term. The only terms that such an induction would miss are the infinite terms in normal form, for which the lemma is trivial. □

Needed reduction is normalising for orthogonal TRSs and finite normal forms [Hue79]. This is not true when infinite normal forms are considered. A simple example is provided by the orthogonal TRS consiting of the single rule: A → B(A,A). The term A can be reduced using this rule to the infinite binary tree with a B at each node. At every finite stage in a reduction starting from A, every redex is needed. However, it is easy to exhibit infinite reductions from A which do not compute the infinite normal form. For example, if we take the leftmost redex at each step, we generate the reduction

$$A \to B(A,A) \to B(B(A,A),A) \to B(B(B(A,A),A),A) \to \ldots$$

Clearly, some notion of fairness with respect to needed redexes is required to ensure that every part of the infinite normal form is generated.

6.2.3. THEOREM. *Let* t *be a term which has a normal form.*

(i) *Any needed reduction starting from t is strongly converging.*

(ii) *Any hyper-needed reduction starting from t is strongly converging, where a hyper-needed reduction is a reduction such that in between any two subsequent needed reductionsteps at most finitely many non-needed redexes are contracted.*

PROOF. (i) If it were not strongly converging, then there would be an occurrence in some term of the reduction sequence where infinitely often a reduction is performed. This means that the subterm at that occurrence does not have a head normal form. Hence subterms at that occurrence can never complete a redex pattern in at a higher occurrence. For (ii) the same proof applies! □

6.2.4. DEFINITION. A converging reduction $t \to_{\leq\omega} t'$ is *needed-fair* if for every term t'' in the reduction, and every needed redex r of t'', there exists some finite part of the remaining reduction starting at t'' that does not preserve r.

6.2.5. THEOREM. *A needed-fair reduction, starting from a term having a normal form, is stable.*

PROOF. Let $\mathcal{R}$ be an needed-fair reduction from a term t. If $\mathcal{R}$ were not top-terminating, then t would not be top-terminating and t would not have a head normal form by Theorem 5.5.7. Contradiction. Hence $\mathcal{R}$ is top-terminating. So, $\mathcal{R}$ is of the form $\mathcal{R}_1 \cdot \mathcal{R}'_1$, where $\mathcal{R}_1$ is finite and $\mathcal{R}'_1$ contains no root reductions. $\mathcal{R}'_1$ is an interleaving of independent reductions, each reduction occurring wholly within a different immediate subterm of t. Because $\mathcal{R}$ is needed-fair, each of those subreductions must be needed-fair. Each such subreduction must also be top-terminating (with respect to its respective subterm). Since there is a finite number of immediate subterms of t, and once more by needed-fairness we can split $\mathcal{R}'_1$ into $\mathcal{R}_2 \cdot \mathcal{R}'_2$, where $\mathcal{R}'_2$ performs no reduction at a depth of 0 or 1. Repeatition of the argument ad infinitum shows that $\mathcal{R}$ is stable. □

We will describe a strategy $\mathcal{DI}$ that presented with a term always generates a needed-fair reduction whenever the given term has a normal form. Let us first be clarify this terminology.

6.2.6. DEFINITION. (i) A *reduction strategy* for a TRS is a function that maps every term t of the TRS to a set of finite reduction starting from t.

(ii) For any strategy S, the strategy *hyper*-S maps each term t_0 to the set of reductions of the form

$t_0 \rightarrow_n t_n \rightarrow_m t_{n+m}$, where $t_n \rightarrow_m t_{n+m} \in S(t_n)$.

(iii) A sequence $(t_\beta)_{\beta<\alpha}$ is *generated by a reduction strategy* S *on a term* t if $t_0 = t$ and $t_{\beta+1} \in S(t_\beta)$ for all $\beta < \alpha$.

6.2.7. DEFINITION. (i) A reduction strategy is *transfinitely normalising*, whenever it generates a strongly converging reduction to normal form for any term which has a normal form.

(ii) A reduction strategy is *normalising*, whenever it generates a stable reduction to normal form for any term which has a normal form.

(iii) A reduction strategy S is (transfinitely) *hyper-normalising*, whenever hyper(S) is (transfinitely) normalising.

An example of a reduction strategy is the strategy of needed reduction, where S(t) is the set of one-step reductions starting from t which reduce a needed redex of t. We can paraphrase Theorem 6.2.3 bty saying that needed reduction is transfinitely normalising.

6.2.8. COROLLARY. *Parallel-outermost reduction is transfinitely hypernormalising.*

PROOF. Consider a reduction $\mathcal{R}$ starting from a term in normal form. If $\mathcal{R}$ always eventually performs a parallel-outermost reduction, then $\mathcal{R}$ is needed-fair, and converging to normal from. Hence parallel-outermost is a transfinitely hypernormalising strategy. (It may not be simply hypernormalising, since a single parallel outermost part of the sequence may itself be infinitely long.)

6.2.9. DEFINITION. *Depth-increasing reduction* is the following strategy $\mathcal{DI}$. Given a term t_0, for each $n \geq 0$ let t_{n+1} be derived from t_n by complete development of all redexes at occurrences of depth no more than n. Then $\mathcal{DI}(t_0)$ is the set whose only member is the sequence $t_0 \rightarrow^* t_1 \rightarrow^* t_2 \rightarrow^* \ldots$

6.2.10. COROLLARY. *Depth-increasing reduction is hypernormalising.*

PROOF. Clearly, $\mathcal{DI}(t)$ is no longer than ω. For $\mathcal{DI}$ to be normalising, it is sufficient to prove that $\mathcal{DI}(t)$ converges to the normal form of t, whenever t has one. But clearly, if there is a needed redex R in t at some depth n, then at the nth stage or earlier in the construction of $\mathcal{DI}(t)$, either some residual of R or some redex containing some residual of R will be reduced. Thus $\mathcal{DI}(t)$ is needed-fair, hence by Theorem 6.2.5 normalising. □

The notions of neededness and normalisation extend to the Böhm reduction $\rightarrow_\perp$ we introduced to compute Böhm trees.

6.2.11. COROLLARY. (i) *Needed* $\rightarrow_\perp$*-reduction is transfinitely normalizing.*

(ii) *Parallel-outermost* $\rightarrow_\perp$*-reduction is transfinitely normalising.*

(iii) *Depth-increasing* $\rightarrow_\perp$*-reduction is normalising.* □

7. APPLICATIONS TO GRAPH REWRITING

Graph rewriting is a common method of implementing term rewrite languages [Pey87]. It relies on the basic insight, that when a variable occurs many times on the right-hand side of a rule, one need only copy pointers to the corresponding parts of the term being evaluated, instead of making copies of the whole subterm. However, the precise relation between term and graph rewriting has some subtleties [Bar87, Far89, Hof88, Sta80]. This is also true when we consider infinite terms.

Infinite graphs will represent infinite terms, but some finite graphs — the cyclic ones — also represent infinite terms. A single reduction in a cyclic graph can correspond to an infinite reduction of reductions in the corresponding term. For example, applying the rule A $\rightarrow$ B to the graph x:F(A,x) (the notation is explained in 7.1) corresponds to applying it to infinitely many redexes in the term F(A,F(A,F(A,...))).

The correspondence between acyclic graphs and terms has been studied in [Bar87]. We will extend this correspondence and its constituent notions of lifting and unravelling to cyclic graphs and infinite terms.

7.1. Graphs and graph rewriting

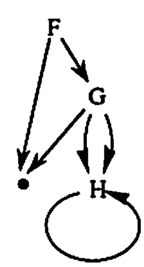
First we define graphs and graph morphisms in a general way.

7.1.1. DEFINITION. A *graph* g over a signature $\Sigma = (\mathcal{F},\mathcal{V})$ is a quadruple (*nodes*(g), *lab*(g), *succ*(g), *roots*(g)), where *nodes*(g) is a (finite or infinite) set of nodes, *lab*(g) is a function from a subset of the nodes of g to $\mathcal{F}$, *succ*(g) is a function from the same subset to tuples of nodes of g, and *roots*(g) is a tuple of (not necessarily distinct) nodes of g. Furthermore, every node of g must be ***accessible*** (defined below) from at least one root. Nodes of g outside the common domain of *lab*(g) and *succ*(g) are called *empty*.

7.1.2. DEFINITION. A *path* in a graph g is a finite or infinite sequence a,i,b,j,... of alternating nodes and integers, beginning and (if finite) ending with a node of g, such that for each m,i,n in the reduction, where m and n are nodes, n is the i'th successor of m. The length of the path is the number of integers in it. If the path starts from a node m and ends at a node n, it is said to be a path from m to n. If there is a path from m to n, then n is said to be *accessible from* m. When this is so, the *distance* of n from m is the length of the shortest path from m to n.

We may write n:F(n_1,...,n_k) to indicate that *lab*(g)(n) = F and *succ*(g)(n) = (n_1,...,n_k). A finite graph may then be presented as a list of such *node definitions*.

x:F(y,z),
z:G(y,w,w),
w:H(w)



In such pictures, we may omit the names x, y, z, etc., as their only function in the textual representation is to identify the nodes. In particular, x, y, z, etc. do not represent variables — these are represented by empty nodes. Different empty nodes need only be distinguished by the fact that they are different nodes; we do not need any separate alphabet of variable names. Multiple references to the same variable in a term are represented in a graph by multiple references to the same empty node.

The tabular description demonstrated above may be condensed, by nesting the definitions; for example, another way of writing the same graph is F(y,z:G(y,w,w:H(w))).

In general a graph may have more than one root. We will only use graphs with either one root (which represent terms) and graphs with two roots (which represent term rewrite rules).

7.1.3. DEFINITION. A *graph homomorphism* from a graph g to a graph h is a function f from the nodes of g to the nodes of h, such that for all nodes n in the domain of *lab*(g), *lab*(h)(f(n)) = *lab*(g)(n), and *succ*(h)(f(n)) = *succ*(g)(n).

Note that a graph homomorphism is not required to map the roots of its domain to the roots of its codomain. The following proposition has a straightforward proof:

7.1.4. PROPOSITION. *A graph homomorphism is determined by its action on the roots of its domain.*

□

On graphs one can define general graph rewrite mechanisms. For our purposes it suffices to study term graph rewriting.

7.1.5. DEFINITION. A *term graph* is a graph with one root.

Our definition of graphs includes infinite graphs. We can also define infinite graphs by the completion of an ultrametric space, as we did for terms.

7.1.6. DEFINITION. Given a term graph g = (N,l,s,r) and an integer n, $\pi_n(g)$ is the *truncation* of g to depth n. It is the term graph (N',l',s',r') defined by:

(i) N' is the set of nodes of g whose minimum distance from any member of r is not more than n.

(ii) r' = r.

(iii) For $p \in N'$, l'(p) and s'(p) are the same as they are for g, if the minimum distance of p from any member of r is less than n. If this distance is equal to n, then $l'(p) = \Omega$ and s'(p) = ( ).

From this notion of truncation, we can define an ultrametric on term graphs in the same way as for terms.

7.1.7. DEFINITION. For term graphs g and h, $d(g,h) = \inf\{\, 2^{-n} \mid \pi_n(g) = \pi_n(h) \,\}$.

Note that for graphs in which nodes may be inaccessible from the root, this is not a metric, as it is independent of the existence of such nodes. Thus there would be distinct graphs at zero distance from each other.

The Cauchy completion of this ultrametric space gives an alternative, but equivalent, definition of infinite term graphs, which then allows us to carry over to graphs the definitions of convergent reduction and strong reduction.

From now on we will consider term graphs and term graph rewriting only, and often we will simply call them graphs.

7.1.8. DEFINITION. A *term graph rewrite rule* is a graph with two, not necessarily distinct, roots (called the *left and right* roots), in which every empty node is accessible from the left root, and the subgraph containing those nodes accessible from the left root is a finite tree. The *left* (resp. *right*) *hand side* of a term graph rewrite rule is the subgraph consisting of all nodes and edges accessible from the left (resp. right) root.

7.1.9. DEFINITION. A *redex* of a term graph rewrite rule r in a graph g is a homomorphism from the left-hand side of r to g. The *occurrence* of the redex is the minimal occurrence of the node of g to which the left root is mapped. The *depth* of a redex is the length of the occurrence.

The result of *reducing* a redex of the rule r in a graph g at occurrence u is the graph obtained by the following construction — with one possible exception, which we shall come to later.

7.1.10. CONSTRUCTION. (i) Construct a graph h by adding to g a copy of all nodes and edges of r not in *left*(r). Where an added edge has one endpoint in *left*(r), in h that edge is connected to the corresponding node of h.

(ii) Let n_l be the node of h corresponding to the left root of r, and n_r the node corresponding to the right root of r. (These are not necessarily distinct.) In h, replace every edge whose target is n_l by an edge with the same sources and target n_r, obtaining a graph k. The root of k is the root of h, unless this is n_l, otherwise it is n_r.

(iii) Remove all nodes which are not accessible from the root of k. The resulting graph is the result of the rewrite.

We have now the ingredients to give the general definition of a Term Graph Rewrite System.

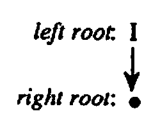
7.1.11. DEFINITION. Let Σ be a signature. A Term Graph Rewrite System (GRS for short) is a pair $(G(\Sigma),\mathbf{R})$ where $G(\Sigma)$ is the set of graphs for the signature Σ, and **R** a set of term graph rewrite rules for the signature Σ.

Having defined term graph rewriting and the notion of depth on term graphs, the concepts of normal form, infinitary rewriting, orthogonality, etc. carry over to term graphs.

7.2. Circular redexes

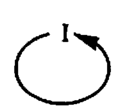
We now consider the exception of which we forewarned the reader in 7.1.9. Consider the following rule, given in both textual and pictorial forms:

$$I(x) \to x$$



and the graph

$$a{:}I(a)$$



It is clear that the graph is a redex of the rule. We call this redex "circular I". What should it be reduced to?

According to the definitions above, it reduces to itself. Other treatments of term graph rewriting differ. According to [Far88], it reduces to a node with no function symbol, but with an edge pointing from itself to itself. According to [Ken90a], it reduces to an empty node. Most other references (e.g. [Sta80, Rao84, Ken87, Bar87]) avoid the problem by excluding cyclic graphs. If one ignores these exclusions, and tries to apply their definitions to circular I, we find that e.g. [Sta80] produces a non-well-formed graph, and [Rao84] and [Ken87] find a match of the rule to the graph, but cannot reduce it.

Circular I is one instance of a class of redexes having the same behaviour, the circular redexes. Circular redexes come into existence via collapsing rules.

7.2.1. DEFINITION. (i) A redex of a rule r is *circular* if the homomorphism from *left*(r) to g maps both roots of r to the same node. (This can only happen if the right root of r is accessible from the left root.)

(ii) A rule is *self-embeddable* if there exists a circular occurrence of the rule.

Note that the subgraph matched by a circular redex is cyclic, but not conversely. A counterexample is the rule F(G(x)) → x and a graph F(y:G(y)). Such a redex is unproblematic.

7.2.2. PROPOSITION. *Every circular redex reduces to itself.* □

7.2.3. DEFINITION. A *collapsing term graph rule* (sometimes called a selector rule) is a term graph rule whose right-hand side is a variable. A *collapsing redex* is a redex of a collapsing (term or graph) rule.

7.2.4. PROPOSITION. *In an orthogonal term graph rewrite system, a rule is self-embeddable iff it is a collapsing rule.* □

An example of a non-collapsing, self-embeddable rule is x:F(y:F(z)) → y. Note that this rule conflicts with itself: it has two overlapping redexes in the graph F(F(F(G))).

Every rule for extracting a component of a structure is self-embeddable, such as the usual rules for breaking up a list: head(cons(x,y)) → x, tail(cons(x,y)) → y. Clearly, these are an essential part of any functional language, and must be properly dealt with by any formal description of term graph rewriting.

We have seen that there is uncertainty in the literature over what a circular redex should reduce to. The choice of definition makes a difference, for the following reasons.

(a) The Church-Rosser property may fail, even for finite graphs and finite rewrite sequences.

(b) The correspondence between term rewriting and graph rewriting is complicated by circular redexes.

As an example of (a), we can adapt example 5.1.1.

Rules: A(x) → x
B(x) → x
C → x:A(B(x))

Initial graph : C

Sequences: $C \to x{:}A(B(x)) \to x{:}B(x) \to x{:}B(x) \to x{:}B(x) \to \ldots$

$C \to x{:}A(B(x)) \to x{:}A(x) \to x{:}A(x) \to x{:}A(x) \to \ldots$

Notice that all the graphs are finite, the first two rules could be found in any functional program, and the third rule is a reasonable optimisation of the term rule $C \to A(B(C))$.

If instead we stipulate that circular redexes reduce to a new symbol $\perp$, not occurring elsewhere in the rule set, then the Church-Rosser property for finitary reduction to hold for termgraph rewriting, just as well as the Church-Rosser and Parallel Moves results we proved for infinitary term rewriting. We will show this in a forth-coming paper on graph rewriting.

Note that the restriction to one collapsing rule bears a similarity to the technique of indirection nodes ([Wad71]). When a rule of the form $C[x] \to x$ (where C is a context) is applied to a subgraph of the form C[g], then it is implemented as if it were the non-collapsing rule $C[x] \to \iota(x)$. ι is a new symbol, the *indirection* symbol, and is considered to be invisible to pattern-matching: $Plus(\iota(1),2)$ is the same redex as Plus(1,2) (of the obvious rules for Plus).

Finally in this section, we note with respect to (b) that reducing circular redexes to themselves corresponds reasonably well to term rewriting with respect to the obvious concept of "unravelling" a graph, although there are certain rough edges in the relationship — several of our proofs have to treat circular redexes as a special case. Alternatively, reducing circular redexes to $\perp$ is related to the Böhm reduction of section 6, since circular redexes unravel to infinite terms which have no head normal form.

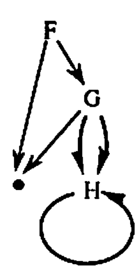
7.3. From graphs to terms

Unravelling is the standard technique to go from graphs to terms.

7.3.1. DEFINITION. The *unravelling* U(g) of a graph g is the forest defined as follows. The nodes of U(g) are the paths of g which start from any of its roots. Given a node a,i,b,j,...y of U(g), if y is a nonempty node of g, then this node of U(g) is labelled with the function symbol *lab*(g)(y), and its successors are all paths of the form a,i,b,j,...y,n,z, where z is the n'th successor of y in g. If y is empty, then it is labelled with a variable symbol, a different symbol being chosen for every empty node of g.

For a node n of g, we define U(n,g) to be the set of nodes of U(g) of the form a,i,b,j,...n.

Note that a cyclic graph will have an infinite unravelling. For example, the unravelling of the graph shown in the next diagram is the term $F(y,G(y,H^{\omega},H^{\omega}))$.



The following simple lemma will be useful when lifting result concerning continuity and strong convergence from terms to graphs.

7.3.2. LEMMA. Every member of U(n,g) is at a depth in U(g) at least equal to the depth of n in g, and at least one is at equal depth.

It is easy to see that for a term graph g, U(g) is a term, and for a graph rewrite rule r, U(r) is a term rewrite rule. We can also apply the notion of unravelling to a whole rewrite system.

7.3.3. DEFINITION. The *unravelling* of a GRS $(G(\Sigma),R)$ is the TRS $(Ter^{\infty}(\Sigma),U(R))$whose rules U(R) are the unravellings of the rules in R. This TRS is also denoted by $U(G(\Sigma),R)$; its set of terms is U(G).

So, given a signature Σ the operator U transforms GRS's over Σ into TRS's over Σ.

7.3.4. PROPOSITION. U *is uniformly continuous on term graphs.*

PROOF. If two graphs have the same truncation to depth n, then it is clear that their respective unravellings will also have the same truncation to depth n. Therefore when $g \neq g'$, the ratio $d(U(g),U(g'))/d(g,g')$ is always less than or equal to 1. Thus U is uniformly continuous. □

7.3.5. PROPOSITION. *There is a homomorphism from* U(g) *to* g *which takes the root of* U(g) *to the root of* g. □

The homomorphism is obtained by mapping each node of U(g) (which is a finite path of g) to its final element. If g is acyclic, this is clearly the only homomorphism from U(g) to g, but if g is cyclic there can be more than one: for example, if g = x:A(A(x)), there are two.

The following proposition is not hard to prove:

7.3.6. PROPOSITION. *A graph g in the GRS* $(G(\Sigma),R)$ *is a normal form iff its unravelling* U(g)*is a normal form in* $(Ter^{\infty}(\Sigma),U(R))$. □

The following alternative representation of graphs is useful.

7.3.7. PROPOSITION. *A graph* g *is determined by the set of those of its maximal paths which begin at any of its roots, together with the following equivalence relation on paths:* $P \cong P' \Leftrightarrow P = P'$ *or* P *and* P' *can be written as* $P_1 \cdot Q$ *and* $P'_1 \cdot Q$, *where* P_1 *and* P'_1 *are finite and end at the same node.*

PROOF. Recall that by definition, every node of a graph is accessible from at least one of its roots. In addition, a path records not just a reduction of nodes, but the edges by which one gets from one node to another. The proposition is then obvious. □

In terms of this representation of a graph as a set of paths and an equivalence relation, the unravelling of a graph is obtained simply by dropping the equivalence relation.

7.3.8. PROPOSITION. *Let* $g \rightarrow g'$ *in a GRS. Then* $U(g) \rightarrow_{\leq\omega} U(g')$ *in the corresponding TRS. Moreover, the depth of every redex reduced in the term sequence is at least equal to the depth of the redex reduced in* g.

PROOF. Let r be the rule that was applied to reduce g to g', and u the occurrence at which it was applied. We need to distinguish two cases.

If the redex is circular, then it reduces to itself, and g'=g. Clearly, U(g) reduces to U(g') by the empty reduction sequence. The condition on depths is trivially satisfied.

Otherwise, we shall show that there is a redex of U(r) at every occurrence in U(g,u), that all these redexes can be reduced by a strongly convergent reduction, and that the limit of this reduction is U(g').

It is clear that there is a redex of U(r) at every occurrence in U(g,u). If U(g,u) is finite, then the theorem holds as shown in [Bar87]. Otherwise, suppose U(g,u) is infinite.

Let the members of U(g,u), ordered by depth, be $u_1, u_2, \ldots$, with depths $d_1, d_2, \ldots$ Consider the effect of reducing the redex at u_1.

Those redexes at occurrences incomparable with u_1 will still be present afterwards, and at the same occurrences. In particular, all redexes previously at the same depth as u_1 will be at the same depth afterwards.

Redexes at occurrences which extend u_1 must be at greater depth. We shall show that after reducing the redex at u_1, the depths of the residuals of such redexes are still greater than d_1.

Suppose this were not the case. If a redex is at $u_i < u_1$, then after reduction at u_1 its residuals must still be within the subterm at u_1. If the redex formerly at u_i has a residual at depth d_1, that residual must therefore be at u_1. But this is only possible if the right-hand side of the rule is a variable, and the subterm matched to that variable by the redex at u_1 is the subterm at u_i. But both redexes originate from the same redex of the original graph. Therefore the graph redex was a cyclic collapsing redex, a case we have already eliminated.

Therefore, after reduction at u_1, for every redex at depth greater than d_1, all its residuals by u_1 are still at depth greater than d_1. Since there can only be finite many redexes at depth d_1, reduction of all of them leaves redexes only at depths greater than d_1. Repeating the argument for the newly shallowest redexes constructs a strongly convergent reduction.

It is immediate from the description of graphs in terms of paths (proposition 7.3.7), that when all the remaining redexes are at depths greater than some depth d, then the term t_d at that point agrees with U(g') down to depth d. Thus the distance between t_d and U(g') is less than 2^{-d}. Therefore the limit of the reduction of terms t_d as d tends to infinity is U(g').

Finally, the condition on the depths of the term reduction steps is immediate from lemma 7.3.2. □

A similar proof establishes the following generalisation.

7.3.9. PROPOSITION. *If* g *reduces to* g' *by complete development of a set* S *of disjoint redexes, then* U(g) *reduces to* U(g') *by complete development of the set of redexes* U(S'), *where* S' *is the set of non-circular members of* S. □

Note that these two propositions show that the term rewrite reduction corresponding to a finite graph rewrite reduction can be chosen to be strongly convergent, not just convergent.

7.3.10. PROPOSITION. *Let* $g \to_\alpha g'$ *in a GRS. Then for some* $\beta \geq \alpha$, $U(g) \to_\beta U(g')$ *in the corresponding TRS.*

PROOF. By applying proposition 7.3.8 to each step in the reduction $g \to_\alpha g'$. This gives a reduction sequence composed of subsequences of the form $U(g_\gamma) \rightarrow_{\leq\omega} U(g_{\gamma+1})$, each of which is strongly convergent. That the concatenation of these is also strongly convergent follows from Lemma 7.3.2.

□

Note that a single reduction of the GRS can correspond to an infinite reduction in the TRS, and that a reduction of two or more steps in the GRS can correspond to a transfinite reduction of the TRS.

7.4. From terms to graphs

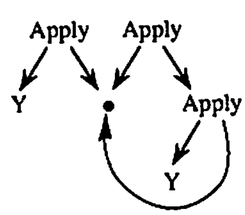
A *lifting* of a TRS $(Ter^\infty(\Sigma),R)$ is a GRS $(G(\Sigma),\mathbf{R})$ whose rules unravel to rules in R.

7.4.1. DEFINITION. The *lifting* $L(Ter^\infty(\Sigma),R)$ of the TRS $(Ter^\infty(\Sigma),R)$ is defined as the GRS $(G(\Sigma),\mathbf{R})$ where the elements of $\mathbf{R}$ are *minimally shared, bi-rooted graphs*, corresponding with the rules in R: reading the left- and right-hand sides of a term rule $T \rightarrow T'$ as trees, and then for each variable identify all leaves of the two trees which bear that variable. The roots of the two trees become the roots of the graph.

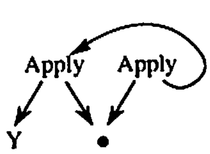
Our definition of the lifting of a TRS picks out one of possibly many GRSs which unravel to the given TRS. We have choosen a version compatible with [Bar87]. We shall briefly consider some other possible liftings, which might result from a certain optimisation in the representation of term rules as graph rules.

Consider the term rule Penultimate(Cons(x,Cons(y,z))) → Penultimate(Cons(y,z)). The graph rule obtained by lifting according to definition 7.4.1 will, when it is applied, create two new nodes. However, we can see that the node Cons(y,z) which it creates has the same contents as a node which was matched by the left-hand side. We can therefore reuse the existing node instead. This amounts to representing the term rule by the graph rule Penultimate(Cons(x,w:Cons(y,z))) → Penultimate(w). In general, we may choose to represent the term rule by any bi-rooted graph which unravels to the term rule, and whose left-hand side is a tree. (Notice the suggestive similarity with the **where**-definitions of Miranda and the as-declarations of ML.)

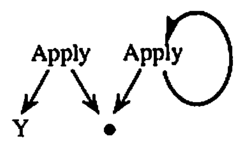
We can go even further. Consider the term rule for the Y combinator: Apply(Y,f) = Apply(f,Apply(Y,f)). The simple lifting to a graph rule is



Notice that again, the rule creates a new node Apply(Y,f) which has the same contents as an existing node. We can again identify the two nodes, like this:



But now the left root of this rule is accessible from the right root. By following the details of the definition of graph rewriting, we can see that when this rule is applied, the created edge corresponding to the edge from the right root to the left root of the rule will in fact go from the node to which the rule is applied, to itself. This is as if the rule were in fact:



This is the rule for "knot-tying" Y — written in textual form, it is Apply(Y,f) → x:Apply(f,x). The unravelling of this rule is not the original term rule, but the rule obtained by repeatedly applying the term rule to the Apply(Y,f) subterm contained in its right-hand side. The graph rule in effect perfoms all those reductions simultaneously. This is an instance of what Farmer and Watro [Far89] call "redex capturing", which they have shown is a sound implementation of the term rewrite rules where the possibility occurs.

Thus to minimise the number of created nodes in a graph rewriting implementation of a TRS, we should take the "smallest" ravelling of the term rules. Any two nodes of the lifting of the rule which are isomorphic may be identified, provided they are not both in the left-hand side. Whenever this results in edges which point to the left root, such edges should be made to point at the right root instead.

Such a minimal lifting is not always unique, because of the condition that the left-hand side of the rule must always be a tree. Consider the rule A(B(x),B(x)) → C(B(x)). There are two minimal liftings of this rule: A(y:B(x),B(x)) → C(y) and A(B(x),y:B(x)) → C(y). We cannot lift it to A(y:B(x),y) → C(y), as the left-hand side is no longer a tree. While the graph rewriting behaviour of such rules is worth studying in its own right, it appears not to correspond to anything in the term rewrite world, and is outside the scope of this paper.

7.5. Neededness and graphs

We now consider the graphical counterpart of needed redexes and necessary sets. The latter is a generalisation by Sekar and Ramakrishnan [Sek90] of the concept of needed redex:

7.5.1. DEFINITION [Sek90]. A set of redexes S of t is *necessary* if in every reduction of t to a finite normal form, a residual of some member of S is rewritten.

7.5.2. THEOREM. *Let* g *be a graph and let* t *be its unravelling. Let* r *be a redex of* g, *and let* R *be the corresponding set of redexes of* t. *Then* r *is needed in* t *iff* R *is necessary in* g.

PROOF. A reduction of t to normal form not reducing any residual of any member of R can be lifted to a reduction of g to normal form not reducing any residual of r. Therefore if r is needed, R is necessary.

A reduction of g to normal form not reducing any residual of r unravels to a reduction of t to normal form not reducing any residual of any member of R. Therefore if R is necessary, r is needed.

□

7.5.3. THEOREM. *Let* g *be a graph and let* t *be its unravelling. Let* R *be a set of redexes of* g. *Then* R *is necessary iff* U(R) *is necessary.*

PROOF. R is necessary	$\Leftrightarrow$ $\exists r \in R$.r is needed	[Sek90], *mutatis mutandis* for graphs
	$\Leftrightarrow$ $\exists r \in R$.U(r) is necessary	Theorem 7.5.2
	$\Leftrightarrow$ U(R) is necessary	[Sek90] □

7.5.4. COROLLARY. *To find a needed redex in a graph in a strongly sequential orthogonal term graph rewrite system, apply the Huet-Lévy algorithm to choose a redex in the unravelled term, and take the corresponding redex of the graph.* □

7.5.5. THEOREM. *Let* g *be a graph with a normal form. Any needed reduction starting from* g *is strongly convergent..*

PROOF. Suppose there exist a non-strongly convergent needed reduction from a graph which has a normal form. If we unravel this reduction, we get a hyper-needed reduction which is not strongly converging. This contradicts 6.2.3. □

7.6. Graph rewriting versus term rewriting

7.6.1. DEFINITION. (i) A TRS (T,R) is *graph reducible* in a lifting (G,**R**) of (T,R) if for every graph g in (G,**R**) it holds that if t is a normal form of U(g), then there is a normal form g' of g in (G,**R**) such that U(g') = t.

(ii) A GRS (G,**R**) is *tree reducible* if there is a TRS (T,R) such that (T,**R**) = U(G,**R**), and such that if g' is a normal form of g in (G,**R**), then U(g') is a normal form of U(g) in (T,**R**).

Note that although this definition is not quite the same as that stated in [Bar87], it is equivalent to the one used there. [Bar87] proves that every acyclic orthogonal GRS is tree reducible, and every weakly orthogonal TRS is graph reducible. These are still true in the presence of infinite terms, graphs and reductions. In fact, the first of these can be strengthened: when we have infinite terms and reductions, every GRS is tree reducible.

The following two theorems extend the results of [Bar87] to the case of cyclic and/or infinite graphs.

7.6.2. THEOREM. *Every GRS is tree reducible.*

PROOF. Immediate from the facts shown in Section 7.3 that the unravelling U preserves reductions and normal forms. □

7.6.3. LEMMA. *A circular redex of a graph having a normal form is not needed.*

PROOF. Let g have a normal form, and let r be a circular redex of g. Consider the sequence g $\to$ g $\to$ g $\to$ ... of infinitely many reductions of r (which by Proposition 7.2.2 reduces to itself). This is not strongly convergent, but needed reduction of graphs having a normal form is strongly convergent (7.5.5). Therefore at least one step in this sequence is not needed. But all steps are reductions of r in g, therefore r is not needed. □

7.6.4. THEOREM. *Every orthogonal TRS is graph reducible in its lifting via* L.

PROOF. The argument is similar to that used in [Bar87] for the finite case. Given a graph g of L(T,R) such that U(g) has a normal form, consider the parallel-outermost reduction originating from g.

$$g \quad \to^{PO} \quad g_1 \quad \to^{PO} \quad g_2 \quad \to^{PO} \quad \ldots$$

Since the system is orthogonal, each parallel-outermost stage performs a complete development of a set of disjoint redexes. We apply proposition 7.3.9 to each stage, obtaining a reduction

$$U(g) \quad \to^{PO'} \quad U(g_1) \quad \to^{PO'} \quad U(g_2) \quad \to^{PO'} \quad \ldots$$

where each $\to^{PO'}$ step is a strongly convergent complete development of a set of redexes which includes at least all the non-circular outermost redexes of the term. Since parallel-outermost-needed reduction is hypernormalising for orthogonal TRSs (corollary 6.2.9), and circular redexes are never needed in terms having a normal form (lemma 7.5.3), the lower reduction must strongly converge to a normal form t. Hence the upper reduction must do likewise, and its limit must be a graph which unravels to t. Therefore the TRS is graph reducible. □

8. RELATIONS WITH OTHER WORK

Infinite terms naturally arise in computing. The ultrametric space of finite and infinite trees is a natural structure to interpret such terms in (cf. [Arn80], [Cou79], [Bak82], [Ber84], [Ber89]). *Infinite reductions* in contexts of recursive program schemes and recursive processes (cf. [Cou79]) or Prolog (cf. [Col82]) however seem to be unwanted: preferably one works with terminating programs. This hesitance to consider infinite reductions is also seen in the fact that infinite reductions are hardly treated in the standard works on Lambda Calculus and Term Rewriting Systems, except for some observations in the context of Böhm trees (cf. [Bar84], [Klo80]).

The work of Dershowitz, Kaplan and Plaisted [Der89a,b, Der90] seems to be the first place where an attempt is made of a theory of infinite reductions. Considering the metric space of the finite and infinite terms of a TRS they introduce the notion of converging reductions. Concentrating on convergence, Dershowitz *c.s.* introduce ω-normal forms: terms that reduce in one step to themselves if they contain a redex. They show, for example, that fair converging derivations result in ω-normal forms.

Farmer and Watro [Far89] realize that it needs strong convergence to compress certain infinitary reductions into strongly converging reductions of length at most ω. In the latest paper [Der90] Dershowitz *c.s.* focus on top-terminating TRSs to force convergent reductions to be strongly convergent, e.g. to prove the compressing lemma for infinitary converging reductions.

Dershowitz *c.s.* [Der89a,b, Der90] also consider algebraic semantics for infinite theories and study constructor TRSs and hierarchical TRSs.

Farmer and Watro [Far89] point out the importance of infinite rewriting for a sound understanding of the graph rewriting on which implementations of functional programming languages are based.

They show that term graph rewriting with arbitrary structure sharing and redex capturing is sound for left linear TRSs.

Chen and O'Donnell [Che90] consider finite cyclic term graphs as finite representations of infinite terms, and accordingly TRSs are extended with those infinite terms (so-called "rational trees") that can be represented by finite term graphs. They define a notion of infinite reduction unrelated to ours, for which they prove results in the line of Dershowitz, Kaplan and Plaisted.

9. REFERENCES

[Arn80] A. ARNOLD and M. NIVAT, *The metric space of infinite trees. Algebraic and topological properties*, Fundamenta Informatica, 4 (1980), 445-76.

[Bak82] J. DE BAKKER and J.I. ZUCKER, *Processes and the denotational semantics of concurrency*, Information and Control, 54:70-120, 1982.

[Bar87] H.P. BARENDREGT, M.C.J.D. VAN EEKELEN, J.R.W. GLAUERT, J.R. KENNAWAY, M.J. PLASMEIJER, and M.R. SLEEP, *Term graph rewriting*, Proc. PARLE Conference vol II, LNCS vol. 259, 141–158 (Springer-Verlag, 1987).

[Bar84] H.P. BARENDREGT, *The Lambda Calculus, its Syntax and Semantics*, 2nd ed., (North-Holland, 1984).

[Ber84] J.A. BERGSTRA and J.W. KLOP, *The algebra of recursively defined processes and the algebra of regular processes*, in: Proceedings ICALP 84, Antwerp (J. Paeredaens, ed.) LNCS 172, Springer-Verlag, pp. 82-95, 1984.

[Ber89] J.A. BERGSTRA and J.W. KLOP, *Process theory based on bisimulation semantics*, in: Proceedings REX School/Workshop on Linear time, branching time and partial order in logics and models for concurrency, Noordwijkerhout (J.W. de Bakker, W. -P. de Roever and G. Rozenberg eds.) LNCS 354, Springer-Verlag, pp. 50-122, 1989.

[Che90] Y. CHEN and M.J. O'DONNELL, *Infinite terms and infinite rewritings*, in [Kap90].

[Col82] A. COLMERAUER, *Prolog and infinite trees*, in Proceedings Logic Programming (K.L. Clark and S.-A. Tärnlund eds.), Academic Press, New York, 1982.

[Cou79] B. COURCELLE, *Infinite Trees in Normal Form and Recursive Equations Having a unique Solution*, Math. Systems Theory 13, 131-180 (1979).

[Der89a] N. DERSHOWITZ and S. KAPLAN, *Rewrite, rewrite, rewrite, rewrite, rewrite*, Principles of programming languages, Austin, Texas, 1989, pp. 250–259.

[Der89b] N. DERSHOWITZ, S. KAPLAN and D.A. PLAISTED, *Infinite Normal Forms (plus corrigendum)*, ICALP 1989, pp. 249–262.

[Der90a] N. DERSHOWITZ and J.P. JOUANNAUD, *Rewrite Systems*, to appear in *Handbook of Theoretical Computer Science* (ed. J. van Leeuwen) vol.B, chapter 15, North-Holland.

[Der90b] N. DERSHOWITZ, S. KAPLAN and D.A. PLAISTED, *Rewrite, rewrite, rewrite, rewrite, rewrite* to appear.

[Far88] W.M. FARMER, J.D. RAMSDELL, and R.J. WATRO, *A correctness proof for combinator reduction with cycles*, Report M88-53, The MITRE Corporation, Massachusetts, 1988.

[Far89] W.M. FARMER and R.J. WATRO, *Redex capturing in term graph rewriting*, in *Computing with the Curry Chip*, Report M89-59, The MITRE Corporation, Massachusetts, 1989.

[Hof88] B. HOFFMANN and D. PLUMP, *Jungle evaluation for efficient term rewriting*, Proc. Int. Workshop on Algebraic and Logic Programming, 1988. *Mathematical Research*, 49. (Akademie-Verlag, Berlin, 1988).

[Hud88] P. HUDAK et al, *Report on the Functional Programming Language Haskell.* Draft Proposed Standard, 1988.

[Hue79] G. HUET and J.-J. LÉVY, *Call by need computations in non-ambiguous linear term rewriting systems*, Report 359, INRIA, 1979.

[Kap90] S. KAPLAN and M. OKADA (eds.), Preliminary proceedings of CTRS 90 (Second International Workshop on Conditional and Typed Rewriting Systems), June 1990, Logic and Formal Method Lab., Dept. of Comp. Sci., Concordia Univ., Montreal, Canada.

[Kel55] J.L. KELLEY, *General Topology*, Graduate Texts in Mathematics 27, Springer-Verlag New York, 1955, Second Printing 1985.

[Ken87] J.R. KENNAWAY, *On 'On graph rewritings'*, Th. Comp. Sci., v.52, 37-58, 1987.

[Ken90a] J.R. KENNAWAY, *Graph rewriting in some categories of partial morphisms*, paper presented at the 4th Workshop on Graph Grammars, Bremen, 1990.

[Ken90b] J.R.KENNAWAY, J.W. KLOP, M.R. SLEEP and F.J. DE VRIES, *An infinitary Church-Rosser property for non-collapsing orthogonal term rewriting systems*, report CS-R9043, CWI, Amsterdam, 1990.

[Klo80] J.W. KLOP, *Combinatory reduction systems*, Mathematical Centre Tracts no. 127, CWI, Amsterdam 1980.

[Klo89] J.W. KLOP and A. MIDDELDORP, *Sequentiality in orthogonal term rewriting systems*, Report CS-R8932, CWI, Amsterdam, 1989.

[Klo91] J.W. KLOP, *Term rewriting systems*, to appear in *Handbook of Logic in Computer Science*, Vol I (eds. S. Abramsky, D. Gabbay and T. Maibaum), Oxford University Press, 1991.

[Oya87] M. OYAMAGUCHI, *Sufficient sequentiality: a decidable condition for call-by-need computations in term rewriting systems*, unpublished draft, Faculty of Engineering, Mie University, Tsu-shi 514, Japan, 1987.

[Pey87] S.L. PEYTON JONES, *The Implementation of Functional Programming Languages*, (Prentice-Hall, 1987).

[Ros73] B.K. ROSEN, *Tree manipulating systems and Church Rosser theorems*, JACM 20 (1973) p. 160-187.

[Sek90] R.C. SEKAR and I.V. RAMAKRISHNAN, *Programming in Equational logic Beyond Strong Sequentiality*, LICS 1990.

[Sta80] J. STAPLES, *Computation on graph-like expressions*, Th. Comp. Sci., v.10, 171–185, 1980.

[Tur85] D.A. TURNER, *Miranda: a non-strict functional language with polymorphic types.* In J.-P. Jouannaud (ed.), Proc. ACM Conf. on Functional Programming Languages and Computer Architecture, Lecture Notes in Computer Science, vol.201, Springer-Verlag, 1985.

[Wad71] C.J. WADSWORTH, *Semantics and pragmatics of the lambda calculus*, PhD. thesis, Oxford University, 1971.