Infinitary Rewriting: From Syntax to Semantics*

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1 Introduction

Rewriting is the repeated transformation of a structured object according to a set of rules. This simple concept has turned out to have a rich variety of elaborations, giving rise to many different theoretical frameworks for reasoning about computation. Aside from its theoretical importance, rewriting has also been a significant influence on the design and implementation of real programming languages, most notably the functional and logic programming families of languages. For a theoretical perspective on the place of rewriting in Computer Science, see for example [14]. For a programming language perspective, see for example [16].

Much of the interest in rewriting paradigms for programming arises from the possibility of a dual reading of a rewrite rule. On the one hand, a rule can be read as a syntactic transformation on a structure. On the other hand, a rule can be read as an equation. For example, the rule:

$$\mathit{fibs} = f(0,1) \quad \text{where} \quad f(m,n) = \mathit{Cons}(m, f(n, m+n))$$

can be read either as an equational definition of a structure which is the infinite list of Fibonacci numbers, or alternatively as instructions for a rewriting machine to construct increasingly better approximations to this infinite list. No real machine can compute the whole of an infinite structure, but by defining suitable finite selectors, we can write programs for rewriting machines which define finite structures in terms of infinite ones. Thus giving the command:

$$\mathit{print}(\mathit{nth}(15, \mathit{fibs}))$$

to a suitable rewriting machine will result in the printing of the 15th Fibonacci number. The rewriting machine has to be careful about how it uses the definitions if it is to achieve a result. From a purely rewriting perspective, the problem is to find a sequence of reductions which is normalising, for example the famous normal order reduction for the lambda calculus [4]. Solutions to this problem are the basis of lazy functional languages. Using implementations of such languages, it is possible to program by devising a suitable set of equations over infinite data structures which can be read as syntactic rewrite rules which deliver an effective means of computing the solution.

* Dedicated in friendship to Jan Willem Klop on the occasion of his 60th birthday.

However, it is rather easy to write down things that look as if they have both equational and rewrite interpretations, but which do not do what one might expect. Here is an example:

$$primesthenfibs = append(primes, fibs)$$

where *append* appends one list to another, and *primes* and *fibs* are the infinite lists of prime and Fibonacci numbers. One can write this program in a lazy functional language such as Haskell, but the result is just the infinite list of primes — the Fibonacci numbers disappear. The problem here is that the first list does not have an end to attach the second list to, so the *append* function seeks forever.

It is clear that some styles of building infinite terms can be computationally useful, whilst others are not. This raises an interesting question for the underlying theory of term rewriting, which is: what happens to various standard results for term rewriting if we allow infinite terms and infinite rewriting sequences, and what should those infinitary concepts be? Do the standard confluence and related results still hold for orthogonal infinitary systems?

We give an account of a theory of infinitary rewriting, beginning with the initial work done with and inspired by Jan Willem Klop, and ending with some recent work on lambda calculus which derives model theoretic notions from the kind of infinite terms which obstruct some traditional theorems of finitary rewriting.

2 Infinite term rewriting systems

In this section we will introduce the basic concepts of infinite term and reduction sequence of transfinite length. We introduce the notion of a strongly convergent reduction sequences for the more general setting of abstract reduction systems. Then we will describe some of the basic theorems that hold for infinite extensions of term rewriting systems. Detailed proofs can be found in [7, 9, 10].

2.1 Infinite terms

By interpreting finite terms as trees, infinite terms can be defined as trees having infinite branches as in Figure 1. There is a decision to be made about whether an infinite path in such a tree may be allowed to have a symbol at its end, which could then have further descendants, allowing paths from the root of a term to its leaves to have any ordinal length. We have taken the view that such terms have no computational meaning. Although we might imagine the limit of a reduction sequence $A \to B(A) \to B(B(A)) \to \ldots$ to be the term $B(B(B(...(A))))$ with infinitely many occurrences of B, there is no corresponding infinite process by which the symbol at the end of such a branch might be brought back up to the root.

We shall also require trees to be finitely branching, that is, that every operator symbol have finite arity. It is not clear whether allowing infinite arities

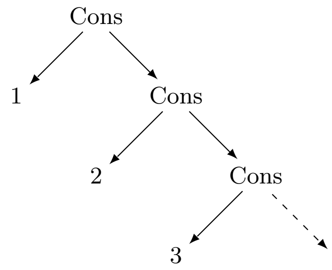

Fig. 1. An infinite term

would significantly change the resulting theory, but it would significantly complicate the exposition for little gain, and the restriction is reasonable on intuitive computational grounds.

Definition 1. *Let $\mathcal{V}$ be a set of variables and let $\Sigma = \bigcup \Sigma^n$ be a signature, i.e. a disjoint union of sets Σ^n of function symbols of arity $n \geq 0$. The set $\mathcal{T}^\infty(\Sigma, \mathcal{V})$ (abbreviated as $\mathcal{T}^\infty$) of finite and infinite terms is defined by coinduction from the grammar:*

$$t := x \mid f(t_1, \ldots, t_n) \text{ for } x \in X \text{ and } f \in \Sigma^n$$

To talk about subterms in a precise way, we introduce the concept of an address. An address is a finite sequence of positive integers $\alpha = i_1 \ldots i_n$, where n is called the *depth* of α. Given a term t, the subterm of t at α (if it exists) is denoted by $t|_\alpha$ and defined as follows. If α is empty then $t|_\alpha$ is t. If $\alpha = i\beta$, $t = f(t_1, \ldots, t_m)$, and $i \leq n$, then $t|_\alpha = t_i|_\beta$. The result of replacing the subterm of t at α by a term t' is denoted by $t[\alpha := t']$.

A term in $\mathcal{T}^\infty$ can equivalently be defined as a function from a set of addresses to function symbols and variables. The set of addresses must satisfy three conditions: it must be prefix-closed; if αi is in the set, so is αj for $j < i$; and for each α there is an upper bound to the i for which αi is in the set. This is Rosen's definition of a "tree domain"[18], generalised to infinite terms with finite arities. Variables may only occur at leaves, i.e. at addresses α for which no αi is in the set.

Note that addresses are always finite, even for infinite terms. In a term such as that of Figure 1, there is nothing at the end of the infinite branch, and no infinite address $222\ldots$. However, we do not require any sort of regularity or computability. Given the symbols *Cons*, 0, and 1, all infinite lists of 0 and 1 are included in the set of infinite terms, even lists which are not recursively enumerable.

2.2 Reduction on infinite terms

A term rewrite rule is defined as usual, that is, a pair of terms, written $t \to t'$, such that t is not a variable and contains every free variable of t'. We allow t' to be infinite, but require t to be finite. The rule is called *left-linear* if no variable occurs more than once in t.

A single reduction step is defined in the same way as for finitary term rewriting. A *substitution* is a function from a set of variables to terms. A substitution σ is applied to a term t by replacing every occurrence in t of every variable x in the domain of σ by $\sigma(x)$. The result is denoted by $\sigma(t)$. Given a rule $p \to q$ and a term t, if $t|_\alpha = \sigma(p)$ for some substitution σ of terms for variables, then t can be reduced to $t[\alpha := \sigma(q)]$. The concept of an orthogonal set of rules is also identical to that for finitary term rewriting: all rules must be left-linear, and for any two rules $p \to q$ and $r \to s$, there is no address α such that $p|_\alpha$ exists and is not a variable and r and $p|_\alpha$ are unifiable (excluding the trivial case where the two rules are the same rule and α is the empty address).

On computational grounds we might restrict the sets of function symbols and of rewrite rules to be finite, but none of our results depend on such a restriction.

2.3 Reduction sequences of transfinite length

To define transfinite rewriting sequences, we must have some notion of the limit of a sequence of terms. The natural notion is one which arises from the standard metric on trees. For distinct terms t and t', define the *tree distance* $d(t, t') = 2^{-n}$, where n is the length of the shortest address α for which t and t' have different symbols at α, or for which α is in the tree domain of one but not the other. For example, $d(x, y) = 1$, $d(A(B), A(C)) = \frac{1}{2}$, and $d(A(B(C)), A(B)) = \frac{1}{4}$.

For an infinite rewriting sequence to be considered to converge to a limit, we might simply require that the sequence of its terms converge in the metric. This type of convergence (called *weak* or *Cauchy* convergence) was first substantially studied in [6].

Example 2. With the rules $A(x, y) \to A(y, x)$ and $B \to C$, the term $A(B, B)$ reduces to itself infinitely often. If one occurrence of B is reduced to C, the standard construction of a confluent diagram for finite orthogonal rewriting cannot be completed, because the bottom row of the diagram does not converge:

$$\begin{array}{cccccccccc} A(B,B) & \to & A(B,B) & \to & A(B,B) & \to & A(B,B) & \to & \ldots & A(B,B) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & & \downarrow \\ A(B,C) & \to & A(C,B) & \to & A(B,C) & \to & A(C,B) & \to & \ldots & ? \end{array}$$

Example 3. With the rule $I(x) \to x$, we can reduce $I^\omega = I(I(I(\ldots)))$ to itself infinitely often, reducing at the root of the term each time:

$$I(I(I(\ldots))) \to I(I(I(\ldots))) \to I(I(I(\ldots))) \to \ldots \; I(I(I(\ldots)))$$

But if we track the identity of the occurrences of I throughout the sequence, we observe something strange:

$$I_1(I_2(I_3(\ldots))) \to I_2(I_3(I_4(\ldots))) \to I_3(I_4(I_5(\ldots))) \to \ldots \; I_?(I_?(I_?(\ldots)))$$

Every redex in the original term is reduced, yet we still have infinitely many in the final term, none of which derive from any part of the initial term.

Example 4. With the rule $A(x) \to A(B(x))$ we have the following convergent sequence:

$$A(C) \to A(B(C)) \to A(B(B(C))) \to \dots \; A(B(B(\dots)))$$

If we again track the identities of subterms, we see that an endless stream of Bs flows down from the root, but none of those occurring in the final term derive from them:

$$A_1(C) \to A_2(B_1(C)) \to A_3(B_2(B_1(C))) \to \dots \; A_?(B_?(B_?(\dots)))$$

Example 5. (Due to Simonsen [24].) With an infinite set of rules:

$$\begin{aligned} A &\to B \\ F(G^k(C), x, y) &\to F(G^{k+1}(C), y, y) \qquad \text{for even } k \\ F(G^k(C), x, y) &\to F(G^{k+1}(C), A, y) \qquad \text{for odd } k \end{aligned}$$

we can construct the following weakly convergent reduction:

$$F(C, A, A) \to F(G(C), A, A) \to F(G(G(C)), A, A) \to \dots \; F(G^\omega, A, A)$$

We also have $F(C, A, A) \to F(C, A, B)$. However, $F(C, A, B)$ and $F(G^\omega, A, A)$ have no common reduct. Thus although the system is orthogonal, it is not confluent.

In all of these examples, what goes wrong is that although the terms of the sequence have larger and larger prefixes in common, rewriting always continues at the root of the term. In order to be able to relate the structure of the limiting term to the structures of the terms of the sequence, we require a stronger notion of convergence, according to which not only must the terms of the sequence converge, but the depths at which rewrites take place must increase without bound, so that larger and larger prefixes of the term remain "stable".

We can capture the essentials of the situation by considering abstract reduction systems equipped with a measure of the depth of a reduction.

Definition 6. *An* abstract reduction system *is a set $\mathcal{A}$ of objects called terms, and a function from a set $\mathcal{L}$ to $\mathcal{A} \times \mathcal{A}$. We write $a \xrightarrow{l} b$ if $l \in \mathcal{L}$ is mapped to (a, b), and call this a* reduction step. *Note that there can be more than one reduction step from a to b, of different sizes.*

A metric abstract reduction system *in addition has a metric on $\mathcal{A}$ and a measure of size s mapping $\mathcal{L}$ to positive real numbers.*

In such a system, a strongly convergent reduction sequence *of length α, for an ordinal α, consists of:*

1. *a sequence of terms t_β for all $\beta \leq \alpha$, and*
2. *for each $\beta < \alpha$, a reduction step $t_\beta \xrightarrow{s_\beta} t_{\beta+1}$,*

such that for every limit ordinal $\lambda \leq \alpha$, the sequence $\{s_\beta | \beta < \lambda\}$ tends to zero.

We write $t \to t'$ for a single reduction step, $t \twoheadrightarrow t'$ for a finite sequence of reductions, and $t \twoheadrightarrow\!\!\!\!\!\rightarrow t'$ for a possibly infinite strongly convergent sequence. $t \to^\alpha t'$ denotes a strongly convergent sequence of length α.

The equality relation generated by the transfinite rewrite relation is the equivalence closure of $\twoheadrightarrow\!\!\!\!\!\rightarrow$, that is, $(\twoheadrightarrow\!\!\!\!\!\rightarrow \cup \leftarrow\!\!\!\!\!\twoheadleftarrow)^*$.

A term rewriting system forms a metric abstract reduction system in an obvious way: the size of a reduction step is 2^{-d} where d is the length of the address of the redex, and the metric is the tree distance.

Metric abstract reduction systems on their own, however, have too little structure to produce interesting theorems. For that we depend on the term structure.

Example 7. With the rule $I(x) \to x$ and the term I^ω as in Example 3, we can reduce every other redex of the initial term, and obtain a limiting term whose subterms all arise from subterms of terms earlier in the sequence:

$$I_1(I_2(I_3(I_4(\dots)))) \to I_2(I_3(I_4(I_5(\dots)))) \to I_2(I_4(I_5(I_6(\dots)))) \to$$

$$I_2(I_4(I_6(I_7(\dots)))) \to^\omega I_2(I_4(I_6(I_8(\dots))))$$

Example 8. With the rule $A \to B(A)$, we can generate an infinite term in a way similar to Example 4:

$$A_1 \to B_1(A_1) \to B_1(B_2(A_1)) \to^\omega B_1(B_2(B_3(\dots)))$$

However in this case the place where reductions happen moves down the term instead of staying at the root.

The movement of reductions to deeper and deeper levels is the crucial property that allows the structure of the limiting term to be related to that of the earlier terms in the sequence.

Note that when a rewrite system is "top-terminating" (having no reduction sequences performing infinitely many reductions at the root), a condition introduced by Dershowitz *et al.* [6], weak convergence and strong convergence are equivalent. However, many systems of interest are not top-terminating.

2.4 Compression of transfinite sequences to length ω

Once we have the concept of an infinite rewriting sequence that converges to a limit term, we cannot avoid opening the door to rewriting sequences of any ordinal length. If the limit term after ω steps contains redexes, we can continue to rewrite, to generate a reduction sequence of length $\omega + \omega$, ω^2, or longer.

From Example 3 we can see that a reduction of at least any countable ordinal length can be constructed. This is in fact the maximum: because arities and addresses are finite, there are no uncountably long strongly convergent sequences.

Theorem 9. *Every strongly convergent sequence has countable length.*

Proof. In a strongly convergent sequence, there can be only finitely many reductions of depth n, for any given finite n. Therefore the total number of steps must be countable. □

For left-linear systems we can prove a much stronger result, which helps to give computational meaning to sequences longer than ω: they are all equivalent to sequences of length at most ω.

Theorem 10 (Compression Lemma). *In a left-linear term rewriting system, for every strongly convergent sequence $t \to^\alpha t'$, there is a reduction from t to t' of length at most ω.*

Proof. This is proved by induction on α.

If $\alpha = \lambda + 1$ for a limit ordinal λ, then the redex reduced by the final step must, by strong convergence and the finiteness of left hand sides, already exist at some point before λ. One can show that it is possible to reduce it at such an earlier point, and to carry out the remainder of the original reduction sequence in no more than λ steps. By repetition, this proves the theorem for $\lambda + n$ for all finite n.

If α is a limit ordinal greater than ω, then we proceed by considering the minimum depth d of any step in the sequence. One can reorder the sequence so as to perform all of the steps at depth d within some finite initial segment of an equivalent sequence no longer than α. The remainder of the sequence performs reductions only at depth at least $d + 1$. Repeating the argument generates a sequence consisting of at most ω finite subsequences, in which the nth subsequence performs reductions only at depth at least n. This sequence must converge to the limit of the original sequence. □

Example 11. The Compression Lemma does not hold for weakly convergent reduction in left-linear systems. Consider the rules $G(x, B) \to G(F(x), B)$ and $B \to C$. $G(A, B)$ reduces by weakly convergent reduction to $G(F^\omega, C)$ in $\omega + 1$ steps but not in any smaller number:

$$G(A, B) \to G(F(A), B) \to G(F(F(A)), B) \to^\omega G(F^\omega, B) \to G(F^\omega, C)$$

Example 12. The Compression Lemma does not hold for strongly converging reductions in non-left-linear systems. Consider the rules $A \to G(A)$, $B \to G(B)$, and $F(x, x) \to C$. Then $F(A, B) \to^\omega F(G^\omega, G^\omega) \to C$, but $F(A, B)$ does not reduce to C in fewer than $\omega + 1$ steps.

2.5 Confluence

One of the fundamental properties of finite rewriting in orthogonal systems is confluence. Surprisingly, this turns out to not quite hold for strongly convergent reductions. A limited version does hold, called the Strip Lemma.

Theorem 13 (Strip Lemma). *If $t_0 \to t_1$ and $t_0 \twoheadrightarrow t_2$, then for some t_3, $t_1 \twoheadrightarrow t_3$ and $t_2 \twoheadrightarrow t_3$.*

Proof. The proof is essentially the same as for finitary term rewriting. We consider the set of residuals of the redex $t_0 \to t_1$ in each term in the reduction of t_0 to t_2. Because the residuals of a subterm are always disjoint from each other (that is, none of them is a subterm of any other), each of these sets of residuals has a strongly convergent complete development. It is a straightforward matter to show that the resulting construction of a tiling diagram can be carried through, and that its bottom side is strongly convergent. □

But confluence fails.

Example 14. Consider the rules $A(x) \to x$ and $B(x) \to x$. In the term $A(B(A(B(\dots))))$, if we reduce all of the A redexes, we obtain $B(B(B(\dots)))$ but if we reduce all of the B redexes, we obtain $A(A(A(\dots)))$. These two terms reduce only to themselves, and have no common reduct.

Example 15. By adding the rule $C \to A(B(C))$ to the previous example, we obtain an example in which all the terms in the two sequences except for the limiting terms are finite.

$$C \to A(B(C)) \to A(C) \to A(A(B(C))) \to A(A(C)) \to^{\omega} A(A(A(\dots)))$$

$$C \to A(B(C)) \to B(C) \to B(A(B(C))) \to B(B(C)) \to^{\omega} B(B(B(\dots)))$$

However, the situation is not lost. Examples similar to the above are essentially the only way in which an orthogonal transfinite term rewriting system can fail to be confluent.

Definition 16. *A* collapsing rule *is a rewrite rule whose right hand side is a variable. A* hyper-collapsing term *is a term whose every reduct reduces to a redex of a collapsing rule. A* collapsing tower *is a term of the form $t_1[\alpha_1 := t_2[\alpha_2 := t_3[\alpha_3 := \dots]]]$, where each term $t_i[\alpha_i := x]$ is a redex of a collapsing rule $t \to x$ such that $t|_{\alpha_i} = x$.*

Theorem 17. *Strongly convergent reduction in an orthogonal term rewriting system is confluent if and only if it contains at most one collapsing rule, and the left hand side of that rule contains only one variable.*

We can also prove restricted versions of confluence for systems not covered by the above theorem, to the effect that if collapsing towers do not arise in the construction of a particular tiling diagram, its construction can be completed. For details we refer to [8].

The types of orthogonal rewriting system that are used to model functional languages almost always contain multiple collapsing rules, for example, to implement selectors for data structures:

$$\mathit{Head}(\mathit{Cons}(x,y)) \to x \qquad \mathit{Tail}(\mathit{Cons}(x,y)) \to y$$

These rules immediately give counterexamples to confluence like that of Example 14.

Instead of proving exact confluence for restricted situations, we can prove approximate versions of confluence for all orthogonal systems. Such theorems can be found by further consideration of the meaning of hyper-collapsing terms.

2.6 A more general way of restoring confluence

The collapsing towers which obstruct confluence do not have an obvious meaning. In domain-theoretic terms, a term such as I^ω with the rewrite rule $I(x) \to x$ suggests the least fixed point of the identity function, which is undefined. The same is true of all the hyper-collapsing terms. If we regard these terms as meaningless, and identify them all with each other, it turns out that the confluence property is recovered for orthogonal systems.

Definition 18. *Given a class of terms $\mathcal{U}$, rewriting is* confluent modulo $\mathcal{U}$ *if, whenever $t_0 \twoheadleftarrow t_1 \overset{\mathcal{U}}{\longleftrightarrow} t_2 \twoheadrightarrow t_3$, there exist t_4 and t_5 such that $t_0 \twoheadrightarrow t_4 \overset{\mathcal{U}}{\longleftrightarrow} t_5 \twoheadleftarrow t_3$.*

Theorem 19. *An orthogonal term rewriting system is confluent modulo $\mathcal{HC}$.*

The next theorem assures us that the identification of all hyper-collapsing terms with each other introduces no new equalities, since they are already provably equal.

Theorem 20. *Any two hyper-collapsing terms t and t' are interconvertible. Specifically, there exist terms t'', s, and s' such that $t \twoheadrightarrow s \twoheadleftarrow t'' \twoheadrightarrow s' \twoheadleftarrow t'$.*

Proof. Since t and t' are hyper-collapsing, they reduce to collapsing towers $C_0[C_1[C_2[\ldots]]]$ and $D_0[D_1[D_2[\ldots]]]$. The term $C_0[D_0[C_1[D_1[C_2[D_2[\ldots]]]$ reduces to each of these towers. □

2.7 Axiomatic treatment of undefinedness

Theorem 19 was proved for orthogonal term rewriting systems in [8], but later work has shown that it does not depend on the details of this particular set of terms. Instead, we can state a set of axioms which any set of "undefined" terms should satisfy, and derive confluence modulo undefinedness from these axioms. Some preliminary definitions are necessary:

Definition 21. *For any set $\mathcal{U}$ of terms, define $t \overset{\mathcal{U}}{\longleftrightarrow} t'$ if t' can be obtained from t by replacing some (finite or infinite) set of subterms of t in $\mathcal{U}$ by terms in $\mathcal{U}$. The transitive closure of $\overset{\mathcal{U}}{\longleftrightarrow}$ is denoted by $\overset{\mathcal{U}}{=\!=}$.*

Let t contain a redex by a rule $p \to q$ at address α, and a subterm at address β. That subterm overlaps *the redex if $\beta = \alpha\gamma$ for some nonempty γ such that $p|_\gamma$ exists and is not a variable.*

A term t is root-active *if every reduct of t can be reduced to a redex. The set of root-active terms is denoted by $\mathcal{RA}$.*

Definition 22. *A set $\mathcal{U}$ satisfying the following axioms will be called a* set of undefined terms*.*

1. *Closure. For all $s \twoheadrightarrow t$, $s \in \mathcal{U}$ if and only if $t \in \mathcal{U}$.*
2. *Overlap. If t is a redex, and some subterm of t overlapping the redex is in $\mathcal{U}$, then $t \in \mathcal{U}$.*
3. *Activeness. $\mathcal{U}$ includes $\mathcal{RA}$.*
4. *Indiscernability. If $t \stackrel{\mathcal{U}}{\longleftrightarrow} t'$ then $t \in \mathcal{U}$ if and only if $t' \in \mathcal{U}$.*

These axioms were first stated in [10], except that we have here strengthened the Closure axiom, which originally required only that $\mathcal{U}$ be closed under reduction. This extra condition ensures that the Compression Lemma continues to hold for an extended form of reduction we shall introduce in section 2.8. In most cases, the Indiscernability axiom is the only axiom requiring any significant effort to prove. An equivalent way of stating it is that the $\stackrel{\mathcal{U}}{\longleftrightarrow}$ and $\stackrel{\mathcal{U}}{=}$ relations are identical.

In [10, 7] it is proved that for any set $\mathcal{U}$ satisfying enough of these axioms, transfinite reduction is confluent modulo $\mathcal{U}$, and also possesses the following genericity property:

Definition 23. *Call a term* totally meaningful *if none of its subterms is in $\mathcal{U}$. $\mathcal{U}$ is* generic *if for every $s \in \mathcal{U}$ and every term t, if $t[x := s]$ reduces to a totally meaningful term t', then for every term r, $t[x := r]$ also reduces to t'.*

Theorem 24. *In an orthogonal sytem, if $\mathcal{U}$ satisfies all the axioms except possibly Activeness, and includes $\mathcal{HC}$, then reduction is confluent modulo $\mathcal{U}$. If $\mathcal{U}$ satisfies Closure and Overlap, it is generic.*

The root-active terms are themselves a class satisfying all the axioms, and the hyper-collapsing terms satisfy all but the Activeness axiom. In the next section we will give some other concrete examples.

2.8 Syntactic domain models from sets of undefined terms

Another way of looking at the concept of reduction modulo undefinedness is to identify all undefined terms with each other by introducing a new symbol $\bot$. Terms which may contain $\bot$ are called *partial terms*, and form the set $\mathcal{T}^{\infty}_{\bot}$. $\mathcal{T}^{\infty}_{\bot}$ is partially ordered by the prefix order $\preceq$, defined as the least partial order for which $\bot$ is the bottom element and all the function symbols are monotonic. The rewrite relation of the original rewrite system $\mathcal{R}$ extends immediately to partial terms. A set $\mathcal{U}$ of undefined terms can be extended to a set $\mathcal{U}_\bot \subseteq \mathcal{T}^{\infty}_{\bot}$ by defining $t \in \mathcal{U}_\bot$ if there is a way of replacing all occurrences of $\bot$ in t by terms in $\mathcal{U}$ to obtain a term in $\mathcal{U}$. (Note that by the Indiscernability property, if one such substitution yields a term in $\mathcal{U}$, then every substitution does.) We then add an additional rule called $\bot_{\mathcal{U}}$-reduction, which allows any undefined subterm to be replaced by $\bot$. Let $\mathcal{R}^{\infty}_{\bot_{\mathcal{U}}}$ denote this extension of the original system $\mathcal{R}$. We write $\to_{\mathcal{R}}$ for rewriting by the original rules, $\to_{\bot_{\mathcal{U}}}$ for rewriting by the new rule, and $\to_{\mathcal{R}\bot_{\mathcal{U}}}$ for the combination.

For any set $\mathcal{U}$ of undefined terms in an orthogonal term rewriting system, the following statements hold.

1. $\mathcal{R}^{\infty}_{\bot_{\mathcal{U}}}$ satisfies the Compression Lemma.
2. $\mathcal{R}^{\infty}_{\bot_{\mathcal{U}}}$ is confluent.
3. $\bot_{\mathcal{U}}$-reductions can always be postponed after ordinary reductions. That is, if $t \twoheadrightarrow_{\mathcal{R}\bot_{\mathcal{U}}} t'$, then for some t'', $t \twoheadrightarrow_{\mathcal{R}} t'' \twoheadrightarrow_{\bot} t'$. This fact serves to connect reductions in the augmented system with plain reductions of ordinary terms.
4. Every term t has a unique normal form $\mathsf{NF}(t)$ by strongly converging $\twoheadrightarrow_{\mathcal{R}\bot_{\mathcal{U}}}$ reduction.

This allows the construction of models of a term rewriting system. The normal forms of $\mathcal{R}^{\infty}_{\bot_{\mathcal{U}}}$ are the values, with the semantics given by the mapping NF of terms to their unique normal forms.

The properties of this interpretation depend on the choice of $\mathcal{U}$. In some pathological cases the normal form function is not monotonic, and the set of normal forms may not be a complete partial order with respect to the prefix order. As a somewhat contrived counterexample, consider a term rewriting system with two unary symbols s and p, and no rewrite rules. Any set of the form $\{t\}$, where t is any term which is not a proper subterm of itself (as infinite terms can be) satisfies all of the axioms of undefinedness. (Closure, Overlap, and Activeness are trivial when there are no rewrite rules.) Take $\mathcal{U} = \{s(p^{\omega})\}$. Then $s(\bot)$ is a normal form for $\to_{\mathcal{R}\bot_{\mathcal{U}}}$, but $s(\bot) \preceq s(p^{\omega}) \to_{\bot_{\mathcal{U}}} \bot$, and so $\mathsf{NF}(s(\bot)) \not\preceq \mathsf{NF}(s(p^{\omega}))$. Furthermore, NF is not continuous at limit points, since every finite prefix of $s(p^{\omega})$ is its own normal form.

2.9 Sets of undefined terms

There may be many different sets of undefined terms in an orthogonal term rewriting system. The set of root-active terms is the smallest set of undefined terms. Trivially, the set of all terms is the largest. We call sets which are smaller than the set of all terms *consistent*. The intersection of any set of sets of undefined terms is a set of undefined terms. This does not necessarily hold for unions: Closure, Overlap, and Activeness all hold for unions, but Indiscernability may not.

An interesting set of undefined terms is the *opaque* terms. A term t is opaque if no reduct of that term can overlap any redex. This is proved to be a set of undefined terms in [10] for the axioms used there. Our stronger form of the Closure axiom can be ensured by extending the set to include every term that reduces to an opaque term, and we shall use this as our definition of opaqueness here. Orthogonality immediately implies that all root-active terms are opaque, but in general there are many others. For example, $\mathit{Head}(\mathit{Nil})$ is opaque in a term rewriting system with just the rule $\mathit{Head}(\mathit{Cons}(x, y)) \to x$.

Of more interest is the concrete term rewriting system for calculating Fibonacci numbers in Figure 2. The opaque terms in this TRS are the terms

$$
\begin{array}{ccc}
0 + y \to y & nth(0, y : z) \to y & fibs \to f(0, s(0)) \\
s(x) + y \to s(x + y) & nth(s(x), y : z) \to nth(x, z) & f(x, y) \to x : f(y, x + y)
\end{array}
$$

Fig. 2. The orthogonal Fibonacci TRS

that cannot reduce to any instance of 0, $s(x)$, or $x : y$. This includes root-active terms like $0 + (0 + (\dots))$, but also some normal forms such as $nth(0, 0)$. The normal forms of $\twoheadrightarrow_{\mathcal{R}\perp_{opaque}}$ reduction are all the terms built from the constructor symbols, i.e. 0, s, and :, together with $\perp$.

3 Infinite lambda calculus

The theory of infinitary term rewriting can be extended to lambda calculus in a straightforward way, allowing us to prove confluence modulo a similar notion of undefined term, and to derive models from sets of undefined terms. The collection of all sets of undefined lambda terms turns out to be much richer than our original collection of three such sets in [9]. We will explain and extend some of the recent developments in [21–23].

3.1 Infinite λ-terms

The concept of an infinite term can be defined for lambda calculus in the same way as for terms, interpreting application as a binary operator and λx as a unary operator for each x.

Definition 25. *The set of $\Lambda^\infty_\perp$ of finite and infinite λ-terms is defined by coinduction from the grammar:*

$$M ::= \perp \mid x \mid \lambda x.M \mid MM$$

The set Λ^∞ consists of the terms in $\Lambda^\infty_\perp$ which do not contain $\perp$.

We ignore the identity of bound variables and do not distinguish alpha-equivalent terms, considering $(\lambda x.x)(\lambda x.x)$ and $(\lambda y.y)(\lambda z.z)$ to be the same term. In particular, the distance between two terms is defined to be the minimum tree distance between any members of their alpha-equivalence classes.

Definition 26. *We will need the following abbreviations of λ-terms:*

1. $\mathbf{\Delta} = \lambda x.xx$, $\mathbf{\Omega} = \mathbf{\Delta\Delta}$, $\mathbf{K} = \lambda x \lambda y.x$, $\mathbf{I} = \lambda x.x$ *and the fixed point combinator* $\mathbf{Y} = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$
2. *The normal form of the fixed point* $\mathbf{YK}$ *of* $\mathbf{K}$ *is* $\mathbf{O} = \lambda x_1 \lambda x_2 \lambda x_3 \dots$, *also known as the* ogre.[1]

[1] Because it eats an unlimited number of arguments.

3.2 Reduction on infinite λ-terms

The rule of β-reduction extends in the obvious way to $\Lambda^{\infty}_{\perp}$. The concept of a strongly convergent reduction sequence in Definition 6 applies to the set $\Lambda^{\infty}_{\perp}$. Since beta reduction is a collapsing rule, it is not surprising that the confluence property fails, for the same reason it fails for term rewriting. In fact, even the Strip Lemma fails. This is because in lambda calculus, unlike term rewriting, the residuals of a redex can be nested within each other, and in the Strip Lemma diagram, it is possible to find examples in which the set of residuals of the initial redex by an infinite sequence form a collapsing tower.

Example 27. We show a simple counterexample to the Strip Lemma which can be found in [2]. Define $W = \lambda x.\mathbf{I}(xx)$. Then the term $\mathbf{\Delta} W$ has a one-step reduction to $\mathbf{\Omega} = \mathbf{\Delta\Delta}$ and an infinite reduction to $\mathbf{I}(\mathbf{I}(\ldots)))$, namely

$$\mathbf{\Delta} W \to_\beta WW \to_\beta \mathbf{I}(WW) \to_\beta \mathbf{I}(\mathbf{I}(WW)) \twoheadrightarrow_\beta \mathbf{I}(\mathbf{I}(\ldots)))$$

Both $\mathbf{\Delta\Delta}$ and $\mathbf{I}(\mathbf{I}(\mathbf{I}(\ldots)))$ reduce only to themselves, and have no common reduct. Note that both terms are examples of root-active terms in the sense of Definition 21 applied to lambda calculus.

Despite the counterexample there are several useful restricted forms of Strip Lemma [7]. For instance:

Theorem 28 ([7]). *If $M_0 \to_\beta M_1$ and $M_0 \twoheadrightarrow_\beta M_2$ then for some M_3, $M_1 \to_\beta M_3$ and $M_2 \twoheadrightarrow_\beta M_3$ provided $M_0 \to M_1$ is a head β-reduction.*

It is interesting to note that although root-active terms may not have common reducts, they are all interconvertible.

Theorem 29. *For every root-active term M, there is a term which reduces to both M and $\mathbf{I}^\omega$.*

Proof. For any term M, define $M^{\mathbf{I}}$ to be the term resulting from replacing every application PQ in M by $\mathbf{I}(PQ)$. Clearly $M^{\mathbf{I}} \twoheadrightarrow M$. We also have $(P[x := Q])^{\mathbf{I}} = P^{\mathbf{I}}[x := Q^{\mathbf{I}}]$ (which is immediate by considering the introduced copies of $\mathbf{I}$ as labels attached to the applications, and applying the technique of labelled reduction [17]). Hence also

$$((\lambda x.P)Q)^{\mathbf{I}} = \mathbf{I}(\lambda x.P^{\mathbf{I}})Q^{\mathbf{I}} \to \mathbf{I}(P^{\mathbf{I}}[x := Q^{\mathbf{I}}]) = \mathbf{I}(P[x := Q])^{\mathbf{I}} \to (P[x := Q])^{\mathbf{I}}$$

This lets us mimic for $M^{\mathbf{I}}$ any reduction of M: if $M \twoheadrightarrow M'$ then $M^{\mathbf{I}} \twoheadrightarrow M'^{\mathbf{I}}$.

If, however, we modify this construction by omitting the reduction of $\mathbf{I}$ whenever it occurs at the root, then we instead reduce $M^{\mathbf{I}}$ to $\mathbf{I}^n(M'^{\mathbf{I}})$ when the reduction of M to M' performs n reductions at the root. This transforms a reduction of M which performs infinitely many such reductions to a strongly convergent reduction of $M^{\mathbf{I}}$ to $\mathbf{I}^\omega$. □

Note that the proof of Theorem 20 does not work for lambda calculus, since a root-active term in lambda calculus (for example, $\mathbf{\Omega}$) need not be reducible to a collapsing tower. This is because in term rewriting, a reduction at the root cannot create new redexes, whereas in lambda calculus it can.

3.3 Undefinedness in lambda calculus

The remedies for the failure of confluence are the same as for term rewriting: we can identify a set of terms as undefined and define rewriting modulo this set,[2] or extend reduction with a $\perp$ rule reducing undefined terms to $\perp$, and prove that these forms of rewriting are confluent.

The Closure, Activeness, and Indiscernability axioms carry over unchanged. Note that since the β rule is a collapsing rule, all root-active terms are hyper-collapsing. The Overlap rule is also unchanged, but can be stated in a simpler and more explicit form. There is also an additional axiom requiring closure under substitution. This last axiom was not necessary for term rewriting, because the variables in a term behave like constant symbols, and are never substituted for by the process of reduction.

Definition 30. *A set $\mathcal{U} \subseteq \mathsf{\Lambda}^\infty$ will be called a* set of undefined terms *if it satisfies the Axioms of* Closure, Activeness *and* Indiscernability *of Definition 22 and the following two axioms:*

1. *Overlap. If $(\lambda x.P) \in \mathcal{U}$ then $(\lambda x.P)Q \in \mathcal{U}$.*
2. *Substitution. $\mathcal{U}$ is closed under substitution.*

Now let $\mathcal{U}$ be a set of terms of $\mathsf{\Lambda}^\infty$ satisfying the axioms. We add the following rewrite rule:

$$\frac{M[\perp := \mathbf{\Omega}] \in \mathcal{U} \quad M \neq \perp}{M \to \perp} \ (\perp_{\mathcal{U}})$$

(Note that by Indiscernability, there is nothing special about the use of the term $\mathbf{\Omega}$ — any other member of $\mathcal{U}$ could be used.) The infinitary lambda calculus over $\mathsf{\Lambda}^\infty_\perp$ with the β and $\perp_{\mathcal{U}}$ rules is denoted $\lambda^\infty_{\beta\perp_{\mathcal{U}}}$, and the combined reduction relation written simply $\to$. Reductions using only one or other of the rules will be denoted $\to_\beta$ or $\to_{\perp_{\mathcal{U}}}$

The $\perp_{\mathcal{U}}$ rule is of course not computable (since $\mathcal{U}$ is not recursively enumerable unless $\mathcal{U} = \mathsf{\Lambda}^\infty$), but it provides a mathematically convenient way of talking about terms modulo undefinedness. The postponement property in the next theorem serves to connect reductions in $\lambda^\infty_{\beta\perp_{\mathcal{U}}}$ with plain beta reduction.

Theorem 31. *Let $\mathcal{U}$ be a set of undefined terms.*

1. *Strongly converging reduction in $\lambda^\infty_{\beta\perp_{\mathcal{U}}}$ is confluent.*
2. *Every term M has a strongly converging reduction to normal form, which by the first part is unique and will be denoted by $\mathsf{NF}_{\mathcal{U}}(M)$.*

[2] Recently Ketema and Simonsen [12, 13] have shown that strongly converging reduction is confluent modulo $\mathcal{HC}$ in fully-extended orthogonal infinitary combinatory term rewriting systems with rules with finite right hand sides. Since the notions root-active and hyper-collapsing coincide in lambda calculus (because the beta rule is hyper-collapsing) their result generalises our results on confluence modulo $\mathcal{HC}$ for orthogonal term rewriting and confluence modulo $\mathcal{RA}$ for lambda calculus with the beta rule.

3. *$\bot$-reduction can be postponed after β-reduction. That is, if $M \twoheadrightarrow N$, then for some term L, $M \twoheadrightarrow_\beta L \twoheadrightarrow_{\bot_\mathcal{U}} N$.*
4. *The Compression Lemma holds for strongly converging reduction in $\lambda^\infty_{\beta\bot_\mathcal{U}}$.*

Thus $\lambda^\infty_{\beta\bot_\mathcal{U}}$ is a complete (normalising and confluent) extension of the finite lambda calculus λ_β.

Theorem 32. *Let $\mathcal{U}$ be a set of undefined terms. For any term M in $\lambda^\infty_{\beta\bot_\mathcal{U}}$ we have $\mathsf{NF}_\mathcal{U}(M) = \bot$ iff $M[\bot := \mathbf{\Omega}] \in \mathcal{U}$.*

Proof. "If"is trivial. "Only if": suppose that for a term M in $\Lambda^\infty_\bot$ we have that its normal form in $\lambda^\infty_{\beta\bot_\mathcal{U}}$ is equal to $\bot$. Hence there is a reduction $M \twoheadrightarrow_{\beta\bot} \bot$. By Theorem 31 this factors as $M \twoheadrightarrow_\beta K \twoheadrightarrow_\bot \bot$. Hence $M[\bot := \mathbf{\Omega}] \twoheadrightarrow_\beta K[\bot := \mathbf{\Omega}] \twoheadrightarrow_\bot K \twoheadrightarrow_\bot \bot$. By definition of $\bot_\mathcal{U}$-reduction and indiscernability it follows that $K[\bot := \mathbf{\Omega}] \twoheadrightarrow_\bot \bot$ implies $K[\bot := \mathbf{\Omega}] \rightarrow_\bot \bot$. Hence $K[\bot := \mathbf{\Omega}] \in \mathcal{U}$. Since $\mathcal{U}$ is closed under β expansion we find that $M[\bot := \mathbf{\Omega}] \in \mathcal{U}$. □

3.4 Sets of undefined lambda terms

In this section we will study the collection $\mathbf{U}$ of all sets of undefined terms. Since $\mathbf{U}$ is closed under intersections (though not under unions), it forms a complete lattice under set inclusion. The top and bottom elements are Λ^∞ and $\overline{\mathcal{TN}}$, and the meet operation is intersection. The join of a set of sets of undefined terms is the intersection of all sets of undefined terms containing their union.

Let us now give some concrete examples of such sets. For a while the following three sets were the only known sets of undefined lambda terms (cf. [8, 1, 10, 7]).

Definition 33. 1. *A term $M \in \Lambda^\infty$ is a* head normal form *if M is of the form $\lambda x_1 \dots x_n.yP_1 \dots P_k$. $\overline{\mathcal{HN}}$ is the set of terms without a finite β-reduction to head normal form.*
2. *A term $M \in \Lambda^\infty$ is a* weak head normal form *if M is a head normal form or $M = \lambda x.N$. $\overline{\mathcal{WN}}$ is the set of terms without a finite β-reduction to weak head normal form.*
3. *A term $M \in \Lambda^\infty$ is a* top normal form *if it is either a weak head normal for or an application (NP) if there is no Q such that $N \twoheadrightarrow_\beta \lambda x.Q$. $\overline{\mathcal{TN}}$ is the set of terms without a finite β-reduction to top normal form.*

Lemma 34. *$\overline{\mathcal{HN}}$, $\overline{\mathcal{WN}}$ and $\overline{\mathcal{TN}}$ satisfy the axioms for undefined terms.*

Proof. Apart from closure under expansion all the axioms have been proved to hold for $\overline{\mathcal{HN}}$, $\overline{\mathcal{WN}}$ and $\overline{\mathcal{TN}}$ in [10]. We show the expansion property for $\overline{\mathcal{HN}}$. Suppose N is a term in Λ^∞ without a β-reduction to head normal form. Suppose also that $M_1 \twoheadrightarrow_\beta N$ and M_1 has a head normal form M_2. Without loss of generality we may assume that there is a finite head reduction from M_1 to M_2. Repeated application of the Restricted Strip Lemma 28 then gives us a common reduct M_4 of both N and M_2. The term M_4 is a head normal form because it is a reduct of the head normal form M_2. This contradicts the assumption that N has no head normal form. Hence M_1 has no head normal form either. Closure under expansion for the other two sets can be proved in a similar way. □

If we now apply the Main Theorem 31 of the previous section to these three sets we find that $\lambda^{\infty}_{\beta\bot_{\overline{\mathcal{HN}}}}$, $\lambda^{\infty}_{\beta\bot_{\overline{\mathcal{WN}}}}$ and $\lambda^{\infty}_{\beta\bot_{\overline{\mathcal{TN}}}}$ are confluent and normalising extensions of finite lambda calculus. The normal form of a term M in $\lambda^{\infty}_{\beta\bot_{\overline{\mathcal{HN}}}}$ (respectively $\lambda^{\infty}_{\beta\bot_{\overline{\mathcal{WN}}}}$ and $\lambda^{\infty}_{\beta\bot_{\overline{\mathcal{TN}}}}$) corresponds to the Böhm tree (respectively the Lévy-Longo tree and Berarducci tree) of M. As a useful corollary of the confluence and normalisation property of $\lambda^{\infty}_{\beta\bot_{\overline{\mathcal{TN}}}}$ we obtain a useful refinement of the old observation of Wadsworth [4] that finite lambda terms are either of the form $\lambda x_1 \dots \lambda x_n.yM_k \dots M_1$ or $\lambda x_1 \dots \lambda x_n.(\lambda y.P)QM_k \dots M_1$ where $n, k \geq 0$.

Lemma 35 ([23]). *A term in $\Lambda^{\infty}_{\bot}$ has one of the following five forms:*

1. $\lambda x_1 \dots \lambda x_n.yM_k \dots M_1$
2. $\lambda x_1 \dots \lambda x_n.(\lambda y.P)QM_k \dots M_1$
3. $\lambda x_1 \dots \lambda x_n.\bot M_k \dots M_1$
4. $\lambda x_1 \dots \lambda x_n.(((\dots M_3)M_2)M_1$
5. $\lambda x_1 \lambda x_2 \lambda x_3 \dots = \mathbf{O}$

Of course, the third option does not apply to terms in Λ^{∞}.

Now the key to constructing other sets of undefined terms lies in finding a definition of these sets in terms of what they include, rather than what they exclude [22, 23]. For doing so we need some terminology.

Definition 36.
1. A term $M \in \Lambda^{\infty}_{\bot}$ is *root-active* (with respect to β) if for all $M \twoheadrightarrow_{\beta} N$ there exists a redex $(\lambda x.P)Q$ such that $N \twoheadrightarrow_{\beta} (\lambda x.P)Q$.
2. A term $M \in \Lambda^{\infty}_{\bot}$ is a *head active form* if $M = \lambda x_1 \dots x_n.RP_1 \dots P_k$ and R is root-active.
3. A term $M \in \Lambda^{\infty}_{\bot}$ is a *strong active form* if $M = RP_1 \dots P_k$ and R is root-active.
4. A term $M \in \Lambda^{\infty}_{\bot}$ is a *strong active form relative to X* if $M = RP_1 \dots P_k$ and R is root-active and $P_1, \dots, P_k \in X$.
5. A term $M \in \Lambda^{\infty}_{\bot}$ is an *infinite left spine form* if $M = \lambda x_1 \dots x_n.((\dots)P_2)P_1$.
6. A term $M \in \Lambda^{\infty}_{\bot}$ is a *strong infinite left spine form* if $M = ((\dots)P_2)P_1$. A term $M \in \Lambda^{\infty}_{\bot}$ is a *strong infinite left spine form relative to X* if $M = ((\dots)P_2)P_1$ and $P_i \in X$ for all i.

Example 37.
1. The term $\mathbf{\Omega}$ is a finite root-active term. The fixed point $\mathbf{YI}$ reduces to the infinite root-active term $\mathbf{I}(\mathbf{I}(\mathbf{I}(\dots)))$.
2. $\mathbf{\Omega}xyz$, $(\mathbf{YI})xyz$ and $\mathbf{I}(\mathbf{I}(\mathbf{I}(\dots)))xyz$ are strong active terms.
3. The finite term $\mathbf{\Omega}_3 = (\lambda x.xxx)(\lambda x.xxx)$ reduces to the strong infinite left spine form $((\dots)\omega_3)\omega_3$, where $\omega_3 = \lambda x.xxx$.

We can now redefine the sets $\overline{\mathcal{HN}}$, $\overline{\mathcal{WN}}$ and $\overline{\mathcal{TN}}$.

Lemma 38.
1. *A term $M \in \Lambda^{\infty}$ has no top normal form if and only if M is root-active.*
2. *A term $M \in \Lambda^{\infty}$ has no weak head normal form if and only if M reduces to a strong head active form, or a strong infinite left spine form.*

3. *A term* $M \in \Lambda^\infty$ *has no head normal form if and only if* M *reduces to a head active form, an infinite left spine form, or the ogre.*

The reformulation of the set $\overline{\mathcal{WN}}$ reveals that the terms without weak head normal form in the lambda calculus are precisely the strong zero terms, terms of which no instance can reduce to an abstraction. Strong zero terms can also be characterised as those terms no reduct of which can overlap any redex, which are exactly the terms that we called *opaque* in section 2.9.

Before we can define a partition of Λ^∞ we need to define some notation.

Definition 39. We define the following subsets of Λ^∞.

$$\begin{array}{ll}
\mathcal{HA} & = \{M \in \Lambda^\infty \mid M \twoheadrightarrow_\beta N \text{ and } N \text{ is head active}\} \\
\mathcal{IL} & = \{M \in \Lambda^\infty \mid M \twoheadrightarrow\!\!\!\!\rightarrow_\beta N \text{ and } N \text{ is an infinite left spine form}\} \\
\mathcal{O} & = \{M \in \Lambda^\infty \mid M \twoheadrightarrow\!\!\!\!\rightarrow_\beta \mathbf{O}\} \\
\mathcal{RA} & = \{M \in \Lambda^\infty \mid M \text{ is root-active}\} \\
\mathcal{SA} & = \{M \in \Lambda^\infty \mid M \twoheadrightarrow_\beta N \text{ and } N \text{ is strong active}\} \\
\mathcal{SIL} & = \{M \in \Lambda^\infty \mid M \twoheadrightarrow\!\!\!\!\rightarrow_\beta N \text{ and } N \text{ is a strong infinite left spine form}\}
\end{array}$$

Theorem 40. Λ^∞ *is the disjoint union of* $\mathcal{HN}$, $\mathcal{HA}$, $\mathcal{IL}$ *and* $\mathcal{O}$.

With these components we can make the sets of undefined terms of Figure 3.

Theorem 41 ([23])**.** *All eight sets of Figure 3 are sets of undefined terms.*

Proof. The proofs for the three sets defined in Definition 33 can be found in [10]. The proofs for all other sets can be found in [23]. □

There are many more sets of undefined terms besides the eight depicted in Figure 3. Although we do not have a complete classification, we can say where these other sets can be found in relation to those eight sets. In the figure we use solid arrows $X \longrightarrow Y$ to express that $X \supset Y$ and there are NO other sets of undefined terms in between X and Y. Dashed arrows $X \dashrightarrow Y$ indicate that $X \supset Y$ and that there are at least $2^{\mathfrak{c}}$ many other sets of undefined terms in between X and Y, where $\mathfrak{c}$ is the cardinality of the continuum. To prove the correctness of these arrows we will first prove a useful lemma.

Lemma 42. *Let* $\mathcal{U}$ *be a set of undefined terms.*

1. *If* $\lambda x.M \in \mathcal{U}$ *then* $M \in \mathcal{U}$.
2. *If* $\lambda x.M \in \mathcal{U}$ *for some* M *then* $\mathcal{HA} \subseteq \mathcal{U}$.
3. *If* $\mathbf{O} \in \mathcal{U}$ *then* $\mathcal{HA} \subseteq \mathcal{U}$.
4. *If* $\lambda x.M \in \mathcal{U}$ *and* $\mathcal{U} \subseteq \mathcal{SA} \cup \mathcal{SIL}$ *then* $\mathcal{U} \subseteq \mathcal{HA} \cup \mathcal{IL}$.
5. *If* $\mathcal{SIL} \subseteq \mathcal{U}$ *then* $\mathcal{SA} \subseteq \mathcal{U}$.
6. *If* $\mathcal{IL} \subseteq \mathcal{U}$ *then* $\mathcal{HA} \subseteq \mathcal{U}$.
7. *If a head normal form is in* $\mathcal{U}$ *then* $\mathcal{U} = \Lambda^\infty$.

Proof. Proofs as in [23]. These are straightforward deductions from the axioms: for example, to prove (1), $\lambda x.M \in \mathcal{U}$ implies $(\lambda x.M)x \in \mathcal{U}$ by indiscernability, therefore $M \in \mathcal{U}$ by Closure. □

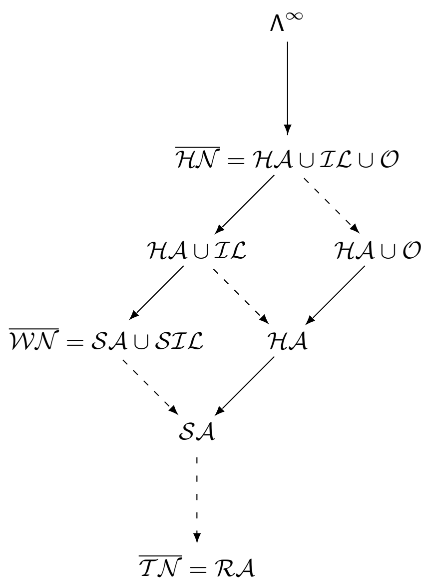

Fig. 3. The eight main sets of undefined terms

Theorem 43. *The set $\overline{\mathcal{HN}}$ is the largest set of undefined terms which is a proper subset of Λ^∞.*

Proof. The first statement follows from Lemma 42(7). The second statement follows directly from the axioms of undefinedness. $\square$

Definition 44. *Let $A \subseteq B$ be two sets of undefined λ-terms. The* (open) interval *$\langle A, B\rangle$ is the set $\{C \mid A \subsetneq C \subsetneq B$ and C is a set of undefined λ-terms$\}$.*

Theorem 45. *1. The interval $\langle \mathcal{SA}, \mathcal{HA} \cup \mathcal{O}\rangle$ contains only the element $\mathcal{HA}$.*
2. The interval $\langle \mathcal{SA} \cup \mathcal{SIL}, \mathcal{HA} \cup \mathcal{IL} \cup \mathcal{O}\rangle$ contains only the element $\mathcal{HA} \cup \mathcal{IL}$.

Proof. These follow from Lemma 42, parts (2) and (4) respectively. $\square$

Theorems 43 and 45 imply that the solid arrows in Figure 3 are correctly drawn. From Theorem 45(2) it follows also that the collection of sets of undefined terms is not closed under unions. The union of the sets of undefined terms $\mathcal{SA} \cup \mathcal{SIL}$ and $\mathcal{HA}$ is $\mathcal{SIL} \cup \mathcal{HA}$, which is not a set of undefined terms. The next theorem will imply the correctness of the dashed arrows in Figure 3.

Definition 46. *Let $X \subset \Lambda^\infty$.*

1. *We say that a term M is a* strongly head active term relative to *X if M reduces to a term of the form $RX_1 \dots X_n$, where $R \in \mathcal{RA}$ and $X_1, \dots, X_n \in X$. We denote the set of strongly head active terms relative to X by $\mathcal{SA}_X$.*

2. *We say that a term M is a* strong infinite left spine term relative to X *if M reduces to a term of the form $\dots X_3 X_2 X_1$ where all $X_i \in X$. We denote the set of strong infinite left spine terms relative to X by $\mathcal{SIL}_X$.*
3. *We say that a term M is an* almost strong infinite left spine term relative to X *if M reduces to a term of the form $\dots X_3 X_2 X_1 N_k \dots N_1$ where all $X_i \in X$ and the $N_i \in \Lambda^\infty_\perp$. We denote the set of almost strong infinite left spine terms relative to X by $\mathcal{SIL}^+_X$.*
4. *The set $\mathcal{IL}^+_X$ of* almost infinite left spine terms relative to X *is defined similarly.*

Lemma 47. *If X is a subset of closed normal forms in Λ^∞ then $\mathcal{SA}_X$ is a set of undefined terms. Moreover if X does not contain subterms which are infinite left spine forms. then also $\mathcal{SA}_X \cup \mathcal{SIL}_X$, $\mathcal{SA}_\cup \mathcal{SIL}^+_X$, $\mathcal{HA} \cup \mathcal{IL}^+_X$ and $\mathcal{HA} \cup \mathcal{IL}^+_X \cup \mathcal{O}$ are sets of undefined terms*

Proof. In [23] we have shown that under their respective conditions $\mathcal{SA}_X$ and $\mathcal{SA}_X \cup \mathcal{SIL}_X$ are sets of undefined terms. The proofs of the similar statements for the other three sets are similar. ☐

Theorem 48. *The cardinality of each of the open intervals $\langle \mathcal{RA}, \mathcal{SA} \rangle$, $\langle \mathcal{SA}, \mathcal{SA} \cup \mathcal{SIL} \rangle$, $\langle \mathcal{HA}, \mathcal{HA} \cup \mathcal{IL} \rangle$, and $\langle \mathcal{HA} \cup \mathcal{O}, \mathcal{HA} \cup \mathcal{IL} \cup \mathcal{O} \rangle$ is at least $2^{\mathfrak{c}}$.*

Proof. There are at least $2^{\mathfrak{c}}$ subsets X of closed normal forms in Λ^∞ that do not contain subterms which are infinite left spine forms. The sets $\mathcal{SA}_X$, $\mathcal{SA}_\cup \mathcal{SIL}^+_X$, $\mathcal{HA} \cup \mathcal{IL}^+_X$ and $\mathcal{HA} \cup \mathcal{IL}^+_X \cup \mathcal{O}$ are element of the respective intervals listed in the theorem. ☐

3.5 Normal form models of the lambda calculus

By Theorem 31, each set $\mathcal{U}$ of undefined terms gives rise to a complete extension $\lambda^\infty_{\beta\perp_\mathcal{U}}$ of the finite lambda calculus λ_β. From each $\lambda^\infty_{\beta\perp_\mathcal{U}}$ we can construct a generalised Böhm model $\mathfrak{M}_\mathcal{U}$ of the finite lambda calculus as follows. As underlying set we take the set $\mathsf{NF}_\mathcal{U}(\Lambda^\infty_\perp)$ of normal forms of terms in $\lambda^\infty_{\beta\perp_\mathcal{U}}$. Let $\mathsf{NF}_\mathcal{U} : \Lambda^\infty_\perp \to \mathsf{NF}_\mathcal{U}(\Lambda^\infty_\perp)$ be the function that maps each M in $\Lambda^\infty_\perp$ to its normal form. On $\mathsf{NF}_\mathcal{U}(\Lambda^\infty_\perp)$ we define application simply by:

$$\mathsf{NF}_\mathcal{U}(M_1) \bullet \mathsf{NF}_\mathcal{U}(M_2) = \mathsf{NF}_\mathcal{U}(M_1 M_2)$$

The applicative structure $\mathfrak{M}_\mathcal{U} = \langle \mathsf{NF}_\mathcal{U}, \bullet \rangle$ extends readily to a syntactic model of the finite lambda calculus along the lines of Definition 5.3.2 in [4]. The construction works because of normalisation and confluence properties of $\lambda^\infty_{\beta\perp_\mathcal{U}}$.

We will call these models *normal form models*. The three well-known models of the Böhm trees [3, 4], the Lévy-Longo [15] trees and the Berarducci trees [5, 9] can be seen as examples of this construction and correspond respectively to $\mathfrak{M}_{\overline{\mathcal{HN}}}$, $\mathfrak{M}_{\overline{\mathcal{WN}}}$ and $\mathfrak{M}_{\overline{\mathcal{TN}}}$.[3] There are many different sets of undefined terms, and

[3] The concept of a Berarducci tree also applies to orthogonal term rewriting, since it is based on the concept of root-active term. Ketema asks in [11] whether the

so there are also many different normal form models. Note that $\mathfrak{M}_{\Lambda^\infty_\perp}$ degenerates to the single element $\perp$. The construction provides non-trivial models for all other sets of undefined terms. We will now examine some properties of these models.

Definition 49. Let $M, N \in \Lambda^\infty_\perp$. We say that M is a prefix of N (we write $M \preceq N$) if M is obtained from N by replacing some subterms of N by $\perp$.

The pair $(\Lambda^\infty_\perp, \preceq)$ is an algebraic cpo and its compact elements are the finite λ-terms. As for term rewriting, the pair $(\mathsf{NF}_\mathcal{U}(\Lambda^\infty_\perp), \preceq)$ may not be a cpo:

Counterexample 50 ([22]) Consider the term $(((\ldots\mathbf{K})\mathbf{K})\mathbf{I}$. The set $\mathcal{U} = \mathcal{RA} \cup \{M \in \Lambda^\infty \mid M \twoheadrightarrow_\beta ((\ldots\mathbf{K})\mathbf{K})\mathbf{I})\}$ is a set of undefined terms. The term $((\ldots\mathbf{K})\mathbf{K})\mathbf{I}$ is a redex in $\lambda^\infty_{\beta\perp_\mathcal{U}}$ but none of its prefixes $(((\perp\mathbf{K})\ldots\mathbf{K})\mathbf{K})\mathbf{I}$ contains a redex. Let X be the set of prefixes of $((\ldots\mathbf{K})\mathbf{K})\mathbf{I}$. Clearly $\bigcup X \neq ((\ldots\mathbf{K})\mathbf{K})\mathbf{I}$. Hence $(\mathsf{NF}_\mathcal{U}, \preceq)$ is not a cpo.

Notwithstanding such counterexamples it is not hard to show that the eight main sets of undefined terms give rise to models whose underlying set is a cpo.

Theorem 51. $(\mathsf{NF}_\mathcal{U}, \preceq)$ *is a cpo for any* $\mathcal{U}$ *chosen from the main sets of undefined terms of Figure 3.*

Next we consider the properties continuity and monotony.

Definition 52. *1. Let $\sqsubseteq$ be a partial order on $\Lambda^\infty_\perp$. A function $F : \Lambda^\infty_\perp \to \Lambda^\infty_\perp$ is called* monotone *in $(\Lambda^\infty_\perp, \sqsubseteq)$, if $F(M) \sqsubseteq F(N)$ for all $M, N \in \Lambda^\infty_\perp$ such that $M \sqsubseteq N$.*
2. Let $(\Lambda^\infty_\perp, \sqsubseteq)$ be a cpo. A function $F : \Lambda^\infty_\perp \to \Lambda^\infty_\perp$ is called continuous *in $(\Lambda^\infty_\perp, \sqsubseteq)$, if $\bigcup_{i\in I} F(M_i) = F(\bigcup_{i\in I} M_i)$ for any directed set $\{M_i \mid i \in I\} \subseteq \Lambda^\infty_\perp$, where a subset X of $\Lambda^\infty_\perp$ is* directed *if for any two elements $M_1, M_2 \in X$ there exists an $M_3 \in X$ such that $M_1 \sqsubseteq M_3$ and $M_2 \sqsubseteq M_3$.*

The function $\mathsf{NF}_\mathcal{U} : \Lambda^\infty_\perp \to \Lambda^\infty_\perp$ is not always continuous, or even monotone:

Counterexample 53 The map $\mathsf{NF}_\mathcal{U} : \Lambda^\infty_\perp \to \Lambda^\infty_\perp$ is not continuous in the cpo $(\Lambda^\infty_\perp, \preceq)$ in the following cases:

1. Case $\mathcal{U} = \overline{\mathcal{TN}}$: the Berarducci trees are not monotone in $(\Lambda^\infty_\perp, \preceq)$. Take $M = \perp y$, $N = (\lambda x.\perp)y$. Then $M \preceq N$ but $\mathsf{NF}_{\overline{\mathcal{TN}}}(M) \not\preceq \mathsf{NF}_{\overline{\mathcal{TN}}}(N)$.
2. Case $\mathcal{U} = \mathcal{HA} \cup \mathcal{IL}$. Now $\mathsf{NF}_{\mathcal{HA}\cup\mathcal{IL}}$ is monotone but not continuous. This can be seen as follows. The infinite sequence of abstractions $\mathbf{O} = \lambda x_1 x_2 \ldots$ is in normal form but the truncations $\mathbf{O}^n = \lambda x_1 \ldots x_n.\perp$ reduce to $\perp$ for all n. Hence $\bigcup_{n\in\omega} \mathbf{O}^n = \mathbf{O} = \mathsf{NF}(\mathbf{O}) \neq \bigcup_{n\in\omega} \mathsf{NF}(\mathbf{O}^n) = \perp$.

The prefix relation behaves well with respect to continuity only for the cases of Böhm and Lévy-Longo trees:

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concepts of Böhm tree and Lévy-Longo tree also apply to term rewriting. Sections 2.9 and 3.4 answer this affirmatively for Lévy-Longo trees, because in lambda calculus, the opaque terms are exactly the terms without weak head normal form.

Theorem 54 ([22]). *$\mathsf{NF}_{\mathcal{U}} : \Lambda^{\infty}_{\perp} \to \Lambda^{\infty}_{\perp}$ is continuous in $(\Lambda^{\infty}_{\perp}, \preceq)$ if and only if $\mathcal{U} = \overline{\mathcal{HN}}$ or $\mathcal{U} = \overline{\mathcal{WN}}$.*

Recall that Barendregt's proof [4] of the fact that the Böhm trees form a model for lambda calculus depends heavily on continuity. The previous theorem implies that this proof technique does not generalise to models other than the Lévy-Longo model.

Theorem 55 ([23]). *$\mathsf{NF}_{\mathcal{U}} : \Lambda^{\infty}_{\perp} \to \Lambda^{\infty}_{\perp}$ is monotone in $(\Lambda^{\infty}_{\perp}, \preceq)$ for any $\mathcal{U}$ chosen among the following: $\mathcal{SA}$, $\mathcal{HA}$, $\mathcal{HA} \cup \mathcal{O}$, $\mathcal{SA} \cup \mathcal{SIL}$, $\mathcal{HA} \cup \mathcal{IL}$ and $\mathcal{HA} \cup \mathcal{IL} \cup \mathcal{O}$.*

3.6 Another proof of incompleteness of the finite lambda calculus

In [23] we have shown that there are at least $2^{\mathfrak{c}}$ many sets $\mathcal{U}$ of undefined terms such that $\mathfrak{M}_{\mathcal{U}}$ cannot be ordered by a partial order with a least element and for which application and abstraction are monotone. The idea was to use sets of undefined terms of the form $\mathcal{U} = \mathcal{SA}_{X \cup \{\mathbf{O}\}}$ for suitable X. Here we will improve this result and use it to obtain another proof of incompleteness of the finite lambda calculus.

Definition 56. We say that $\langle \mathfrak{M}_{\mathcal{U}}, \sqsubseteq \rangle$ is a po$^{\bullet}$ model if $\sqsubseteq$ is a partial order on $\mathsf{NF}(\Lambda^{\infty}_{\perp})$ with a least element (which may be different from $\perp$), and application is monotone wrt $\sqsubseteq$, i.e. whenever $M_1 \sqsubseteq N_1$ and $M_2 \sqsubseteq N_2$ then $M_1 \bullet M_1 \sqsubseteq N_1 \bullet N_2$.

Theorem 57. *If $\langle \mathfrak{M}_{\mathcal{U}}, \sqsubseteq \rangle$ is a po$^{\bullet}$ model then:*

1. *Either $\perp$ is the least element of $\sqsubseteq$ and $\perp P \to_{\perp} \perp$ for all $P \in \Lambda^{\infty}_{\perp}$, or*
2. *$\mathbf{O}$ is the least element of $\sqsubseteq$.*

Proof. Suppose that $M \in \mathsf{NF}(\Lambda^{\infty}_{\perp})$ is the least element. Then $M \sqsubseteq \lambda x.M$ for some x free in M. If application is monotone then $M \bullet P \sqsubseteq (\lambda x.M) \bullet P =_{\mathsf{NF}} M$ and hence $MP =_{\mathsf{NF}} M$ for all P for all $P \in \mathsf{NF}(\Lambda^{\infty}_{\perp})$. Now either $M = \perp$ in which case $\perp P \to_{\perp} \perp$ for all $P \in \Lambda^{\infty}_{\perp}$. Or $M \neq \perp$ and then $Mx = M$ for all x. Hence M is the solution of the recursive equation $M = \lambda x.M$ and so $M = \mathbf{O}$. □

We can now strengthen Theorem 47 in [23]:

Theorem 58. *The interval $\langle \mathcal{RA}, \mathcal{SA} \rangle$ contains at least $2^{\mathfrak{c}}$ many sets $\mathcal{U}$ of undefined terms for which there exist no partial order such that $\langle \mathfrak{M}_{\mathcal{U}}, \sqsubseteq \rangle$ is a po$^{\bullet}$ model.*

Proof. Take a non-empty subset X of closed terms in $\mathsf{BerT}(\Lambda^{\infty}_{\perp})$ without $\perp$. Clearly there are $2^{\mathfrak{c}}$ many choices for this X. Let $\mathcal{U}$ be the set of terms in Λ^{∞} with a beta reduction (not necessarily finite) to a term of the form $RN_0N_0N_1N_1 \ldots N_kN_k$ where $k \geq 0$, $R \in \mathcal{RA}$ and all $N_i \in X$.

Suppose there exists a partial order $\sqsubseteq$ on $\mathsf{NF}(\mathcal{U})$ such that $\langle \mathfrak{M}_{\mathcal{U}}, \sqsubseteq \rangle$ is a po$^{\bullet}$ model. By Theorem 57 we have that $\mathbf{O}$ is the least element of $\sqsubseteq$. Choose $M \in X$

such that $\mathbf{O} \neq M$. Then $\mathbf{\Omega} MM \in \mathcal{U}$. Consider also $\mathbf{\Omega OO}$. Since $\mathbf{O}$ is the least element wrt to $\sqsubseteq$ we have $\mathbf{O} \sqsubseteq M$.

On one hand, $\bot\mathbf{OO}$ and $\mathbf{\Omega} MM$ reduce both to $\bot$, as they are elements of $\mathcal{U}$. On the other hand, $\bot M\mathbf{O}$ does not reduce to $\bot$, because $\bot M\mathbf{O} \notin \mathcal{U}$. But $\bot = \bot\mathbf{OO} \sqsubseteq \bot M\mathbf{O} \sqsubseteq \bot MM = \bot$ implying that $\bot = \bot M\mathbf{O}$. Contradiction.

Hence there is no partial order such that $\langle \mathfrak{M}_{\mathcal{U}}, \sqsubseteq \rangle$ is a $\text{po}^\bullet$ model.

□

For each model $\mathfrak{M}_{\mathcal{U}}$ there is a corresponding lambda theory, namely the collection of pairs of closed finite lambda terms with the same interpretation in the model. As a corollary we obtain an alternative proof for Salibra's theorem that any semantics of lambda calculus given in terms of a partially ordered model with a bottom element is incomplete.

Corollary 59 (Salibra [19]). *There are at least continuum many lambda theories that cannot be ordered with a* $\text{po}^\bullet$ *model.*

Proof. Restrict in the previous proof the collection X to closed finite normal forms in Λ. There are continuum many such X. Clearly for any two different such sets, the corresponding lambda theories are different. □

Salibra's proof is different. He considers first the enumerable lambda theory Π axiomatised by $\mathbf{\Omega} xx = \mathbf{\Omega}$ to prove with the help of a nice idea by Plotkin that any semantics of lambda calculus given in terms of $\text{po}^\bullet$-models with a bottom element is incomplete (cf. [19]). Then he uses a theorem by Visser [26, 4] to obtain a continuum of distinct unorderable enumerable lambda theories satisfying the conditions: $\mathbf{\Omega} xx = \mathbf{\Omega}$ and $\mathbf{\Omega}(\mathbf{\Omega KI})\mathbf{\Omega} \neq \mathbf{\Omega}$. Note that in the proof of Theorem 58 none of the constructed models $\mathfrak{M}_{\mathcal{U}}$ is a model of Salibra's theory Π, because they do not validate $\mathbf{\Omega\Omega\Omega} = \mathbf{\Omega}$.

This section demonstrates that infinitary lambda calculus can be a convenient tool for proving facts about finite lambda calculus.

3.7 Extensional infinite lambda calculus

Far less is known about *extensional* lambda calculus. The collection of normal form models of extensional lambda calculus is still waiting to be explored. Our earlier work [9] on infinite lambda calculus depended heavily on the Compression property, which does not hold for extensional lambda calculus. An anonymous referee of this paper suggested us an elegant counterexample, simpler than the one we gave in [9].

Counterexample 60 *Let* M *be* $\lambda x.(\lambda y.\mathbf{K}xy(\mathbf{K}xy(\ldots)))x$. *Then neither* M *nor its finite* β*-reducts contain any* η*-redexes. However,* M *can* β*-reduce in* ω *steps to* $\lambda x.(\lambda y.y(y(\ldots)))x$, *which can* η*-reduce further to* $(\lambda y.y(y(\ldots)))$. *This reduction clearly cannot be compressed to a shorter one.*

The transfinite tiling diagram used in [7] to prove confluence of $\lambda^{\infty}_{\beta\perp\mathcal{U}}$ opens the way to confluence proofs of $\lambda^{\infty}_{\beta\perp\mathcal{U}}$ extended with extensionality for certain $\mathcal{U}$. In [20, 21] we have shown confluence and normalisation of $\lambda^{\infty}_{\beta\perp\eta}$ and $\lambda^{\infty}_{\beta\perp\eta!}$ for $\mathcal{U} = \overline{\mathcal{HN}}$. Here $\eta!$ is a strengthened version of the η rule, defined with the help of the concept of strongly converging η-expansion:

$$\frac{x \notin FV(M)}{\lambda x.Mx \to M}\,(\eta) \qquad \frac{x \notin FV(M)}{M \to \lambda x.Mx}\,(\eta^{-1}) \qquad \frac{x \twoheadrightarrow_{\eta^{-1}} N \qquad x \notin FV(M)}{\lambda x.MN \to M}\,(\eta!)$$

In $\lambda^{\infty}_{\beta\perp\eta}$ and $\lambda^{\infty}_{\beta\perp\eta!}$ we also have that extensionality postpones over both β reduction and $\perp$ reduction. Despite the above counterexample against the Compression lemma, there is still a weaker form of compression: any strongly converging reduction in $\lambda^{\infty}_{\beta\perp\eta}$ and $\lambda^{\infty}_{\beta\perp\eta!}$ can be compressed to a strongly converging reduction of length at most $\omega + \omega$.

We are currently working to extend these results to other sets $\mathcal{U}$ and to other forms of extensionality.

4 Summary and conclusions

The application of rewriting theory to functional languages leads naturally to the consideration of infinite rewriting sequences and their limits. Our theory of transfinite rewriting puts this intuitive concept on a sound footing through the concept of a strongly convergent rewriting sequence, of which the crucial property is that not only does the sequence of terms tend to a limit, but the sequence of redex positions tends to infinite depth.

This notion allows us to demonstrate some classical results for orthogonal systems. However, it transpires that the most important of these, confluence, fails in certain cases. These cases can be precisely characterised, and confluence can be re-established modulo the equality of the set of terms which obstruct exact confluence. The offending terms are of a form that can reasonably be viewed as representing infinite computations that produce no result, and in this sense are undefined.

Further consideration of this set of terms reveals that any set which satisfies certain natural axioms can serve as the class of undefined terms. Not only does confluence hold relative to any such class, but we can immediately construct a semantic model of the rewrite system in which the undefined terms are all mapped to the same element.

For the lambda calculus this has yielded a new uniform characterisation of several known models, and a construction of several classes of new ones.

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