TYPE THEORETICAL TOPICS IN TOPOS THEORY

TYPETHEORETISCHE THEMA'S IN DE TOPOSTHEORIE

(MET EEN SAMENVATTING IN HET NEDERLANDS)

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Introduction, Abstract and Summary

Before giving a technical introduction to this thesis, I would like to present an introduction aimed at the more innocent reader of this thesis.

Suppose we have an arbitrary topos $\mathbb{E}$ that we imagine to be the universe of a certain mathematician Mr. $\mathbb{E}$ doing mathematics within the logic dictated by the topos. In general Mr. $\mathbb{E}$'s logic will not resemble ordinary classical logic of everyday. The principle, that for every statement A either A is true or $\neg$A is true, which is Aristotle's principle of the excluded middle, is likely to fail. Mr. $\mathbb{E}$'s logic will be of an intuitionistic kind.
For instance, he will state that there exist a number with a particular property if and only if he has a genuine construction of such a number.
If this principle does not hold then certain proofs will be blocked. A well-known example of such a proof (cf. [Troelstra and van Dalen 88]) is the following.

Theorem. There exist two irrational numbers a and b such that a^b is a rational number.

Tentative proof. Knowing that $\sqrt{2}$ is irrational, we consider the number $(\sqrt{2})^{\sqrt{2}}$. Now $(\sqrt{2})^{\sqrt{2}}$ is rational, or it is not. If $(\sqrt{2})^{\sqrt{2}}$ happens to be rational, we choose $a=b=\sqrt{2}$. If $(\sqrt{2})^{\sqrt{2}}$ is irrational, then we choose $a=(\sqrt{2})^{\sqrt{2}}$ and $b=\sqrt{2}$, and we calculate that $a^b=((\sqrt{2})^{\sqrt{2}})^{\sqrt{2}}=(\sqrt{2})^2=2$. Hence also in this case we can find irrational numbers a and b such that a^b is rational.

□

This proof is blocked if the principle of excluded third fails. Note that this proof is far from constructive. The proof does not provide us with a recipe to construct two irrational numbers a and b such that a^b is rational. It has been the Dutch mathematician L.E.J. Brouwer who criticized classical mathematics, and instead proposed intuitionistic mathematics.

If the principle of the excluded middle fails then the principle that if not A is not true, then A is true, fails as well, for an arbitrary statement A. In his Brouwer memorial lecture (1987), Manin has given the following beautiful metaphor, why this last principle ought to fail: "*Negation of a particular dogma opens up before the mind such a universe of alternative possibilities that no subsequent negation can return it to the status quo.*"

Now suppose that our Mr. $\mathbb{E}$ has become aware of the double negation, $\neg\neg$ as a logical operator. And suppose that at a particular night he has a dream about classical natural numbers. On the following day he tries to reconstruct the classical natural numbers in his world. One attempt of Mr. $\mathbb{E}$ could be Lifschitz' singleton construction on natural numbers (cf. [Lifschitz], or chapter 3 and 7 of this thesis). Instead of his own natural numbers, $\mathbb{N}$, Mr. $\mathbb{E}$ considers in this reconstruction the collection of non empty subsets S of $\mathbb{N}$ that contain at most one element of $\mathbb{N}$, i.e., subsets S such that $\neg\neg\exists n{\in}\mathbb{N}\ n{\in}S$. Indeed, modulo a translation he can now recapture the first order properties of classical natural numbers.

If, from a distance, we look at the accomplishments of Mr. $\mathbb{E}$, then we see that by his singleton construction Mr. $\mathbb{E}$ has lifted his own natural numbers to an object that also is an object of the topos of sheaves for the double negation. This sheaf topos is a universe in which the principle of the excluded middle holds, i.e., a universe of classical logic. The new object even (modulo an isomorphism) is the object of natural numbers in this sheaf topos.

Abstract.

In this thesis we consider in several contexts general constructions to make sheaves for unary logical operators, that are Lawvere-Tierney topologies. Then we give a translation directly from the sheaf topos to the base topos. With help of this translation we then define a general Gödel-Friedman translation for type theories. We give a general criterion that a type theory H should satisfy in order that for instance H+principle of excluded middle proves a statement A if and only if H itself proves the Gödel-Friedman translation of A. Finally we consider Dedekind real numbers, and show what kind of real number objects can be obtained by applying sheaf constructions or by Gödel-Friedman translating classical Dedekind reals with respect to arbitrary Lawvere-Tierney topologies.

Summary.

Chapter 1. The language rules, axioms and models, toposes of type theory are described, following the presentation of [Lambek and Scott]. We extend this system with explicit types of functions in order to facilitate the description of mathematical proofs and constructions in the languages. As an example of the use of the internal type theory of a topos for proving facts on that topos we introduce natural numbers and give an entirely constructive proof that the recursion principle that, in the internal type theory of a topos, is the interpretation of Lawvere's elegant categorical description is equivalent with the axioms of Peano. A second example is an elementary, direct, internal proof of a well-known categorical theorem of Mikkelsen, that states the equivalence of the presence of a natural number object to the constructability of free monoids.

Chapter 2. We give an entirely internal treatment of the following notions from elementary topos theory: Lawvere-Tierney topologies on the type of truth values Ω, Grothendieck topologies on Ω, dense and closed subtypes and sheaves. The internal treatment of Grothendieck topologies appears to be new, but is implicit in [Johnstone 77]. The operations that make the Grothendieck topologies on Ω as well as the dense Grothendieck topologies into a complete Heyting algebras are easy to describe. Internal proofs do not necessarily follow the same strategy as categorical proofs. An example is an internal proof of the well-known fact that Ω_j is a sheaf.

Chapter 3. We present a systematic internal treatment of the construction of the associated sheaf functor. By a careful investigation of the various different ways of expressing the notion of singleton subset in intuitionistic type theory we find twelve different, but related constructions including the well known constructions of Lawvere and Tierney, and Grothendieck and Johnstone.

Chapter 4. Given a topos $\mathbb{E}$ we consider the category $\mathbb{E}_{\mathbb{T}}$ of $\mathbb{T}$-algebras in $\mathbb{E}$, for internal finitary algebraic theory $\mathbb{T}$. We show that the object of subalgebras of the free $\mathbb{T}$-algebra generated by $\mathbb{1}$ acts as a subalgebra classifier for a suitable class of characteristic morphisms. For commutative theories we show that our notion of characteristic morphism is equivalent to a notion of Borceux and Van den Bossche. We present all this within the constructive setting of type theory.

Chapter 5. Internalizing ideas of Borceux, Van den Bossche and Veit we give an internal treatment of the one-one correspondence of universal closure operations, Lawvere-Tierney topologies and Grothendieck topologies. For commutative algebras we can give a notion of sheaf with a corresponding internal associated sheaf construction à la Grothendieck-Johnstone. Although a different, entirely categorical construction of associated sheaves for a larger class of algebras recently has been given we still think that it is of interest to see how the Grothendieck-Johnstone method can be adapted to this algebraic context.

Chapter 6. Given a topology $j:\Omega \to \Omega$ the topos of sheaves $\mathcal{S}h_j\mathbb{E}$ is a subcategory of the base topos $\mathbb{E}$. We extract this well-known phenomenon a translation $j: L_{\mathcal{S}h_j\mathbb{E}} \to L_{\mathbb{E}}$ satisfying $\mathcal{S}h_j\mathbb{E} \models \phi \Leftrightarrow \mathbb{E} \models \phi^j$. Let $\mathsf{Hu}\{j\}$ denote the type theory H extended with a unary logical operator j that satisfies the axioms of a Lawvere-Tierney topology. An interpretation of $\mathsf{Hu}\{j\}$ in $\mathbb{E}$ is easy to transform into a interpretation of the language of H in the $\mathcal{S}h_j\mathbb{E}$. Combining this with the j-translation, we find a generalization of both the Gödel negative translation and the Friedman translation for type theories H, that basically is nothing but an endo-translation of the types and function symbols of the language of H. Modulo the inhabitedness of the types of the free variables involved, for this G-translation we can prove that $\mathsf{Hu}\{j\} \vdash \mathsf{H}^{\mathsf{G}}$ if and only if $\mathsf{Hu}\{j\} \vdash \phi^{\mathsf{G}} \Leftrightarrow \mathsf{H} + \mathsf{PEM}_j \vdash \phi$, where PEM_j stands for $\forall \omega \in \Omega (j\omega \to \omega)$, the generalization with respect to j of a principle equivalent to the principle of excluded middle. Of interest are type theories that prove their own translation: $\mathsf{Hu}\{j\} \vdash \mathsf{H}^{\mathsf{G}}$. The interpretation of such type theories is preserved under the associated sheaf functor. Examples are Heyting's Arithmetic, Geometric Logic and Existential Fixed Point Logic of Gurevich and Blass.

Chapter 7. Inside a type theory with natural numbers we can construct Dedekind reals. In the constructive context there are many different definitions and constructions that eventually can be seen (modulo isomorphisms) linguistic variations of the standard definition. Dedekind reals, MacNeille reals, Troelstra's extended reals, Troelstra's classical reals, Staples reals and van Dalen's singleton reals. We give a systematic and general treatment of these notions and the three methods (linguistic variations, application of the associated sheaf functor and application of the general Gödel-Friedman translation) using again an arbitrary Lawvere-Tierney topology instead of the double negation topology.

Chapter 8. References.

Chapter 1

Type theory with natural numbers

In this chapter we describe language, rules, axioms and models of typed intuitionistic set theories with natural numbers or, shortly, *type theory*. The expressive power of the language of such formal systems is great: after a little practice one gets the feeling that mathematics can be formalized in them.

As basis for our definition of type theory we take the formal system of [Lambek and P.J. Scott], and we extend it with explicit types of functions. The system of [Lambek and P.J. Scott] is related to the formal system of [Boileau and Joyal] and to the Mitchell-Bénabou language described in [Johnstone 77]. Formal systems of this kind are closer to the categorical definition of toposes then systems in the style of [Fourman], [D.S. Scott 79] and [Fourman and D.S. Scott] which are designed to incorporate extra notions like designators and existence predicates.

The natural models for such intuitionistic set theories with natural numbers are toposes with natural number objects. Toposes are categories with finite limits, a subobject classifier and exponentials. Standard textbooks on topos theory are [Johnstone 77] and [Barr and Wells]. Mac Lane's *Categories for the Working Mathematician* is a standard reference for category theory.

Proving facts about a topos can be done in category theory or with help of the internal intuitionistic set theory of a topos with an appeal to the soundness theorem of intuitionistic set theory for toposes. It is customary to use the terminology "*internal* proof" for type theoretical proofs in contrast to "*external* proof" for categorical proofs of facts about a topos. Occasionally results concerning categorical notions of toposes, that can be seen as interpretations of meta type theoretical statements, are more conveniently proved via the internal constructive type theory of a topos followed by an appeal to the soundness theorem of intuitionistic set theory for toposes. We will use the terminology internal proof for such proofs as well.

We will present an example of such a result concerning natural numbers. In mathematical practice natural numbers are axiomatized set theoretically via the axioms of Peano. [Lawvere 64] has given an elegant definition of natural numbers in simple categorical terms. This definition is the recursion principle in a categorical setting. It is equivalent with Peano's axioms, which is among the oldest known facts in topos theory. Now, there exists a well-known theorem of Mikkelsen expressing the equivalence of the presence of a natural number object to the constructability of free monoids. Using type theory it is possible to give a direct internal proof of this theorem. Only indirect external proofs of this theorem seem to exist.

Finally, when we are proving facts about categories by categorical means we will speak of *objects* and *morphisms*. If we give a proof in which we use type theory, we will speak of *types* and *functions*. Some statements that we will be prove internally will be just formulas in type theory. Other statements will be in meta type theory, i.e. they are statements in which we quantify over types and morphisms we will prove. A simple example: different ways of expressing that "the successor function $s: \mathbb{N} \to \mathbb{N}$ is injective", are

(i) $\forall x,y:\mathbb{N}\ [s(x)=s(y) \to x=y]$
(ii) for all types A and all functions $f,g: A \to \mathbb{N}$ it holds that
$\forall n:\mathbb{N}\ [s(f(n))=s(g(n))] \to \forall n:\mathbb{N}\ [f(n)=g(n)]$.

1.1 Type theory without function types

Our aim is to give a definition of type theory in which we have exponential types containing functions. In this section we will give a definition of type theory as provided by [Lambek and P.J. Scott]. More precisely, we take over their notion of type theory generated by a graph $\mathcal{G}$ as occurs in example 1.2 at page 132 of [Lambek and P.J. Scott] omitting at first the natural numbers. In the next section we will present our (conservative) extension of this definition with function types.

The language of a type theory will be defined in two steps. First we define a kernel, then we extend this kernel to the full language. The interpretation of type theory in the models will follow this two step pattern: a full type theory will be translated into a kernel type theory, and the kernel type theory will be interpreted in the topos models.

There are three pieces of information from which one constructs a kernel type theory without function types:
(i) a set of basic type symbols,
(ii) a set of function symbols, where each function symbol has basic type symbols as domain and codomain, notation $f: A \to B$.

(iii) a set of axioms formulated in the language constructed from the first two data.

1.1.1 Types

The set of *types* of the theory is the smallest set containing the given basic type symbols and the type $\mathbb{1}$, and closed under products and powerset formation. Respective notations: $A \times B$, $P(A)$. The power of $\mathbb{1}$ will contain the truth values and is denoted by Ω.

1.1.2 Terms and formulas of kernel type theory

Terms of kernel type theory are constructed as usual:

(i) $*$ is a term of type $\mathbb{1}$, and for each type A we have countably many variables $x_1, x_2, \ldots$.

(ii) terms are closed under the following constructions:

$\langle s,t\rangle : A \times B$	for all terms $s:A$ and $t:B$
$s=t:\Omega$	for all terms $s,t:A$
$\{x:A \mid \phi\}: P(A)$	for each $\phi:\Omega$
$t \in s:\Omega$	for $t:A$ and $s:P(A)$
$f(t):B$	for each $t:A$ and function symbol $f:A \to B$.

Formulas are terms of type Ω. This can cause confusion, which can be avoided by writing (ϕ) to indicate that we use a formula ϕ as term of type Ω.

1.1.3 Terms and formulas of full type theory

A full type theory is obtained by extending the language of a kernel type theory with the following symbols: $\top, \bot, \neg, \to, \wedge, \vee, \forall, \exists$ and $\exists!$. The translation of these symbols into the language of the kernel type theory is given in the following list (we will sometimes use square brackets [and] to facilitate reading of the formulas):

$$\begin{aligned}
\top &:= (* = *) \\
\phi \wedge \psi &:= \langle \phi, \psi \rangle = \langle \top, \top \rangle \\
\forall x{:}A\ \phi &:= \{x:A \mid \phi\} = \{x:A \mid \top\} \\
\phi \to \psi &:= (\phi \wedge \psi) = \phi \\
\phi \leftrightarrow \psi &:= \phi = \psi \\
\phi \vee \psi &:= \forall \omega{:}\Omega\ [(\phi \to \omega) \wedge (\psi \to \omega) \to \omega] \\
\bot &:= \forall \omega{:}\Omega\ \omega = \top
\end{aligned}$$

$$
\begin{aligned}
\neg\phi &:= \phi\to\bot \\
\exists x{:}A\ \phi &:= \forall\omega{:}\Omega\ [\forall x{:}A\,(\phi\to\omega)\to\omega] \\
\exists! x{:}A\ \phi &:= \exists x{:}A\ [\phi(x)\wedge\forall y{:}A\,(\phi(y)\to x=y)]. \\
\exists x\in B\ \phi &:= \exists x{:}A\ [x\in B\wedge\phi(x)],\ \text{where B is a term of type } P(A) \\
\forall x\in B\ \phi &:= \forall x{:}A\ [x\in B\to\phi(x)],\ \text{where B is a term of type } P(A) \\
x\in A &:= x=x,\ \text{where A is the type of } x.
\end{aligned}
$$

1.1.4 Axioms and rules

Having constructed the terms and formulas of type theories, we now can give the axioms and rules for entailment $\vdash$. Precision is needed here, entailment $\vdash_X$ is defined for each finite set X of variables between terms of type Ω, whose free variables are contained in X. We tacitly assume that the customary conventions concerning renaming bound variables are observed in order to avoid clashes which substitution of terms for variables can cause.
The set of axioms and rules of a type theory contains the axioms of the data defining this specific type theory together with the following standard set of axioms and rules:

1.1.4.1. Structural rules.

(i) $\phi \vdash_X \phi$

(ii) $\dfrac{\Gamma\vdash_X\phi \quad \Gamma\cup\{\phi\}\vdash_X\psi}{\Gamma\vdash_X\psi}$

(iii) $\dfrac{\Gamma\vdash_X\phi}{\Gamma\cup\{\psi\}\vdash_X\phi}$

(iv) $\dfrac{\Gamma\vdash_X\phi}{\Gamma\vdash_{X\cup\{y\}}\phi}$

(v) $\dfrac{\Gamma(y)\vdash_{X\cup\{y\}}\psi(y)}{\Gamma(b)\vdash_X\psi(b)}$, provided that b and y are of the same type and b is free for y, i.e., no free variable of b becomes bound in $\psi(b)$.

1.1.4.2. Pure equality rules

(i) $\vdash_X t=t$

(ii) $\phi(x/s),\ s=t \vdash_X \phi(x/t)$
provided that t and s are free for x in ϕ,
and the free variables of s appear free in the conclusion.

(iii) $$\frac{\Gamma\cup\{\phi\} \vdash_X \psi \quad \Gamma\cup\{\psi\} \vdash_X \phi}{\Gamma \vdash_X \phi=\psi}$$

1.1.4.3. Other rules

(i) $\langle x,y\rangle=\langle x',y'\rangle \vdash_X x=x'$

(ii) $\langle x,y\rangle=\langle x',y'\rangle \vdash_X y=y'$

(iii) $\vdash_X x\in\{x:A\,|\,\phi\}\leftrightarrow\phi$, where $x\in X$.

(iv) $$\frac{\Gamma\vdash_{X\cup\{x\}} \phi(x)=x\in B}{\Gamma\vdash_X \{x:A\,|\,\phi(x)\}=B}$$, where x is not free in Γ

(v) $\vdash_Z z=*$, where z is of type $\mathbb{1}$

(vi) $$\frac{\Gamma, z=\langle x,y\rangle \vdash_{X\cup\{x,y,z\}} \phi(z)}{\Gamma \vdash_{X\cup\{z\}} \phi(z)}$$, where x and y do not occur free in Γ or $\phi(z)$.

1.1.4.4. Notations

(i) We will often write $\Gamma\vdash\phi$ for $\Gamma\vdash_X\phi$, when X happens to be the empty set.

(ii) We will often write ϕ for $\vdash_X\phi$ when X contains exactly the types of the free variables occurring in ϕ.

1.1.5 Theorem. The following are consequences of the foregoing rules and axioms:

(i) the usual propositional axioms of intuitionism, for example:

(a) $\phi\to(\psi\to\phi)$
(b) $[\phi\to(\psi\to\sigma)]\to[(\phi\to\psi)\to(\phi\to\sigma)]$
(c) $(\phi\wedge\psi)\to\phi$
(d) $(\phi\wedge\psi)\to\psi$
(e) $\phi\to[\psi\to(\phi\wedge\psi)]$

(ii) the axioms for = concerning symmetry and transitivity:
(g) $\forall x,y{:}A\,(x=y\to y=x)$
(h) $\forall x,y,z{:}A\,(x=y\wedge y=z\to x=z)$

(iii) three rules:

(i) $\dfrac{\phi \quad \phi\to\psi}{\psi}$ provided that the free variables of ϕ appear free in ψ

(j) $\dfrac{\phi\to\psi}{\phi\to\forall x{:}A\,\psi}$ provided that x does not occur free in ϕ

(k) $\dfrac{\phi\to\psi}{\exists x{:}A\,\phi\to\psi}$ provided that x does not occur free in ψ

(iv) Two more axioms for quantifiers
(l) $(\forall x{:}A\,\phi)\to\phi$
(m) $(\phi\wedge\exists x{:}A\,\top)\to\exists x{:}A\,\phi$
where $\exists x{:}A\,\top$ expresses that the type A is inhabited.

1.2 Type theory with function types

We are not satisfied with a formulation of type theory in which there are no explicit types for functions. We would like to have available an internal notion of function, such that we can introduce a function whenever we encounter a predicate $\phi(x,y)$ for which we can prove that it represents a function, i.e., for which we can prove $\forall x{:}A\,\exists!y{:}B\,\phi(x,y)$.

Thereto we extend the above given definition of type theory in the following way.

(1.1.1') Types.
We add to the rules that define the formation of types the new rule: closure under exponentials, notation B^A or just $A\to B$.

(1.1.2') Terms.

We add *closure under application* to the term formation rules: if f is a term of type B^A and t is a term of type A then f(t) is a term of type B.

(1.1.4') Axioms and rules.
We add the following axiom concerning extensionality of functions and the axiom scheme that formulates the unique choice of functions

(g) $\forall f,g{:}B^A\,(f{=}g \leftrightarrow \forall x{:}A\, f(x){=}g(x))$
(h) $[\forall x{:}A\, \exists! y{:}B\ \ \phi(x,y)] \rightarrow \exists! f{:}B^A\, \forall x{:}A\ \ \phi(x,f(x))$

Notation: we will write $\Gamma \vdash_{fun} \phi$ for entailment in this theory of types with functions.

Observe that one can define a translation $(\ )^{\#}$ from a language $\mathcal{L}$ of a type theory H with functions into a subset of $\mathcal{L}$ that is type theory without function types, such that the following properties hold:

(i) $\vdash_{fun} \phi^{\#} \leftrightarrow \phi$

(ii) $H \vdash_{fun} \phi \Leftrightarrow H^{\#} \vdash \phi^{\#}$.

The definition of the translation $\phi^{\#}$ of ϕ will go by induction to the structure of ϕ, the idea is to replace subterms of the form $\{x{:}B^A \mid \psi\}$ by

$\{z{:}P(A,B) \mid \psi^{\#} \wedge \forall x{:}A\, \exists! y{:}B\ \ z(x,y)\}$

and to conjugate subformula with formulas $\forall a{:}A_x \exists! b{:}B_x z_x(a,b)$ for each free variable x of exponential type $A_x \longrightarrow B_x$ that occurs in the subformula ϕ.
However we have to be careful as exponential types can have been used in the construction of other types.

1.2.1 Definition Consider some type theory with function types. For each type A we define a corresponding type $A^{\#}$ constructed without exponentials by induction to the structure of A:
(i) $A^{\#}{:=}A$ for basic types
(ii) $(AxB)^{\#}{:=}A^{\#}xB^{\#}$ for product types
(iii) $(P(A))^{\#}{:=}P(A^{\#})$ for power types
(iv) $(B^A)^{\#}{:=}P(A^{\#},B^{\#})$ for exponential types.

It is clear that $A^{\#}$ does not need to be isomorphic with A. The next predicate standard will cut out the subtype of $A^{\#}$ isomorphic with A.

1.2.2 Definition Consider some type theory with function types. For each type A we define a corresponding formula $\mathrm{standard}(x):\Omega$ with $x:A^{\#}$ by induction to the structure of A:

(i) $\mathrm{standard}(x)=\top$, for basic types A

(ii) $\mathrm{standard}\langle x,y\rangle=\mathrm{standard}(x)\wedge\mathrm{standard}(y)$, for product types $B\times C$

(iii) $\mathrm{standard}(x)=\forall y{:}A^{\#}\,(y\in x\rightarrow\mathrm{standard}(y))$, for power types $P(B)$

(iv) $\mathrm{standard}(x)=\forall z{:}C^{\#}\,\exists!y{:}B^{\#}x(z,y)$, exponential types B^{C}.

Now we have sufficient tools to define the translation $^{\#}$ of a formula ϕ into an equivalent formula $\phi^{\#}$ in which no types occur in whose definition somewhere exponentials are used.

1.2.3 Definition Consider some type theory with function types. For formula ϕ we define a corresponding formula $\phi^{\#}$, in which no terms occur of exponential type, by induction to the structure of ϕ:

(i) for all types A replace all variables of type A by variables of type $A^{\#}$.

(ii) replace each subformula $\psi(x_1,\ldots,x_n)$ by $\psi(x_1,\ldots,x_n)\wedge\bigwedge_{1\leq i\leq n}\mathrm{standard}(x_i)$

The proofs of the two properties are easy proofs by inductions to structure.

1.3 Two examples of type theories

(i) *Higher order Heyting arithmetic.*

One basic type $\mathbb{N}$, two function symbols $0:\mathbb{1}\rightarrow\mathbb{N}$ and $s:\mathbb{N}\rightarrow\mathbb{N}$ are given, plus the axioms of Peano (cf. 1.6.1). This system is sometimes called "Type Theory".

(ii) *The canonical intuitionistic set theory of a topos* $\mathbb{E}$.

The internal language $L_{\mathbb{E}}$ has as basic type symbols the objects of $\mathbb{E}$ and as function symbols the morphisms of $\mathbb{E}$. Note that the collection of types of $L_{\mathbb{E}}$ described by the syntax is larger than the collection of objects of $\mathbb{E}$. The axioms of $H_{\mathbb{E}}$ will correct this again.

The axioms of the type theory $H_{\mathbb{E}}$ are

(i) expressions stating for function symbols which corresponding arrows are compositions of (two - which suffices! -) others.
(ii) expressions stating that type constructions correspond to particular basictypes.

The usual indirect way of describing the axioms is (cf. [Fourman]) that the axioms of the canonical type theory of a topos $\mathbb{E}$ are exactly those formulas that are true in $\mathbb{E}$ under the canonical interpretation, which we will define in section (1.5).

1.4 List of additional notations

(i) Two types A and B are *isomorphic*, notation $A \approx B$,
if $\exists f{:}A \to B\ [\forall x,y \in A\ (f(x)=f(y) \to x=y) \wedge \forall x \in B\ \exists y \in A\ f(y)=x]$

(ii) In case of a family of subtypes of a type A indexed by a type I, and given to us by a function $\mathrm{index}{:}I \to PA$, we will employ the following notation:

$A_i := \mathrm{index}(i)$
$\bigcap_{i \in I} A_i := \{a \in A \mid \forall i \in I\ a \in A_i\}$
$\bigcup_{i \in I} A_i := \{a \in A \mid \exists i \in I\ a \in A_i\}$

(iii) For $f{:}X \to A$ and $g{:}X \to B$ we write $\langle f,g \rangle{:}X \to A \times B{:}x \mapsto \langle f(x),g(x) \rangle$

(iv) For $f{:}A \to B$ and $g{:}C \to D$ we write $f \times g{:}A \times C \to B \times D{:}\langle a,c \rangle \mapsto \langle f(a),g(c) \rangle$

(v) $R \subseteq A \times B$ is called a *graph* (of a function) from A to B if
$\forall a \in A\ \exists! b \in B\ \langle a,b \rangle \in R$.

(vi) Definition of a function by cases. Assume for subtypes $A,B \subseteq C$ and functions $f{:}A \to X$ and $g{:}B \to X$ that $\forall c \in A \cap B\ f(c)=g(c)$ and $A \cup B = C$ then

$R := \{(c,x) \in C \times X \mid (c \in A \to x=f(c)) \wedge (c \in B \to x=g(c))\}$

is a graph from C to X.
We are justified to use the following notation to define the function corresponding to R:

$$f{:}C \to X{:}c \mapsto \begin{cases} f(c) & \text{if } c \in A \\ g(c) & \text{if } c \in B \end{cases}$$

(vii) For $f{:}A \to B$ and $m{:}C \rightarrowtail B$ we write $f^{-1}(C)$ for $\{c \in C \mid \exists a \in A\ m(c)=f(a)\}$.

1.5 Toposes, models for intuitionistic type theory

Toposes are categories with extra properties which makes them into suitable models for constructive set theory. In the early sixties Grothendieck and Giraud tried to capture the properties of sheaves over topological spaces and discovered the concept of Grothendieck topos (cf. [Grothendieck and Verdier]). The more general notion of an elementary topos has been introduced by Lawvere and Tierney around 1970 in an attempt to characterize in an categorical axiomatic way the properties of the elementary theory of categories of sheaves.
It has been considered as a surprising fact that toposes happen to be the natural models for intuitionistic set theories. In retrospect hints in this direction have been:

(i) the old result of [Stone] and [Tarski] that topological spaces are models for intuitionistic propositional logic (cf. [Rasiowa and Sikorski]).

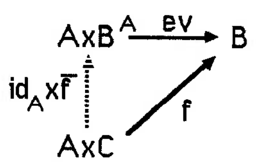
(ii) [Scott 68]'s topological interpretation to intuitionistic analysis.

Since then the definition of an elementary topos has been simplified. Ample information and further references to topos theory can be found in [Johnstone 77]. A short introduction is provided by [Lambek and P.J. Scott 86].

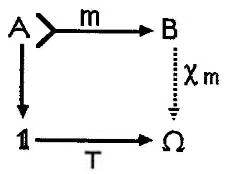
1.5.1 Definition. An *(elementary) topos* is a cartesian closed category with finite limits and a subobject classifier.

In [Mac Lane] and [Barr and Wells] one can find good introductions to the notions of category, (finite) limit, cartesian closedness and other categorical notions that we will use.
Recall that in order to prove that a category $\mathcal{C}$ has finite limits it suffices to show that $\mathcal{C}$ has equalisers and products or, equivalently, that $\mathcal{C}$ has a terminal object $\mathbb{1}$ and pullbacks.
We repeat the definitions of cartesian closedness and subobject classifier.

1.5.2 Definition. A category $\mathcal{C}$ that has products is *cartesian closed* if for all objects A and B in $\mathcal{C}$ there is an exponent B^A in $\mathcal{C}$ and an evaluation $ev: A \times B^A \to B$ such that for each $f: A\times C \to B$ there is a unique $\overline{f}: C \to B^A$ such that the following diagram commutes:



1.5.3 Definition. A category C with finite limits has a *subobject classifier* $\top: \mathbb{1} \to \Omega$ if for each monomorphism $m: A \rightarrowtail B$ there is a unique morphism $\chi_m: B \to \Omega$ such that the following diagram is a pullback.



$\chi_m: B \to \Omega$ is then called the characteristic morphism of $m: A \rightarrowtail B$.

1.5.4 Interpretation of type theories in toposes

An interpretation of a type theory H into a topos **E** maps types into objects, constants into morphisms. As no confusion can arise, we will denote both maps by the same symbol $[\![\ ]\!]$. Since terminal object, products and exponentials are only uniquely determined upto isomorphisms, an interpretation of types in a topos has to make specific choices for constructs in a coherent way such that the following conditions on types and constants hold:

(a) $[\![\mathbb{1}]\!] \approx \mathbb{1}$
(b) $[\![\Omega]\!] \approx \Omega$
(c) $[\![A \times B]\!] \approx [\![A]\!] \times [\![B]\!]$
(d) $[\![P(A)]\!] \approx \Omega^{[\![A]\!]}$
(e) $[\![B^A]\!] \approx [\![B]\!]^{[\![A]\!]}$
(f) a function $f: A \to B$ is mapped to a morphism $[\![f]\!]: [\![A]\!] \to [\![B]\!]$

From such an interpretation of the types and functions of type theory an interpretation of terms and formulas - with respect to finite sequences of variables X in which all free variables of the term that has to be interpreted occur - can be constructed by induction on the structure of the terms with help of the following equations (where the sequence of types corresponding with the sequence of variables in X is $A_1 \times \ldots \times A_n$):

(g) $[\![*]\!]_X = (A_1 \times \ldots \times A_n)_{\mathbb{1}}: A_1 \times \ldots \times A_n \to \mathbb{1}$,
(h) $[\![x_i]\!]_X = \pi_i: A_1 \times \ldots \times A_n \to A_i$, if the type of x_i is A_i,
(j) $[\![f(t)]\!]_X = [\![f]\!] \circ [\![t]\!]_X: A_1 \times \ldots \times A_n \to A \to B$,
if the types of function symbol f and term t are respectively $A \to B$ and A,
(k) $[\![g(t)]\!]_X = ev_A \circ \langle [\![g]\!]_X, [\![t]\!]_X \rangle: A_1 \times \ldots \times A_n \to B^A \times A \to B$,
if the types of term g and term t are respectively B^A and A,

(l) $[\![t=s]\!]_X = =_A \circ \langle [\![t]\!]_X, [\![s]\!]_X \rangle : A_1 \times \ldots \times A_n \to A \times A \to \Omega$,
if the type of both s and t is A,and $=_A = \chi_{\langle id_A, id_A \rangle} : A \times A \to \Omega$

(m) $[\![t \in s]\!]_X = ev_A \circ \langle [\![t]\!]_X, [\![r]\!]_X \rangle : A_1 \times \ldots \times A_n \to A \times \Omega^A \to \Omega$,
if the types of s and t are Ω^A and A,

(n) $[\![\langle t,s \rangle]\!]_X = \langle [\![t]\!]_X, [\![s]\!]_X \rangle : A_1 \times \ldots \times A_n \to A \times B$,
if the types of s and t are B and A,

(o) $[\![\{x:A \mid \phi\}]\!]_{X \setminus x} = [\![\phi]\!]_X \circ \sigma : A_1 \times \ldots \times A_{n-1} \to \Omega^A$,
where σ is the permutation $A_1 \times \ldots \times A_i \times A \times A_{i+1} \times \ldots \times A_n \to A \times A_1 \times \ldots \times A_n$.

1.5.5 Definition.
(i) An interpretation $[\![\ ]\!]$ of a type theory H into a topos **E** *satisfies* ϕ (notation $[\![\ ]\!] \models \phi$) if for each finite sequence X containing all free variables of ϕ the morphism $[\![\phi]\!]_X$ factors through $\top : \mathbb{1} \to \Omega$.

(ii) An interpretation $[\![\ ]\!]$ of a type theory H into a topos **E** *satisfies* $\phi \vdash \psi$ (notation $[\![\ ]\!], \phi \models \psi$) if for each finite sequence X containing all free variables of ϕ and ψ the morphism $[\![\psi]\!]_X$ factors through $\top : \mathbb{1} \to \Omega$ whenever the morphism $[\![\phi]\!]_X$ factors through $\top : \mathbb{1} \to \Omega$.

1.5.6 Soundness Theorem. Let **H** be a type theory. If $H \vdash \phi$ then for any topos **E** that satisfies H it holds that it satisfies ϕ

1.5.7 Completeness Theorem. Let **H** be a type theory. Then $\mathbf{H} \vdash \phi$ if and only if for any interpretation in a topos **E** that satisfies **H** it holds that it satisfies ϕ.

1.5.8 Theorem. Let **E** be a topos.
(i) Let $f,g : A \to B$ be two morphisms in **E**. Then f and g are identical in **E** if and only if $\mathbf{E} \models f=g$.
(ii) *Description holds in a topos*: if $\mathbf{E} \models \forall x{:}A\, \exists! y{:}B\ \phi(x,y)$ then there is a unique morphism $g : A \to B$ such that $\mathbf{E} \models \forall x{:}A\, \exists y{:}B\ \phi(x,f(y))$.

Proofs. Detailed proofs can be found in [Lambek & Scott 86].

□

1.5.9 Lemma. A morphism $f : A \to B$ is a monomorphism in a topos **E** if and only if $\mathbf{E} \models \forall x,y \in A\ (f(x)=f(y) \to x=y)$.

Proof.

$f{:}A \longrightarrow B$ is a monomorphism in $\mathbb{E}$

$\Leftrightarrow$

for all objects C and morphisms $g,h{:}C \longrightarrow A$ in $\mathbb{E}$ we have that if $f \circ g = f \circ h$ then $g=h$

$\Leftrightarrow$

for all types C and constants $g,h{:}C \longrightarrow A$ in $L_{\mathbb{E}}$ it holds that if $\mathbb{E} \models f \circ g = f \circ h$ then $\mathbb{E} \models g=h$.

$\Leftrightarrow *$

$\mathbb{E} \models \forall x,y \in A\, (f(x)=f(y) \rightarrow x=y)$.

($\Rightarrow *$) We prove for arbitrary type theory H: if for all types C it holds that $\forall g,h{:}C \longrightarrow A\, (f \circ g = f \circ h \rightarrow g=h)$ then $\forall x,y \in A\, (f(x)=f(y) \rightarrow x=y)$.

Proof. Assume for all types C and $\forall g,h{:}C \longrightarrow A\, (f \circ g = f \circ h \rightarrow g=h)$. Now suppose for $x,y \in A$ that $f(x)=f(y)$. Construct function $x'{:}\mathbb{1} \longrightarrow A{:}\, * \mapsto x$, and a similar function y'. Because $f \circ x'(*) = f(x) = f(y) = f \circ y'(*)$ we get $\forall z{:}\mathbb{1}\; f \circ x'(z) = f \circ y'(z)$. Hence $f \circ x' = f \circ y'$. Applying the assumption we get $x'=y'$, and so $x = x'(*) = y'(*) = y$.

□

1.6 The natural number type

One can play the game of doing mathematics without having recourse to a type of natural numbers, like Bénabou did in his lecture *Imperial Logic* given at the Category conference in Louvain-la-Neuve 1987. However, we will need a type of natural numbers to work with algebras, and therefore we will introduce them here.
A type $\mathbb{N}$ and functions $0{:}\mathbb{1} \longrightarrow \mathbb{N}$ and $s{:}\mathbb{N} \longrightarrow \mathbb{N}$ is a *natural number type* if the axioms of Peano hold for them:

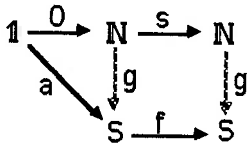
1.6.1 The axioms of Peano

(i) $\forall n,m \in \mathbb{N}\, (s(n)=s(m) \rightarrow n=m)$
(ii) $\forall n \in \mathbb{N}\, (0=s(n) \rightarrow \bot)$
(iii) $\forall X \subseteq \mathbb{N}\, [0 \in X \wedge \forall n (n \in X \rightarrow s(n) \in X) \rightarrow X = \mathbb{N}]$.

1.6.2 Recursion principle. Let S be a type. Let $f{:}S \longrightarrow S$ be a function and let $a \in S$. Then there is a unique $g{:}\mathbb{N} \longrightarrow S$ such that

(i) $g(0)=a$
(ii) $\forall n \in \mathbb{N}\;\; g(s(n))=f(g(n))$

This principle can be represented by the diagram:



The interpretation of the recursion principle in a topos is precisely Lawvere's categorical definition of a natural number object (cf. [Lawvere 64]).

In the context of toposes the equivalence of the Peano axioms with the recursion principle is well-known. [Freyd], [Johnstone 77], [Osius] and [Goldblatt] are sources for categorical proofs. In classical logic the equivalence is older and belongs to folklore (cf. [Hatcher]). For instance, [Henkin] gives a proof for classical logic of ($\Rightarrow$) and credits it to Lorentzen, Hilbert and Bernays.
For sake of completeness and because we will need a similar proof later on, we have given an elementary intuitionistic proof. We have followed [Osius] to prove ($\Leftarrow$), and we have rewritten [Henkin]'s classical proof into an intuitionistic proof of ($\Rightarrow$).

1.6.3 Theorem. A triple $\langle \mathbb{N}, 0{:}\,\mathbb{1}\to\mathbb{N}, s{:}\,\mathbb{N}\to\mathbb{N}\rangle$ satisfies the axioms of Peano if and only if the recursion principle holds for it.

Proof. ($\Rightarrow$) Assume the Peano axioms hold for $\langle \mathbb{N}, 0{:}\,\mathbb{1}\to\mathbb{N}, s{:}\,\mathbb{N}\to\mathbb{N}\rangle$. Let $\langle A, a{:}\,\mathbb{1}\to A,\ f{:}\,A\to A\rangle$ be another triple.We will construct the unique $g{:}\,\mathbb{N}\to A$ that makes the recursion diagram commute by approximating graphs on segments of $\mathbb{N}$.
Define:

$$\mathrm{Segments}:=\{S\subseteq\mathbb{N}\mid 0\in S\wedge\forall n\in\mathbb{N}(s(n)\in S\to n\in S)\}$$

$$\mathrm{ApproxGraphs}:=\{R\subseteq\mathbb{N}\times A\mid \exists S\in\mathrm{Segments}\ \exists g{:}\,S\to\mathbb{N}\ [\mathrm{graph}(g)=R\wedge g(0)=a\wedge\forall n\in\mathbb{N}\ (s(n)\in S\to g(s(n))=f(g(n))]\}$$

$$\mathrm{ext}:=[\mathbb{N}\times A]\to[\mathbb{N}\times A]{:}\,R\mapsto\{(0,a)\}\cup\{(s(n),f(b))\mid(n,b)\in R\}$$

Claim (a) For $R\in\mathrm{ApproxGraphs}$ it holds that $\mathrm{ext}(R)\in\mathrm{ApproxGraphs}$.

Claim (b) For $R,T\in\mathrm{ApproxGraphs}$ it holds that

$$\forall n\in\mathbb{N}\ \forall b,c\in A\ [((n,b)\in R\wedge(n,c)\in T)\to b=c]$$

Now define:

$$G:=\bigcup\{R\subseteq\mathbb{N}\times A\mid R\in\mathrm{ApproxGraphs}\}.$$

Claim (c) $\forall n{\in}\mathbb{N}\ \exists! b{\in}A\ (n,b){\in}G$.

By the axiom of unique choice and the construction of G it follows that $G \in \text{ApproxGraphs}$ and is the graph of a function, say, $g: \mathbb{N} \to A$. Hence, g fits in the diagram of the recursion principle. By induction it follows that it is the unique function with this property. Hence the recursion principle holds for $\langle \mathbb{N}, 0: \mathbb{1} \to \mathbb{N}, s: \mathbb{N} \to \mathbb{N} \rangle$, modulo a proof of the claims.

Proof of claim (a). Let $R \in \text{ApproxGraphs}$. Using induction, it is easy to see that $\text{ext}(R)$ is a graph, i.e., $\forall n{\in}\mathbb{N}\ \forall b,c{\in}A[((n,b){\in}\text{ext}(R) \wedge (n,c){\in}\text{ext}(R)) \to b=c]$. Let S be the domain of $\text{ext}(R)$. Clearly $S \in \text{Segments}$. Let $g: S \to A$ be the function corresponding to the graph. It is not difficult to see that for g it holds that $g(0)=a$ and $\forall n{\in}\mathbb{N}\ (s(n){\in}S \to g(s(n))=f(g(n)))$.
Therefore $\text{ext}(R) \in \text{ApproxGraphs}$.

Proof of claim (b). Let R,T be elements of ApproxGraphs.
Suppose for $b,c{\in}A$ it holds that $((0,b){\in}R \wedge (0,c){\in}T)$. Then $b=c=a$.
Next, suppose that for $n{\in}\mathbb{N}$ we have $\forall b,c{\in}A\ [((n,b){\in}R \wedge (n,c){\in}T) \to b=c]$. Assume for some $b,c{\in}A$ we have $[((s(n),b){\in}R \wedge (s(n),c){\in}T) \to b=c]$. Since R and T belong to ApproxGraphs it follows that there are $b',c'{\in}A$ such that $(n,b'){\in}R$ and $(n,c'){\in}T$. But then $b'=c'$ and hence $b=c$.
By induction we get $\forall n{\in}\mathbb{N}\ \forall b,c{\in}A\ [((n,b){\in}R \wedge (n,c){\in}T) \to b=c]$.

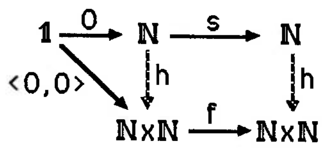
Proof of claim (c). By induction it follows that for each $n{\in}\mathbb{N}$ there exists an R in ApproxGraphs such that n belongs to the domain of R. Hence there is a $b{\in}A$ such that $(n,b){\in}R \subseteq G$. By claim (b) there is only one such b in A.

($\Leftarrow$) Assume the recursion principle holds for $\langle \mathbb{N}, 0: \mathbb{1} \to \mathbb{N}, s: \mathbb{N} \to \mathbb{N} \rangle$. First we will construct the predecessor function $p: \mathbb{N} \to \mathbb{N}$ satisfying $p(0)=0$ and $\forall n{\in}\mathbb{N}\ p(s(n))=n$. Then we prove the three axioms of Peano.

Consider the functions:

$$\langle 0,0 \rangle : \mathbb{1} \to \mathbb{N}\times\mathbb{N} : * \mapsto (0,0),$$
$$f: \mathbb{N}\times\mathbb{N} \to \mathbb{N}\times\mathbb{N} : (n,m) \mapsto (m,s(m)).$$

Because of the recursion principle there is a unique function $h: \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ making the following diagram commute:



It follows by the product axioms of type theory that h is of the form $\langle p,q \rangle$ where $p,q\colon \mathbb{N} \to \mathbb{N}$ satisfy:

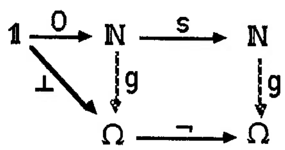
$p(0)=0$ and for each $m \in \mathbb{N}$ $p(s(m))=q(m)$
$q(0)=0$ and for each $m \in \mathbb{N}$ $q(s(m))=s(q(m))$.

By the recursion principle q is the identity on $\mathbb{N}$. Hence, for $m \in \mathbb{N}$ we have $p(s(m))=m$.
Therefore p is the desired predecessor function.

The proof of Peano's axioms now has become easy:

(i) Assume that for some $n \in \mathbb{N}$ we have $s(n)=s(m)$. Then $n=p(s(n))=p(s(m))=m$ by the second property of the predecessor p. Hence $\forall n,m \in \mathbb{N}\ (s(n)=s(m) \rightarrow n=m)$.

(ii) Assume that for some $n \in \mathbb{N}$ we have $s(n)=0$. Then $n=p(s(n))=p(0)=0$. Hence $s(0)=s(n)=0$. By the recursion principle there is now a unique function $g\colon \mathbb{N} \to \Omega$ such that the following diagram commutes.

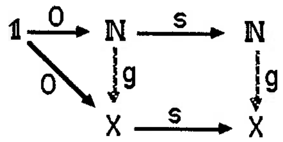


This implies:

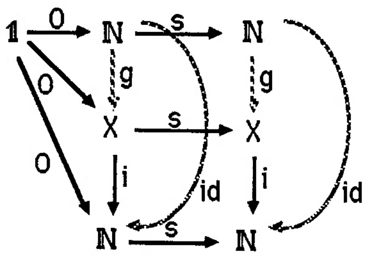
$$\begin{aligned} \top &= \neg\bot \\ &= g(s(0)) \\ &= g(0) \\ &= \bot. \end{aligned}$$

Since we know that $\top$ holds we get $\bot$. I.e., we have shown that $\forall n \in \mathbb{N}\ (s(n)=0 \rightarrow \bot)$.

(iii) Assume that for some $X \subseteq \mathbb{N}$ we have $0 \in X$ and $\forall n (n \in X \rightarrow s(n) \in X)$. By the recursion principle there is a unique (epi-) arrow $g\colon \mathbb{N} \to X$ that makes the following diagram commute.



If we compose it with the embedding $i:X \rightarrowtail \mathbb{N}$, we again get a commuting diagram.



The recursion principle now tells us that $i \circ g$ is the identity on $\mathbb{N}$. Hence $g:\mathbb{N} \rightarrow X$ is also mono. Therefore $X=\mathbb{N}$. And so $\forall X \subseteq \mathbb{N}\ [0 \in X \wedge \forall n(n \in X \rightarrow s(n) \in X) \rightarrow X=\mathbb{N}]$.

□

We need the recursion principle to add all kind of useful notions to the language of type theory, such as addition and multiplication on natural numbers, finite sums: $\Sigma_{0 \leq i \leq n} a_i$ and finite products $\Pi_{0 \leq i \leq n} a_i$. For example:

1.6.4 Lemma. In typed intuitionistic set theory with natural numbers there exists a unique $\Sigma:\mathbb{N} \rightarrow ((\mathbb{N} \rightarrow \mathbb{N}) \rightarrow \mathbb{N})$ such that

(i) $\Sigma(0)(f)=f(0)$

(ii) $\Sigma(n+1)(f)=\Sigma(n)(f)+f(n+1)$

□

With help of natural numbers one can construct a type $FinSeq(X)$ containing all finite sequences of elements of a given set X. We will represent a finite sequence $a_0,\ldots,a_{n-1}$ by a function $\mathbb{N} \rightarrow X \sqcup \mathbb{1}$ together with a number n indicating the length of the sequence. On input $0,\ 1,\ldots,\ (n-1),\ n,\ldots$ the function will give output $a_0,\ \ldots,\ a_{n-1},\ *,\ *,\ldots$.

1.6.5 Definition. Let X be a set.

(i) $FinSeq(X)=\{(n,f):\mathbb{N}\times(\mathbb{N}\rightarrow X\sqcup\mathbb{1}) \mid \forall m{<}n\ f(m)\in X\ \wedge \forall m{\geq}n\ f(m)\in\mathbb{1}\}$.

(ii) We define *concatenation* on $FinSeq(X)$ as:

$\circ:FinSeq(X)\times FinSeq(X)\rightarrow FinSeq(X):((n,f),(m,g))\mapsto(n+m,f\circ g)$, where

$$f\circ g:\mathbb{N}\to X\sqcup\mathbb{1}:k\mapsto\begin{cases}f(k) & \text{if } k\leq n\\ g(k-n) & \text{if } k>n\end{cases}$$

(Note that we have this definition by cases, because the axioms of Peano imply that equality of natural numbers is decidable.)

(iii) The empty sequence, of course, is $\varepsilon=(0,\mathbb{N}\to X\sqcup\mathbb{1}:n\mapsto *)$.

Even if we don't have a natural number type around, there still is a form of induction possible. With every diagram of the form

$$\mathbb{1}\xrightarrow{\ a\ }A\xrightarrow{\ f\ }A$$

(such diagrams are called Peano-structures in [Barr and Wells]) corresponds a principle of, let us say, $\langle A,f,a\rangle$-induction.

Define $M=\bigcap\{X\subseteq A\mid a\in X\wedge f(X)\subseteq X\}$. Then $a\in M$ and $f(M)\subseteq M$, and therefore $M=f(M)\cup\{a\}$. Observe that M is the smallest fixed point of the operation $P(A)\to P(A):X\to f(X)\cup\{a\}$. From this follows:

1.6.6 Lemma (principle of $\langle A,f,a\rangle$-induction) Let A,a,f and M be as above, then $\forall B\subseteq M\,(a\in B\wedge\forall b\in M[b\in B\to f(b)\in B]\to B=M)$.

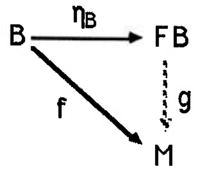
□

The following result belongs to the folklore of topos theory (cf. [Johnstone 77]), but we are able to give a direct and constructive proof of the type theoretic companion of the categorical version for which only an indirect, categorical proof seems to exist by [Johnstone 77].

1.6.7 Theorem. (Mikkelsen, cf. [Johnstone 77].) Let $\mathbb{E}$ be a topos and let $\mathrm{mon}(\mathbb{E})$ be the subcategory of monoids in $\mathbb{E}$. Then $\mathbb{E}$ has a natural number object $\mathbb{N}$ if and only if the forgetful functor $U:\mathrm{mon}(\mathbb{E})\to\mathbb{E}$ has a left adjoint F.

1.6.8 Theorem.; In the type theory H there is a type satisfying the axioms of Peano if and only if there are free monoids in H, i.e., if for each type A and each subtype $B:PA$ there is a type C and monoid $\langle FB\in PC,\circ\in FB\times FB\to FB,e\in FB\rangle$ and function

$\eta_B: B \to FB$ such that for any monoid $\langle M.\circ,e\rangle$ and function $f: B \to M$ there is a unique monoid morphism $g: FB \to M$ such that the following diagram commutes:

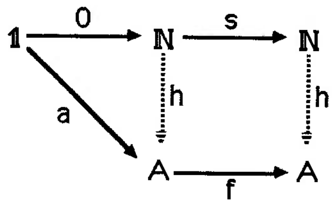


Proof of theorem 1.6.8. (only if) Assume $\langle \mathbb{N},0,s\rangle$ satisfies Peano's axioms. Let A be a type of H. Let FA be the type FinSeq(A) of all finite sequences of elements of A. If for a monoid $\langle M,\circ,\varepsilon\rangle$ there is a function $\phi: A \to M$, then ϕ can be uniquely extended to

$$\overline{\Phi}: FA \to M: (n,f) \mapsto \begin{cases} \varepsilon & \text{if } n=0 \\ \Sigma_{0\leq i\leq n}\phi(f(i)) & \text{otherwise} \end{cases}$$

From a meta point of view this construction gives us the left adjoint of the forgetful functor.

(if) Assume free monoids exist. Consider the free monoid $\langle F\mathbb{1},\circ,\varepsilon\rangle$ generated by $\mathbb{1}$, where $F\mathbb{1}$ are the words based on alphabet $\{*\}$, $\circ$ is concatenation and ε denotes the empty word. Of course, $F\mathbb{1}$ together with $0: \mathbb{1} \to F\mathbb{1}: * \mapsto \varepsilon$ and $s: F\mathbb{1} \to F\mathbb{1}: w \mapsto w\circ *$ ought to be a natural number type, such that for any Peano structure A,a,f in $\mathbb{E}$ we have the following diagram:



(diagram I)

The idea of this part of the proof is to construct the following monoid $\langle M,m,a\rangle$ in A without mentioning natural numbers: $\{a,f(a), (a), ...\}$ with unit a and multiplication $f^n(a)\circ f^m(a) := f^{n+m}(a)$. Then one has to invoke the universal property of the free monoid $F\mathbb{1}$.

Define $M := \bigcap\{X\subseteq A \mid a\in X \wedge \forall b\in X\ f(b)\in X\}$. The construction of $m: M\times M \to M$ is more involved. The idea is to approximate m by functions on growing domains, first on $\{a\}$, then on $\{a,f(a)\}$, $\{a,f(a),f(f(a))\}$ and so on.

Let us call a relation $R\subseteq M\times M\times M$ *good* if

$$\begin{aligned} &\exists X\subseteq M\ \exists m: X\times X \to M\ (a\in X \wedge \\ &\qquad \exists b\in X\ (fX\cup\{a\}=X\cup\{f(b)\}) \wedge \\ &\qquad R=\text{graph}(m) \wedge \end{aligned}$$

$\forall c,d \in X\ (m(c,d)=m(d,c)) \wedge$
$\forall c \in X\ (m(a,c)=c) \wedge$
$\forall c,d,e \in X\ [m(d,e) \in X \wedge m(c,d) \in X \rightarrow m(c,m(d,e))=m(m(c,d),e)] \wedge$
$\forall c,d \in X\ [f(d) \in X \rightarrow m(c,f(d))=f(m(c,d))] \wedge$
$\forall c,d,e \in X\ [f(d) \in X \wedge f(e) \in X \rightarrow [m(c,f(d))=f(e) \rightarrow m(c,d)=e]])$.

Note that $\{(a,a,a)\}$ is a good graph. From good graphs we can construct others.

Define $()_{ext}: P(M\times M\times M) \longrightarrow P(M\times M\times M): R \mapsto R\ \cup\{(f(c),f(d),f(f(e)))\,|\,(c,d,e)\in R\}$
$\cup\{(c,f(d),f(e))\,|\,(c,d,e)\in R\}$
$\cup\{(f(c),d,f(e))\,|\,(c,d,e)\in R\}$.

Now we continue with the construction, skipping the proofs of a number of claims, until we have reached the end of the construction.

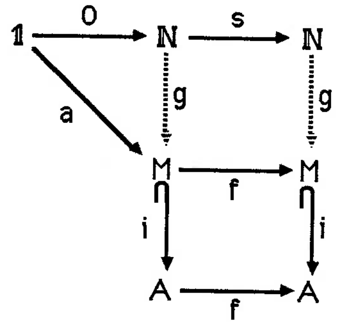
Claim (a): If $R\subseteq M\times M\times M$ is a good graph, then so is R_{ext}.

Define $B:=\{R\subseteq M\times M\times M\,|\,R \text{ is good}\}$. Then $\langle B,ext,\{(a,a,a)\}\rangle$ is a Peano-structure for which we have a principle of induction on
$Approx:=\bigcap\{\Sigma\subseteq B\,|\,\{(a,a,a)\}\in\Sigma\wedge ext(\Sigma)\subseteq\Sigma\}$.

Claim (b): $\forall R,S\in Approx\,(R\subseteq S \vee S\subseteq R)$.

Finally we define $G:=\bigcup\{R\subseteq M\times M\times M\,|\,R\in Approx\}$.

Claim (c): $\forall c,d\in M\ \exists! e\in M\ (c,d,e)\in G$.
It follows from the definition of G and the axiom of unique choice that G is the good graph of a function $m: M\times M \longrightarrow M$. Hence $\langle M,m,a\rangle$ is a monoid, and we can construct the following diagram:



(diagram II)

Since $s\circ 0: 1 \longrightarrow F1$ is the insertion of the generator $*$ into $F1$ we get a unique monoid morphism $g: F1 \longrightarrow M$ such that $g\circ s\circ 0=f\circ a$. Hence $g(*)=f(a)$. Clearly also $g\circ 0=a$.

For $w\in F1$ it holds that $g(s(w)) = g(w\times *)$

$$= m(g(w),g(*))$$
$$= m(g(w),f(a)$$
$$= f(m(g(w),a)$$
$$= f(g(w)).$$

And so we see that $g\circ s=f\circ g$. That is, diagram II commutes. The composition $i\circ g:F\mathbb{1}\to A$ is the required missing arrow h in the diagram I. Uniqueness of h is trivial, suppose there is another $k:F\mathbb{1}\to A$ that makes diagram I commute. If we perform the same monoid construction on $A\cap k(F\mathbb{1})$, instead of A, we get the same M. Hence k factors through i. It follows that k is identical to h.

Thus we see that $\langle F\mathbb{1},\circ,\varepsilon\rangle$ is a natural number type, provided we have proved the claims (a), (b) and (c).

Proof of claim (a). Suppose $R\subseteq M\times M\times M$ is good. Then there is $X\subseteq M$ and $b\in X$ such that $\{a\}\cup f(X)=X\cup\{f(b)\}$, and there is $m:X\times X\to M$ such that $R=graph(m)$. Clearly, R_{ext} is represented by a function $m_{ext}:X_{ext}\times X_{ext}\to M$, where $X_{ext}= X\cup\{f(b)\}$, and m_{ext} is defined as follows:

$$m_{ext}:X_{ext}\times X_{ext}\to M:\begin{cases}(c,d)\mapsto m(c,d) & \text{for } c\in X \text{ and } d\in X\\ (f(b),c)\mapsto f(m(b,c)) & \text{for } c\in X\\ (c,f(b))\mapsto f(m(c,b)) & \text{for } c\in X\\ (f(b),f(b))\mapsto f(f(m(b,b))) & \end{cases}$$

Then
$$\{a\}\cup f(X_{ext}) = \{a\}\cup f(X\cup\{f(b)\})$$
$$= \{a\}\cup f(X)\cup\{f(f(b))\}$$
$$= X\cup\{f(b)\}\cup\{f(f(b))\}$$
$$= X_{ext}\cup\{f(f(b))\}.$$

It is straightforward to check the other properties of goodness for R_{ext}.

Proof of claim (b). This follows with a double $\langle B,ext,\{(a,a,a)\}\rangle$-induction.
Clearly, for $\{(a,a,a)\}$ we have

$$\forall S\in Approx\ [\{(a,a,a)\}\subseteq S\vee S\subseteq\{(a,a,a)\}].$$

So assume for $R\in B$ we know already

$$\forall S\in Approx\,(R\subseteq S\vee S\subseteq R).$$

Then for $S=\{(a,a,a)\}$ we have $R_{ext}\subseteq\{(a,a,a)\}\vee\{(a,a,a)\}\subseteq R_{ext}$.
If $S=T_{ext}$ for $T\in Approx$ we have $R\subseteq T\vee T\subseteq R$, hence $R_{ext}\subseteq S\vee S\subseteq R_{ext}$.
If we apply induction we obtain

$$\forall S\in Approx\,(R_{ext}\subseteq S\vee S\subseteq R_{ext}).$$

And so, with a second appeal to induction we get

$$\forall R,S\in Approx\,(R\subseteq S\vee S\subseteq R).$$

Proof of claim (c). A proof with double <A,f,a>-induction will show $\forall c,d\in M\,\exists e\in M\ (c,d,e)\in G$. Let $c=d=a$. Then $(a,a,a)\in G$ as $\{(a,a,a)\}\in \mathrm{Approx}$.
Let $c=a$, and assume for $d\in M$ we have $(c,d,e)\in G$ for some $e\in M$. Then there is good $R\in \mathrm{Approx}$ such that $(c,d,e)\in R$. Let $m:X\times X\rightarrow M$ be the representing function of R, then $m(c,d)=e$, hence $m_{ext}(f(c),d)=f(m(c,d))=f(e)$. I.e., $(f(c),d.f(e))\in R_{ext}$, which is another good graph in Approx. Hence $(f(c),d,f(e))\in G$.
Assume now that for $c\in M$ we have proved that for all $d\in M$ there is an $e\in M$ such that $(c,d,e)\in G$. In a similar fashion as in the former step one proves that the same is true for $f(c)$.
Uniqueness follows from claim (b). Assume $(c,d,e)\in G$ and $(c,d,e')\in G$. That is for some R and R' we have $(c,d,e)\in R$ and $(c,d,e')\in R'$. As $R\subseteq R'$ or $R'\subseteq R$ it follows that $(c,d,e')\in R$ or $(c,d,e)\in R'$. In both cases we get $e=e'$.

□

Proof of theorem 1.6.7.
(⇒) If $\mathbb{E}$ has a natural number object, then in the internal language $L_{\mathbb{E}}$ of the topos we have a type for natural numbers with which we can perform the free monoid construction on any type of $L_{\mathbb{E}}$, i.e., any object of $\mathbb{E}$. By an appeal to the slogan *"description holds in a topos"* (1.5.8.ii) we see that the interpretation of the subtype constructed by the free monoid construction is a free monoid in the topos $\mathbb{E}$.
(⇐) If B is an object of $\mathbb{E}$ and FB the corresponding free monoid in $\mathbb{E}$, then also in the internal type theory of the topos the type FB is the free monoid corresponding to B. In the internal logic $F\mathbb{1}$ is the natural number type. Again with an appeal to *"description holds in a topos"* we get that $F\mathbb{1}$ is a natural number object in the topos $\mathbb{E}$.

□

Chapter 2

Truth Values, Topologies and Sheaves

In this and later chapters we work inside some type theory H in a rather informal way. The proofs we will give are presented in a loose set-theory-like formalism, but can be formalized entirely in the rigid language of type theory.

We will give an entirely internal treatment of a number of crucial notions from elementary topos theory: Lawvere-Tierney topologies on the type of truth values Ω, Grothendieck topologies on Ω, dense and closed subtypes and sheaves. Most results belong to topos theory, and, for instance, can be found in [Fourman and Scott].
The internal treatment of Grothendieck topologies appears to be new, but is implicit in [Johnstone 77]. The Grothendieck topologies on Ω form themselves a complete Heyting algebra whose operations (join, meet and implication) can be naturally and explicitly described. A corollary of this result is the well-known fact that the type of Lawvere-Tierney topologies on Ω is a complete Heyting algebra.
Internal proofs do not necessarily follow the same strategy as the external categorical proofs. As an example we present an internal proof of the otherwise well-known fact that Ω_j is a sheaf.

2.1 The object of truth values

In type theory Ω is the object of truth values. The (higher order) properties of Ω reflect the (higher order) propositional fragment of its type theory. On Ω we have a lattice structure: take $\rightarrow$ as partial order $\leq$ on Ω, $\wedge$ as finite meet and $\bigvee : P(\Omega) \longrightarrow \Omega : \Sigma \longmapsto (\exists \omega \in \Sigma\ \omega)$ as arbitrary join.
One easily proves that Ω is complete lattice satisfying the infinite distributive law:

$$\omega \wedge \bigvee \Sigma = \bigvee \{\omega \wedge \phi \mid \phi \in \Sigma\}, \text{ for all } \omega \in \Omega \text{ and } \Sigma \subseteq \Omega.$$

In general such lattices are called *complete Heyting algebras*. Note that logical implication $\rightarrow$ now has a double function in the complete Heyting algebra Ω: it serves both as order relation and lattice implication.

The completeness of Ω can be shown in two ways:

(i) in higher order propositional calculus

(ii) a intuitionistic set theoretic way, via the isomorphism of Ω with $P\mathbb{1}$, given by the following functions:

$$F:\Omega\to P\mathbb{1}:\omega\mapsto\{*\in\mathbb{1}\mid\omega\}$$

and

$$G:P\mathbb{1}\to\Omega:R\mapsto *\in R.$$

$\langle P\mathbb{1},\subseteq,\cap,\cup\rangle$ is easily seen to be complete Heyting algebra. Clearly for $\Sigma\subseteq\Omega$ we have

$$G(\bigcup\{F(\omega)\subseteq\mathbb{1}\mid\omega\in\Sigma\}) = *\in\bigcup\{F(\omega)\subseteq\mathbb{1}\mid\omega\in\Sigma\} = \exists\omega\in\Sigma\ *\in F(\omega) = \exists\omega\in\Sigma\ \omega.$$

The prime example in classical mathematics of a complete Heyting algebra is the set of opens of a topological space. For a continuous map $f:X\to Y$ between topological spaces it holds that the inverse image $f^{-1}: O(Y)\to O(X)$ preserves finite meets and arbitrary supremas.
Thus, one is led to the idea that a good notion of morphism between complete Heyting algebras is a $\wedge,\vee$-preserving function. The category of complete Heyting algebras and $\wedge,\vee$-preserving functions is called the category of frames; its dual is called the category of locales (cf. [Johnstone 82]).
Related to the continuous map $f:X\to Y$ is also another $\wedge,\vee$-preserving function $f_*: O(X)\to O(Y)$: $U\mapsto \mathrm{int}((f(U^c))^c)$, which can seen as the right adjoint of f^{-1}. The combination $f_*f^{-1}: O(Y)\to O(Y)$ has a number of interesting properties (these properties are well known, for instance, one can find details in [Fourman and Scott]).

For U,V opens in Y one has:

(i) $f_*f^{-1}(Y)=Y$

(ii) $U\subseteq f_*f^{-1}(U)$

(iii) $f_*f^{-1}f_*f^{-1}(U)=f_*f^{-1}(U)$

(iv) $f_*f^{-1}(U\cap V)=f_*f^{-1}(U)\cap f_*f^{-1}(V)$

In the history of topos theory it has turned out that the notion of an endofunction from a complete Heyting algebra in itself satisfying these four properties is very fruitful. For instance, the reflective subcategories of a topos are in one-one correspondence with the such endofunctions on its object of truthvalues.
In the literature there exist different names for these functions. In the context of toposes they are usually called topologies. Working with complete Heyting algebra's [Fourman and Scott] give them the name J-operators. In the general

context of locales [Johnstone 82] names them nuclei, whereas [Joyal and Tierney] call them local operators.
The name topology is motivated by the observation that every closure operation $j: PX \to PX$ satisfying the four properties determines a topology on X.

It is one of the surprising impacts of topos theory on logic that from this notion of topology an important tool -associated sheaf functor- can be built, which provides a uniform approach to all kind of important forcing technics and translations in logic. Well-known forcing technics in logic correspond with associated sheaf functors in particular toposes (cf. [Tierney]).
On the level of type theory the associated sheaf functor relates to Friedman and Gödel translations. This will be the subject of chapter 6. In this chapter we will give a systematic type theoretical treatment of topologies and associated sheaf constructions.

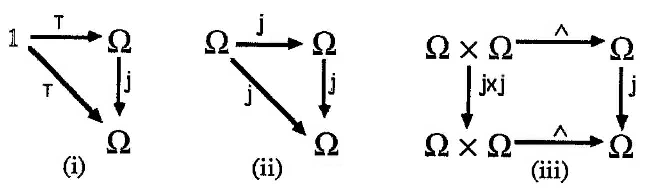
Notationally we will treat elements of the type $\Omega \to \Omega$ as unary modal operators on Ω, that bind stronger than any other logical connective.

2.2 Topologies

2.2.1 Definition A (Lawvere-Tierney) topology on Ω is a (internal) function $j:\Omega \to \Omega$ such that

(i) $j\top = \top$
(ii) $\forall \omega \in \Omega \; jj\omega = j\omega$
(iii) $\forall \omega_1, \omega_2 \in \Omega \; j(\omega_1 \wedge \omega_2) = (j\omega_1 \wedge j\omega_2)$

Or, equivalently, the following diagrams commute:



2.2.2 Examples.
The following are topologies (for more examples see [Fourman and Scott]):

(i) $id: \Omega \to \Omega: \omega \mapsto \omega$ (the minimal topology)
(ii) $j_{max}: \Omega \to \Omega: \omega \mapsto \top$ (the maximal topology)
(iii) $\neg\neg: \Omega \to \Omega: \omega \mapsto \neg\neg\omega$ (the double negation topology)
(iv) $j_p: \Omega \to \Omega: \omega \mapsto p \vee \omega$ for $p \in \Omega$ (the closed topology, e.g. cf. [Johnstone 77])

(v) $j^p:\Omega \to \Omega: \omega \mapsto p \to \omega$ for $p \in \Omega$ (the open topology, e.g. cf. [Johnstone 77])

In the context of the topos of Kripke models on a partial order P of *possible worlds* it makes sense to read $\neg\neg\phi$ as "whatever happens in the future we will always keep the possibility to prove ϕ." Note, that this is stronger than "ϕ ought to happen" and weaker than "it is possible that ϕ happens," which are much studied modal operators in modal logic. Lawvere has suggested a geometric reading of $j\phi$: "it is locally the case that ϕ holds."

The next lemma describes the interaction of topologies with logical connectives.

2.2.3 Lemma.

(i) $\forall \omega \in \Omega\ (\omega \to j\omega)$
(ii) $\forall \omega_1, \omega_2 \in \Omega\ [j(\omega_1 \to \omega_2) \to (j\omega_1 \to j\omega_2)]$
(iii) $\forall \omega_1, \omega_2 \in \Omega\ [(j\omega_1 \to j\omega_2) \leftrightarrow (\omega_1 \to j\omega_2)]$
(iv) $\forall \omega \in \Omega\ (\neg j\omega \to j\neg\omega)$
(v) $\forall \omega_1, \omega_2 \in \Omega\ [(j\omega_1 \vee j\omega_2) \to j(\omega_1 \vee \omega_2)]$
(vi) $\forall \phi: A^\Omega\ [j\forall a \in A\ \phi(a) \to \forall a \in A\ j\phi(a)]$
(vii) $\forall \phi: A^\Omega\ [\exists a \in A\ j\phi(a) \to j\exists a \in A\ \phi(a)]$

For some topologies a property stronger than (ii) holds:

(viii) $\forall \omega_1, \omega_2 \in \Omega\ [\neg\neg(\omega_1 \to \omega_2) \leftrightarrow (\neg\neg\omega_1 \to \neg\neg\omega_2)]$
(ix) $\forall \omega_1, \omega_2 \in \Omega\ [j^p(\omega_1 \to \omega_2) \leftrightarrow (j^p\omega_1 \to j^p\omega_2)]$

(x) The following are equivalent
(a) $j\bot \to \bot$
(b) $\forall \omega \in \Omega\ (\neg j\omega \leftrightarrow j\neg\omega)$
(c) $\forall \omega \in \Omega\ (j\omega \to \neg\neg\omega)$

Proof.
Straightforward. For instance, an informal proof of (i) is: suppose for $\omega \in \Omega$ we have that ω holds. But then $\omega = \top$, and so $j\omega = j\top = \top$. Hence $j\omega$ holds. Thus we have shown $\omega \to j\omega$ for $\omega \in \Omega$.

Or, a bit more formally in the style of a natural deduction argument:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\not\omega}{\top\to\omega}\quad\dfrac{\top}{\omega\to\top}}{\omega=\top}\qquad\dfrac{j\top=\top\quad\top}{j\top}}{j\omega}}{\omega\to j\omega}}{\forall\omega\in\Omega(\omega\to j\omega)}$$

□

There are equivalent formulations of the notion 'topology', one that says that a topology is increasing, idempotent and order preserving, and two others that are closely related to the definition of Grothendieck topology, that we will introduce in (2.2.7).
Note, that because $\to$ is both the order $\leq$ and the implication $\Rightarrow$ of the complete Heyting algebra Ω, the following two expressions for a complete Heyting algebra

$$\forall\omega_1,\omega_2\in\Omega\ (\omega_1\leq\omega_2)\to(j\omega_1\leq j\omega_2)$$
$$\forall\omega_1,\omega_2\in\Omega\ (\omega_1\Rightarrow\omega_2)\leq(j\omega_1\Rightarrow j\omega_2)$$

are both interpreted in Ω by

$$\forall\omega_1,\omega_2\in\Omega\ (\omega_1\to\omega_2)\to(j\omega_1\to j\omega_2).$$

2.2.4 Lemma. The following are equivalent for a function $j:\Omega\to\Omega$:
(i) $j:\Omega\to\Omega$ is a topology

(ii) $j:\Omega\to\Omega$ satisfies the conditions for a closure / local operator on Ω (cf. [Joyal and Tierney]):
- (a) $\forall\omega\in\Omega\ \omega\to j\omega$
- (b) $\forall\omega\in\Omega\ jj\omega\to j\omega$
- (c) $\forall\omega_1,\omega_2\in\Omega\ (\omega_1\to\omega_2)\to(j\omega_1\to j\omega_2)$

(iii) $j:\Omega\to\Omega$ satisfies the internal conditions for a Grothendieck topology:
- (d) $j\top=\top$
- (e) $\forall\omega_1\in\Omega\ (j\omega_1\to\forall\omega_2\in\Omega[(\omega_1\to j\omega_2)\to j\omega_2])$

(iv) $j:\Omega\to\Omega$ satisfies conditions related to the notion of Gabriel-Grothendieck topology:
- (f) $j\top=\top$
- (g) $\forall\omega_1\in\Omega\ (j\omega_1\to\forall\omega_2\in\Omega[(\omega_1\to j(\omega_1\wedge\omega_2))\to j\omega_2])$

Proof. (i)$\Rightarrow$(ii) has already been proved in the preceding lemma (2.2.3).

(ii)$\Rightarrow$(iii) We only have to show (e). Assume $j\omega_1$ and $\omega_1 \to j\omega_2$ for $\omega_1, \omega_2 \in \Omega$. Then $j\omega_1 \to jj\omega_2$ by (c) and so $j\omega_1 \to j\omega_2$ by (b). Hence we get $j\omega_2$. Ergo (e) holds.

(iii)$\Rightarrow$(iv) We prove (g). First we note that (e) together with the equivalence of (a) and (d) implies (b). Assume for $\omega_1 \in \Omega$ such that $j\omega_1$, and assume for $\omega_2 \in \Omega$ that $\omega_1 \to j(\omega_1 \wedge \omega_2)$ holds. Then $j\omega_1 \to jj(\omega_1 \wedge \omega_2)$ by (e). Hence, we can conclude $j(\omega_1 \wedge \omega_2)$. Again applying (e) we derive $j\omega_1$. And so,we see that (g) follows from (iii).

(iv)$\Rightarrow$(ii) First (a) follows by (f). Again, (g) and the fact (a)$\Leftrightarrow$(f) imply (b). Remains (c) which can be derived from (g) as follows. Assume $\omega_1 \to \omega_2$ and $j\omega_1$ for $\omega_1, \omega_2 \in \Omega$. If ω_1 holds then we get ω_2 and thus $\omega_1 \wedge \omega_2$. Hence $\omega_1 \to j(\omega_1 \wedge \omega_2)$, and we conclude $j\omega_2$ by (g).

(ii)$\Rightarrow$(i) The only interesting case to show is that for ω_1, ω_2 in Ω we have

$$j(\omega_1 \wedge \omega_2) \leftrightarrow (j\omega_1 \wedge j\omega_2).$$

($\to$) Since for $i=1,2$ $\omega_1 \wedge \omega_2 \to \omega_i$ we get $j(\omega_1 \wedge \omega_2) \to j\omega_i$. Hence

$$j(\omega_1 \wedge \omega_2) \to (j\omega_1 \wedge j\omega_2).$$

($\leftarrow$) This requires more work. Suppose we have shown:

(h) $\forall \omega_1, \omega_2 \in \Omega \ [j(\omega_1 \to \omega_2) \to (j\omega_1 \to j\omega_2)]$.

With help of (h) we can prove ($\leftarrow$).

For $\omega_1, \omega_2 \in \Omega$ we argue: $\omega_2 \to (\omega_1 \to (\omega_1 \wedge \omega_2))$ and (b) imply

$$j\omega_2 \to j(\omega_1 \to (\omega_1 \wedge \omega_2)).$$

So, $j\omega_1 \wedge j\omega_2$ implies

$$j\omega_1 \wedge j(\omega_1 \to (\omega_1 \wedge \omega_2)).$$

Then (h) gives us $j(\omega_1 \wedge \omega_2)$. Therefore

$$(j\omega_1 \wedge j\omega_2) \to j(\omega_1 \wedge \omega_2).$$

It remains to prove (h) from (ii). For $\omega_1, \omega_2 \in \Omega$ we know

$$(\omega_1 \to \omega_2) \to (j\omega_1 \to j\omega_2)$$

by (c). Hence by common intuitionistic propositional logic:

$$j\omega_1 \to ((\omega_1 \to \omega_2) \to j\omega_2).$$

Again by application of (c) we get

$$j\omega_1 \to (j(\omega_1 \to \omega_2) \to jj\omega_2),$$

and so we have

$$j(\omega_1 \to \omega_2) \to (j\omega_1 \to j\omega_2)$$

by logic and (b). And so we conclude (h).

□

Define the type $\mathsf{TOP} := \{j:\Omega \to \Omega \mid j \text{ is a Lawvere-Tierney topology}\}$. We have seen that Ω is a locale. In classical mathematics as well as in intuitionistic mathematics it is well-known that the set of topologies on a locale forms a locale (cf. [Dowker and Papert], [Fourman and Scott], [Johnstone 82], or [Joyal and Tierney].) We define cHa operations on TOP as follows, slightly different from [Johnstone 82] and [Fourman and Scott]:

(i) $j_1 \leq j_2 := \forall \omega \in \Omega\ j_1\omega \leq j_2\omega$
(ii) $\top = j_{max}: \omega \mapsto \top$
(iii) $\bot = id_\Omega: \omega \mapsto \omega$
(iv) $\bigwedge_{i \in I} j_i := \omega \mapsto \forall i \in I\ j_i\omega$
(v) $\bigvee_{i \in I} j_i := \omega \mapsto \forall j \in \mathsf{TOP}\ [(\forall i \in I\ j_i \leq j) \to j\omega]$, or $\bigwedge\{j \in \mathsf{TOP} \mid \forall i \in I\ j \geq j_i\}$
(vi) $j_1 \to j_2 := \omega \mapsto \forall \sigma \in \Omega\ [(\omega \to \sigma) \to (j_1\sigma \to j_2\sigma)]$

We observe the following useful equivalence

$$[\forall i \in I\ j^\omega \leq j_i] \leftrightarrow \forall j \in \mathsf{TOP}\ [(\forall i \in I\ j_i \leq j) \to j\omega]$$

that we can prove as follows:

($\leftarrow$) Assume for $\omega \in \Omega$ that $\forall i \in I\ j^\omega \leq j_i$. Suppose for $j \in \mathsf{TOP}$ that $\forall i \in I\ j_i \leq j$. Then we have to prove $j\omega$. Now $j^\omega \leq j_i$ implies in particular that $j^\omega(j\omega) \leq j_i(j\omega)$, that is $(\omega \to j\omega) \to j_i(j\omega)$. Hence $\forall i \in I\ j_i(j\omega)$. The assumption $\forall i \in I\ j_i \leq j$ can be rewritten as $\forall \sigma \in \Omega\ \forall i \in I\ [j_i(\sigma) \to j(\sigma)]$. Thus we get $j(j\omega)$, and so $j\omega$.

($\to$) It is easy to see that for all $i \in I$ $j_i(\omega)$ gives $\forall i \in I\ j^\omega \leq j_i$, i.e.,

$$\forall i \in I\ \forall \sigma \in \Omega\ [(\omega \to \sigma) \to j_i(\sigma)].$$

Now, suppose $j_i(\omega)$ and assume $(\omega \to \sigma)$ for $i \in I$ and $\omega, \sigma \in \Omega$. Then $(j_i\omega \to j_i\sigma)$ follows, and so we get $j_i\sigma$ by modus ponens.

2.2.5 Theorem. The type TOP of Lawvere-Tierney topologies on Ω, is a locale.

Proof. Either check one of the afore mentioned references or give a direct proof. The latter is rather simple compared with the more general case that is treated in the references: the topologies on an *arbitrary* locale Ω form a cHa. In our particular case of the object of truth values matters simplify. The difficult part is always to prove that the following part of the infinite distributive law:

$$(j \wedge \bigvee_{i \in I} j_i) \leq \bigvee_{i \in I} (j \wedge j_i).$$

We argue as follows:

$$\begin{aligned}(j \wedge \bigvee_{i \in I} j_i)(\omega) &\to (j\omega \wedge \bigvee_{i \in I} j_i\,\omega)\\ &\to j\omega \wedge \forall i \in I\ j^\omega \leq j_i\\ &\to \forall i \in I\ (j\omega \wedge j^\omega \leq j_i)\\ &\to \forall i \in I\ (j\omega \wedge \forall \sigma \in \Omega\ [(\omega \to \sigma) \to j_i\sigma])\end{aligned}$$

$$\to \forall i\in I\ \forall\sigma\in\Omega\ (j\omega\wedge[(\omega\to\sigma)\to j_i\sigma])$$
$$\to \forall i\in I\ \forall\sigma\in\Omega\ [(\omega\to\sigma)\to(j\sigma\wedge j_i\sigma)]$$
$$\to \forall i\in I\ \forall\sigma\in\Omega\ [(\omega\to\sigma)\to(j\wedge j_i)(\sigma)]$$
$$\to \forall i\in I\ j^{\omega}\leq(j\wedge j_i)$$
$$\to \bigvee_{i\in I}(j\wedge j_i)(\omega)$$

□

We can interpret this result in elementary toposes. Let $\mathbb{E}$ be a topos with arbitrary disjoint unions of $\mathbb{1}$, and $H_{\mathbb{E}}$ the corresponding type theory. In $H_{\mathbb{E}}$ we can prove that TOP is a locale. Externally this means for $\mathbb{E}$ that there is an object TOP together with some morphisms that is a locale in $\mathbb{E}$. The global elements (morphisms from $\mathbb{1}\longrightarrow$ TOP) of the object TOP correspond with the Lawvere-Tierney topologies of the topos $\mathbb{E}$. Hence we get as corollary:

2.2.6 Corollary. The Lawvere-Tierney topologies of an elementary topos with arbitrary disjoint unions of $\mathbb{1}$ (that is, in particular, the Lawvere-Tierney topologies of any topos over *Sets*) form a locale.

Proof. The arbitrary disjoint unions are needed to get an internal copy of an external given set I as an I-indexed disjoint union of $\mathbb{1}$.

□

The notion of *Grothendieck topology* is related to the Lawvere-Tierney topologies.

Grothendieck topologies on a category C are usually introduced pointwise: for each object U a set of covering sieves is specified, such that some conditions are fulfilled (see for instance cf. [Johnstone 77]). In the topos $Sets^{C^{op}}$ this gives rise to corresponding Lawvere-Tierney topologies on Ω. And the definition of the latter can be internalized in the general case of elementary toposes. But then there exist an internal definition of the notion of Grothendieck topology: a subobject J of Ω classified by a topology $j:\Omega\longrightarrow\Omega$. Or equivalently, such a subobject J ought to satisfy the conditions:

(i) $\top\in J$
(ii) $\forall\omega_1,\omega_2\in\Omega\ [(\omega_1\wedge\omega_2)\in J\leftrightarrow(\omega_1\in J\wedge\omega_2\in J)]$
(iii) $\forall\omega\in\Omega\ [\omega\in J\leftrightarrow(\omega\in J)\in J]$.

However, the interpretation (cf. [Goldblatt] or [Johnstone 77]) of these conditions in the topos $Sets^{C^{op}}$ bears no *prima facie* resemblance to the usual definition of a

Grothendieck topology. It can be proven to be equivalent with ths standard definition. However, it is more natural to take another equivalent formulation of Lawvere-Tierney topology, of which the corresponding subobject without much calculations satisfies conditions that interpreted in $\mathcal{S}ets^{\mathcal{C}^{op}}$ immediately give the usual definition of Grothendieck topology.

2.2.7 Definition. A *Grothendieck topology* on Ω is a subobject $J \subseteq \Omega$, such that

(i) $\top \in J$

(ii) $\forall \omega_1 \in J \, \forall \omega_2 \in \Omega \, [(\omega_1 \to (\omega_2 \in J)) \to (\omega_2 \in J)]$

from this it follows that

(iii) $\forall \omega_1, \omega_2 \in \Omega \, [(\omega_1 \to \omega_2) \to (\omega_1 \in J \to \omega_2 \in J)]$.

(iv) $\forall \omega_1, \omega_2 \in \Omega \, [(\omega_1 \wedge \omega_2) \in J \leftrightarrow (\omega_1 \in J \wedge \omega_2 \in J)]$.

2.2.8 Lemma. For $j: \Omega \to \Omega$ and $J \subseteq \Omega$, such that $J := \{\omega \in \Omega \mid j\omega\}$ is the subtype of Ω characterized by j, the following are equivalent:

(i) $j: \Omega \to \Omega$ is Lawvere-Tierney topology,

(ii) $J \subseteq \Omega$ is a Grothendieck topology.

Proof.
Trivial. The properties of (2.2.7) are equivalent to (2.2.4.iii) by writing $\omega \in J$ instead of $j\omega$.

□

For sake of completeness we mention another class of subobjects of Ω that frequently occurs in literature, cf. [Fourman and Scott], [Johnstone 82] and [Joyal and Tierney].
Given a topology $j: \Omega \to \Omega$, the $\{\omega \in \Omega \mid \omega = j\omega\}$ is a quotient or sublocale of Ω (cf. [Fourman and Scott] or [Joyal and Tierney]. In [Joyal and Tierney] one finds a characterization of such quotients of the object of truth values Ω:
quotients of Ω correspond with subsets $Q \subseteq \Omega$ such that

(i) $\top \in Q$

(ii) $\forall \omega_1, \omega_2 \in \Omega \, [(\omega_1 \in Q \wedge \omega_2 \in Q) \to (\omega_1 \wedge \omega_2) \in Q]$.

(iii) $\forall \omega_1 \in \Omega \, \forall \omega_2 \in Q \, [(\omega_1 \to \omega_2) \in Q]$.

Again there is a one-one correspondence between quotients and Lawvere-Tierney topologies. Note that this correspondence is order reversing.

Now consider the type $\mathsf{GTOP} := \{J \subseteq \Omega \mid J \text{ is a Grothendieck topology}\}$.

Define:

(i) $J_1 \leq J_2 := J_1 \subseteq J_2$
(ii) $J_1 \wedge J_2 := J_1 \cap J_2$
(iii) $\bigvee_{i \in I} J_i := \bigcap \{J \in \mathsf{GTOP} \mid \forall i \in I\ J_i \leq J\}$
(iv) $J_1 \Rightarrow J_2 := \bigvee \{J \in \mathsf{GTOP} \mid J \wedge J_1 \leq J_2\}$
(v) $J_{top} := \Omega$
(vi) $J_{bottom} := \{\top\}$.

2.2.9 Theorem. The type GTOP of Grothendieck topologies on Ω is a locale.

Proof. Either one gives a direct proof, or one shows that the morphisms

$$(-)_j := \mathsf{GTOP} \to \mathsf{TOP} : J \mapsto (\Omega \to \Omega : \omega \mapsto \omega \in J)$$

$$(-)_J := \mathsf{TOP} \to \mathsf{GTOP} : j \mapsto \{\omega \in \Omega \mid \top = j\omega\}$$

are inverses to each other, and preserve the lattice structure, whence GTOP is a locale because TOP is one, cf. (2.2.5).

□

As a corollary we will derive that dense topologies (cf. [Johnstone 82]) also form a locale .

2.2.10 Definition.

(i) A Lawvere-Tierney topology $j : \Omega \to \Omega$ is dense if $j\bot = \bot$.
(ii) A Grothendieck topology is dense if $\neg(\bot \in J)$.

Hence a Lawvere-Tierney topology is dense if and only if the corresponding Grothendieck topology is dense.
Note also that $j : \Omega \to \Omega$ is dense if and only if $j \leq \neg\neg$, i.e., $\forall \omega \in \Omega\ j\omega \to \neg\neg\omega$.
It is easy to see that arbitrary meets of *inhabited* families of dense topologies are dense. Similar, the join in TOP of an *inhabited* family $(j_i)_{i \in I}$ of dense topologies is again dense:

$$\begin{aligned}
(\bigvee_{i \in I} j_i)(\bot) &\to \forall_{i \in I}\ j_\bot \leq j_i \\
&\to \forall_{i \in I}\ \forall \sigma \in \Omega [(\bot \to \sigma) \to j_i(\sigma)] \\
&\to \forall_{i \in I}\ \forall \sigma \in \Omega\ j_i(\sigma) \\
&\to \forall_{i \in I}\ j_i(\bot) \\
&\to \forall_{i \in I}\ \bot \\
&\to \bot. \qquad\qquad (\text{I is inhabited})
\end{aligned}$$

If we have a Grothendieck topology J in the topos $\mathcal{S}ets^{\mathcal{C}^{op}}$ then:

$\mathcal{S}ets^{\mathcal{C}^{op}} \models$ J is dense

if and only if

J contains no empty covers as a Grothendieck topology on $\mathcal{C}$

(for a definition of a Grothendieck topology on $\mathcal{C}$, cf. p. 13 in [Johnstone 77]).

2.2.11 Corollary. Let DTOP and DGTOP be the subtypes of respectively TOP and GTOP containing the dense topologies. Then:

$(-)_j$ and $(-)_J$ restrict to locale isomorphisms between DTOP and DGTOP.

Proof.

Only the definition of the locale operations on DTOP is tricky. Note that $\neg\neg$ is top element of DTOP.

$\top^d : \Omega \to \Omega : \omega \mapsto \neg\neg\omega$

$\bot^d : \Omega \to \Omega : \omega \mapsto \omega \quad (=\omega \wedge \neg\neg\omega)$

$\bigwedge^d_{i\in I} j_i : \Omega \to \Omega : \omega \mapsto \neg\neg\omega \wedge \bigwedge_{i\in I} j_i\omega$

$\bigvee^d_{i\in I} j_i : \Omega \to \Omega : \omega \mapsto \neg\neg\omega \wedge \bigvee_{i\in I} j_i\omega$

One can perform the necessary calculations to show that DTOP is a complete Heyting algebra.

Or note that the function d: TOP→TOP defined by

$d(j) : \Omega \to \Omega : \omega \mapsto \neg\neg\omega \wedge j\omega$

is a multiplicative operator (i.e., $d(j_1 \wedge j_2) = d(j_1) \wedge d(j_2)$) on the complete Heyting algebra TOP, and the fixed points of d are precisely the dense topologies of DTOP, and so we get by lemma (2.13) of [Fourman and Scott] that DTOP is a complete Heyting algebra.

□

Grothendieck topologies provide another way of understanding open topologies (cf. (2.2.2). Open topologies can be characterized as Lawvere-Tierney topologies such that the corresponding Grothendieck topology contains a smallest element. This old result of Tierney gets a trivial constructive proof.

2.2.12 Lemma. (Tierney, cf. [Johnstone 77]).

(i) The following are equivalent for a topology $j : \Omega \to \Omega$:

(a) $\exists! p \in \Omega\ \forall \omega \in \Omega\ ((jp \wedge j\omega) \to (p \to \omega))$

(b) $\exists p \in \Omega\ \ j = j^p$

(ii) The following are equivalent for a Grothendieck topology J:

(c) $\exists! p\in J\, \forall \omega\in J\, (p\to\omega)$
(d) $\exists p\in\Omega\; J=(j^p)_J$

Proof. We prove (ii). First observe that $(j^p)_J=\{\omega\in\Omega \mid j^p\omega\}=\{\omega\in\Omega \mid p\to\omega\}$
(c)⇒(d). Assume there exist a unique $p\in J$ such that for all $\omega\in J$ we have $p\to\omega$. Then $J\subseteq(j^p)_J$. If we have $\omega\in(j^p)_J$, then $p\to\omega$. But $p\in J$, so $\omega\in J$. Hence (d) holds.
(d)⇒(c). Assume (d) holds, that is we have some $p\in\Omega$ such that $J=\{\omega\in\Omega \mid p\to\omega\}$. Then trivially $\exists p\in J\,\forall\omega\in J\,(p\to\omega)$. Uniqueness follows immediately: if we have $p,q\in J$ such that $\forall\omega\in J\,(p\to\omega)$ and $\forall\omega\in J\,(q\to\omega)$, then $p\leftrightarrow q$, i.e., $p=q$. Hence we have (c).

□

In the language of type theory it is easy to describe the smallest Grothendieck topology J_D containing a certain subobject $D\subseteq\Omega$. For a categorical construction see for instance [Johnstone 77].

2.2.13 Definition.
(i) For a subobject D of Ω define
$J_D=\bigcap\{J\subseteq\Omega \mid J \text{ is a Grothendieck topology} \wedge D\subseteq J\}$.
(ii) For a function $k:\Omega\to\Omega$ define
$j_k:\Omega\to\Omega:\omega\mapsto[\forall j\in TOP\,(k\leq j\to j\omega)]$

2.2.14 Lemma.
(i) J_D is the smallest Grothendieck topology containing D, for all $D\subseteq\Omega$.
(ii) j_k is the smallest Lawvere-Tierney topology larger than k, for all $k:\Omega\to\Omega$.

Proof. Trivial.

□

2.3 Sheaves and related notions

We now give a number of useful definitions related to the concept of *sheaf*. We formulate the definitions in type theory, in contrast to the usual categorical way with diagrams of functions. It is not difficult to prove in intuitionistic type theory that these two formulations are equivalent. All of this is more or less explicit in [Fourman and Scott], the difference lies in the formal system we use. Fourman and

Scott use a higher order logic with an existence predicate **E** and description operator **I**.

2.3.1 Definition. Let A be a type, $m: B \rightarrowtail A$ a mono and $j: \Omega \to \Omega$ a Lawvere-Tierney topology.

(i) The j-closure B^j of B in A is the subtype $\{a \in A \mid j\exists b \in B\,(m(b)=a)\}$

(ii) B is (j-)closed if $B^j = A$.

(iii) $m: B \rightarrowtail A$ is (j-)dense if $\forall a \in A\; j\exists b \in B\,(m(b)=a)$.

(iv) A is (j-)separated if $\forall a,b \in A\,[j(a=b) \to a=b]$.

(v) A is a (j-)sheaf if $\forall X \subseteq A\,[j\exists! a \in A\; a \in X \to \exists! a \in A\; j(a \in X)]$.

2.3.2 Lemma. Let A be a type and B a subtype of A.

(i) B^j is the subobject of A classified by $j \circ \phi_{B \subseteq A}$, where $\phi_{B \subseteq A}: A \to \Omega : a \mapsto (a \in B)$.

(ii) B is closed if and only if $m: B \to B^j$ is an iso.

(iii) B is dense if and only if $B^j = A$.

(iv) A is separated if and only if for any dense mono $m: B \rightarrowtail C$ and any $f,g: C \to A$ it holds that whenever $fm = gm$ then $f = g$.

(v) A is a sheaf if and only if for any dense mono $m: B \rightarrowtail C$ and any $f: B \to A$ there is a unique $g: C \to A$ such that $gm = f$.

Proof.

(i) $\{a \in A \mid j \circ \phi_{B \subseteq A}(a) = \top\} = \{a \in A \mid j(\exists b \in B\; m(b)=a) = \top\}$
$= \{a \in A \mid j\exists b \in B\; m(b)=a\}$
$= B^j$.

(ii) $m: B \to B^j$ is an iso $\Leftrightarrow \forall a \in B^j\; \exists b \in B\; m(b)=a$
$\Leftrightarrow \forall a \in A\,(j\exists b \in B\; m(b)=a \to \exists b \in B\; m(b)=a)$

(iii) $B^j = A \Leftrightarrow \forall a \in A\,(a \in B^j \leftrightarrow a \in A)$
$\Leftrightarrow \forall a \in A[(j\exists b \in B\;\; m(b)=a) \leftrightarrow a \in A]$
$\Leftrightarrow \forall a \in A\; j\exists b \in B\; m(b)=a$.

(iv) ($\Rightarrow$) Suppose A is separated. Consider an arbitrary dense mono $m: B \rightarrowtail C$ and arbitrary functions $f,g: C \to A$. Suppose $fm = gm$.
For an $a \in A$ it now holds that: $f(m(a)) = g(m(a))$, and since B is dense in C we get $j(f(a) = g(a))$. Hence, using separatedness of A we get $f(a) = g(a)$. Thus $f = g$.

($\Leftarrow$) Suppose for any dense mono $m: B \rightarrowtail C$ and any $f,g: C \to A$ it holds that whenever $fm = gm$ then $f = g$. And suppose for $a,b \in A$ we have $ja = b$. Then the mono

$m:\{a\}\hookrightarrow\{a,b\}$ is dense. Now consider the morphisms $f:\{a,b\}\to A:c\mapsto c$ and $g:\{a,b\}\to A:c\mapsto a$. Since $m(f(a))=a=m(g(a))$ we get $mf=mg$ and by assumption $f=g$. But then $a=g(b)=f(b)=b$. Thus A is separated.

(v) ($\Rightarrow$) Suppose A is a sheaf. For each dense mono $m:B\rightarrowtail C$ and each function $f:B\to A$, we have to show the unique existence of a function $g:C\to A$ such that $gm=f$.
First we prove existence. Define $G:=\{(c,a)\in C\times A \mid j\exists b\in B\,(m(b)=c\wedge f(b)=a)\}$.
For $c\in C$ consider $X:=\{a\in A \mid \exists b\in B\,(m(b)=c\wedge f(b)=a)\}$. Since $j\exists!x\in A\; x\in X$ and A is a sheaf, we get $\exists!x\in A\; jx\in X$. Thus we see $\forall c\in C\;\exists!a\in A\;(c,a)\in G$. Now apply the axiom of unique choice to get $g:C\to A$.
Next, to prove unicity it suffices to show that the sheaf A is separated. This is easy: suppose for $a,b\in A$ we have $ja=b$. Consider $X=\{a,b\}$. Clearly $j\exists!x\in A\; x\in X$. Hence $\exists!x\in A\; jx\in X$. Therefor $a=b$.

($\Leftarrow$) Suppose for any dense mono $m:B\rightarrowtail C$ and any $f:B\to A$ there is a unique $g:C\to A$ such that $g\circ m=f$. Let $X\subseteq A$ be such that $j\exists!x\in A\; x\in X$.
Clearly $B:=\{Y\subseteq A \mid \exists!y\in A\,(y\in Y\wedge X=Y)\}$ is a dense subset of $C:=\{X\}$.
With an appeal to the axiom of unique choice we get a function $f:B\to A:\{a\}\mapsto a$.
By the assumption we obtain $g:C\to A$. Since $jX\in B$ we get $jg(X)\in X$.
Hence, whenever $j\exists!a\in A\; a\in X$ holds, we get $\exists a\in A\; ja\in X$.

It remains to show the uniqueness of a in A with the property $ja\in X$. It suffices to show that A is separated. To see this, consider for $a,b\in A$ with $ja=b$ the dense mono $m:\{a\}\hookrightarrow\{a,b\}$ together with $f:\{a\}\to A:a\mapsto a$. Define $g:\{a,b\}\to A:c\mapsto c$ and $h:\{a,b\}\to A:c\to a$. Clearly $g\circ m=f=h\circ m$. By the assumption it follows that $g=h$ and hence $a=b$.

□

As an illustration we will present a constructive set theoretical proof of the well known fact in topos theory that for a topos $\mathbb{E}$ and topology $j:\Omega\to\Omega$ in $\mathbb{E}$ the object Ω_j, the equalizer of id_Ω and j, is a j-sheaf in $\mathbb{E}$. In [Freyd 72] and [Johnstone 77] one finds categorical proofs, that essentially are calculations with closure operations.

2.3.3 Lemma. $\Omega_j:=\{\omega\in\Omega \mid \omega=j\omega\}$ is a j-sheaf.

Proof.
Let $X\subseteq\Omega$. Assume we have a unique element $\phi\in\Omega_j$ such that $\phi\in X$. Define $\psi:=(\top\in X)$. Then the following formulas are equivalent: ϕ, $(\phi=\top)$, $\top\in X$ and ψ.

Hence $\psi=\phi$. So, $\exists!\omega\in\Omega_j\, \omega\in X\rightarrow \psi\in X$. Hence, $j\exists!\omega\in\Omega_j\, \omega\in X \rightarrow j(\psi\in X)$. However $j(\psi\in X)$ implies $j((\top\in X)\in X)$ and also $\exists\omega\in\Omega_j\, j(\omega\in X)$. Such an ω has to be unique, for suppose we have $j\omega_1\in X$ and $j\omega_2\in X$, then $j(\omega_1=\omega_2)$ by (*). Hence, $j\omega_1=j\omega_2$, i.e., $\omega_1=\omega_2$.
We have shown that $j\exists!\omega\in\Omega_j\, \omega\in X\rightarrow\exists!\omega\in\Omega_j\, j(\omega\in X)$.
and can that conclude Ω_j is a sheaf.

□

The next lemma warns us that in general the object Ω is not a j-sheaf.

2.3.4 Lemma. The following are equivalent for arbitrary topology $j:\Omega\rightarrow\Omega$:
(i) Ω is a j-sheaf
(ii) Ω is j-separated
(iii) $\forall\omega\in\Omega\, (j\omega\rightarrow\omega)$
(iv) $\Omega=\Omega_j$
(v) $id_{\Omega}=j$.

Proof.
(i)⇒(ii) is trivial.
(ii)⇒(iii) Assume that Ω is j-separated. Then for all $\omega\in\Omega$ we have $j\omega=\top\rightarrow\omega=\top$, i.e., $j\omega\rightarrow\omega$.
(iii)⇒(i) is trivial, since the definition of j-sheaf trivializes under the assumption that $\forall\omega\in\Omega\, (j\omega\rightarrow\omega)$.
(iv) and (v) are just other ways of expressing (iii).

□

The following lemma will be useful.

2.3.5 Lemma. A closed subset of a sheaf is again a sheaf.

Proof. Let B be a closed subset of the sheaf A. Suppose for $X\subseteq B$ that $j\exists!x\in B\, x\in X$. Then also $j\exists!x\in A\, x\in X$. It follows from the fact that A is a sheaf that there is a unique $x\in A$ such that $jx\in X$. Hence $jx\in B$, because $X\subseteq B$. The assumption that B is closed now implies $x\in B$. I.e., $\exists!x\in B\, jx\in X$. Therefor B is a sheaf.

□

2.3.6 Corollary. Let A be a sheaf.

Then $\Omega_j{}^A$ is isomorphic with $\{X \subseteq A \mid X \text{ is a j-sheaf}\}$.

Proof.

$$\begin{aligned}\Omega_j{}^A &= \{f{:}A \to \Omega_j \mid T\} \\ &\approx \{f{:}A \to \Omega \mid \forall a \in A\, f(a) \in \Omega_j\} \\ &= \{f{:}A \to \Omega \mid j \circ f = f\} \\ &\approx \{B \subseteq A \mid ja \in B \leftrightarrow a \in B\} \\ &= \{B \subseteq A \mid B \text{ is j-closed}\} \\ &= \{B \subseteq A \mid B \text{ is a j-sheaf}\}.\end{aligned}$$

□

2.3.7 Theorem. Let A be any type.
Then $\Omega_j{}^A$ is a sheaf.

Proof. By the previous lemma we know that $\Omega_j{}^A$ is isomorphic with $\{X \subseteq A \mid X \text{ is a j-sheaf}\}$. Now let $Y \subseteq \Omega_j{}^A$ and assume $j\exists! X \in \Omega_j{}^A\, X \in Y$. We have to prove $\exists! X \in \Omega_j{}^A\, jX \in Y$ in order to conclude that $\Omega_j{}^A$ is a sheaf. Define

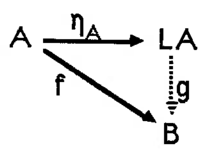
$$X^* = \{x \subseteq A \mid j\exists X \in Y\, x \in X\}.$$

Clearly, if there is a unique X in $\Omega_j{}^A$ such that $X \in Y$ then $X = X^*$, using the j-closedness of X. Hence from the assumption $j\exists! X \in \Omega_j{}^A\, X \in Y$ we can conclude $jX^* \in Y$.
That is, we have shown $\exists! X \in \Omega_j{}^A\, jX \in Y$, as desired.

Chapter 3

Singletons and Associated Sheaf Functors

In this section we will present systematically twelve different ways to construct associated sheaves for a fixed topology $j:\Omega \rightarrow \Omega$. Our descriptions of the constructions will be of a type theoretical nature and will encompass the well known categorical constructions of associated sheaf functors that occur in literature.
Recall that the *associated (j-)sheaf* for a type A is a (j-)sheaf LA together with a function $\eta_A: A \rightarrow LA$ such that for each type B and function $f: A \rightarrow B$ there is a unique function $g: LA \rightarrow B$ such that $f \circ \eta_A = g$. In diagram:



Two main expositions of topos theory, [Johnstone 77] and [Barr & Wells], present three different categorical methods to construct associated sheaves.
(i) The simplest construction is the one of Lawvere-Tierney (cf. [Gray], [Freyd], or [Johnstone 77]): construct LA as the closure of the image of A in $\Omega_j{}^A$ via the function $\{\,\}^j: A \rightarrow \Omega_j{}^A: a \mapsto \{a\}^j$.
(ii) Grothendieck's two-step construction has been extended by [Johnstone 77] from the context of Grothendieck toposes to elementary toposes.
(iii) The method of [Barr & Wells] consists of the application of one step of the Grothendieck construction to $A/_{\approx}$, where $a \approx b \leftrightarrow j(a=b)$ for $a,b \in A$.

Internal descriptions of the associated sheaf functor have lingered in the folklore of topos theory. This is - as far as we know - the first systematic internal treatment of the various constructions of associated sheaves. As a result we obtain some syntactic variations of these constructions.
(i) [Veit] has given an early, mixed internal - external account of the Lawvere-Tierney construction.
(ii) In the masters thesis of [De Vries] one finds an internal account of an associated sheaf construction following the Grothendieck construction.

(iii) [Rosolini] presents an entirely internal one step construction, at first sight unrelated to both Lawvere-Tierney's and Grothendieck's approach.
(iv) [Dupont & Loiseau] is a very recent example of an almost internal account of the three categorical constructions. With help of the internal logic they give uniform descriptions of the constructions, to prove that their constructions are well designed they mimic the categorical proofs mixing logical and categorical arguments. Their treatment could benefit from the Fourman-Scott use of the topology as a modal operator.

3.1 Various notions of singleton

The crucial device in type theory to describe the associated sheaf functors with is the notion of *singleton*. In classical mathematics the concept of a singleton subset of A is well-known:

$S \subseteq A$ is called a *singleton* if $\exists! a \in A\ a \in S$.

In intuitionistic mathematics [Lifschitz] and [van Dalen] have introduced an intuitionistic weakening of this idea for, respectively, the type $\mathbb{N}$ of natural numbers and the type $\mathbb{R}$ of Dedekind reals:

$S \subseteq A$ is called a $\neg\neg$*-singleton* of A if $\neg\neg\exists! a \in A\ a \in S$ for $A \in \{\mathbb{N}, \mathbb{R}\}$.

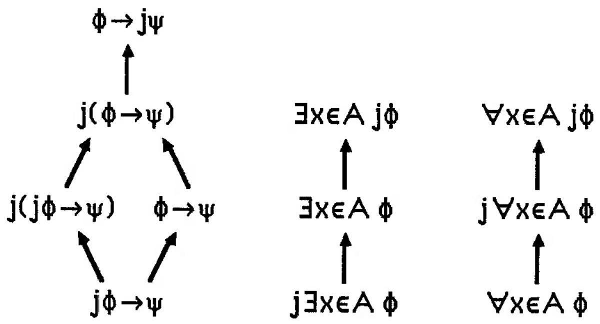
Since both $\mathbb{N}$ and $\mathbb{R}$ are $\neg\neg$-separated, an equivalent definition is for $A \in \{\mathbb{N}, \mathbb{R}\}$

$S \subseteq A$ is a $\neg\neg$*-singleton* of A if $\neg\neg\exists a \in A\ a \in S \wedge \forall a,b \in A\ (a \in S \wedge b \in S \rightarrow a = b)$.

In classical logic there are several equivalent ways of defining singleton, for instance: a subset $S \subseteq A$ is a singleton if

(i) $\exists a \in A\ a \in S \wedge \forall a,b \in A\ (a \in S \wedge b \in S \rightarrow a = b)$
(ii) $\exists a \in A\ \forall b \in A\ [(b \in S \rightarrow a = b) \wedge (a = b \rightarrow b \in S)]$

In type theory with a modal operator each logical symbol gives rise to its own family of variations (logical implication going upwards):



It follows that in type theory there is a multitude of singleton notions. It will turn out to be essential in the construction of the associated sheaf functors, that for a given notion of singleton we can construct a singleton S_a containing a given $a\in A$. From the constructive point of view there are strong and weak forms of containment. Possible candidates for S_a are $\{a\}$, $\{a\}^j=\{b\in A \mid j(a=b)\}$ and $\bigcap\{S\subseteq A \mid S$ is a singleton containing $a\}$. The weakest notion of containing at most one element is *containing locally at most one element*, i.e., $\forall a,b\in A\,(a\in S \wedge b\in S \rightarrow j(a=b))$. Hence we will require that S_a contains locally at most one element. This requirement implies that $S_a\subseteq\{a\}^j$. We will now systematically list all variations. From the resulting list with variations we will discard the notions of singleton for which neither $\{a\}$, nor $\{a\}^j$ is a singleton. Taking care of equivalent notions we will end up with the final list in definition 3.1.3. The impatient reader is advised to go immediately to (3.1.3).

First we will consider formulas related to $\exists a\in A\; a\in S \wedge \forall a,b\in A\,(a\in S \wedge b\in S \rightarrow a=b)$. This results in the following complete list of possible candidates for the notion of locally inhabited singleton:

(1) $j\exists a\in A\; a\in S \wedge \forall a,b\in A\,(ja\in S \wedge j(b\in S)\rightarrow a=b)$
(2) $j\exists a\in A\; a\in S \wedge j\forall a,b\in A\,(ja\in S \wedge j(b\in S)\rightarrow a=b)$
(3) $j\exists a\in A\; a\in S \wedge \forall a,b\in A\; j(ja\in S \wedge j(b\in S)\rightarrow a=b)$
(4) $j\exists a\in A\; a\in S \wedge \forall a,b\in A\,(a\in S \wedge j(b\in S)\rightarrow a=b)$
(5) $j\exists a\in A\; a\in S \wedge j\forall a,b\in A\,(a\in S \wedge j(b\in S)\rightarrow a=b)$
(6) $j\exists a\in A\; a\in S \wedge \forall a,b\in A\; j(a\in S \wedge j(b\in S)\rightarrow a=b)$
(7) $j\exists a\in A\; a\in S \wedge \forall a,b\in A\,(a\in S \wedge b\in S \rightarrow a=b)$
(8) $j\exists a\in A\; a\in S \wedge j\forall a,b\in A\,(a\in S \wedge b\in S \rightarrow a=b)$
(9) $j\exists a\in A\; a\in S \wedge \forall a,b\in A\; j(a\in S \wedge b\in S \rightarrow a=b)$
(10) $j\exists a\in A\; a\in S \wedge \forall a,b\in A\,(a\in S \wedge b\in S \rightarrow j(a=b))$

Notation: we will denote by (n)-singletons A subsets of A satisfying the n^{th} condition.

Note the vertical linear logical dependence; (1)→(2)→... .

For the first six notions there is no singleton S_a of that kind containing a given element $a\in A$. For example: for (1)-singletons the existence of such S_a's would imply that the type A is separated, which, of course, does not need to be the case. This follows from the next lemma.

3.1.1 Lemma.

(i) $\forall a\in A\,\exists S_a\subseteq A\,[a\in S_a \wedge S_a \text{ is a (n)-singleton}]$
$\leftrightarrow \forall a,b\in A\,(j(a=b)\to a=b)$, for $n\in\{1,4\}$

(ii) $\forall a\in A\,\exists S_a\subseteq A\,[a\in S_a \wedge S_a \text{ is a (n)-singleton}]$
$\leftrightarrow \forall a\in A\, j\forall b\in A\,(j(a=b)\to a=b)$, for $n\in\{2,5\}$

(iii) $\forall a\in A\,\exists S_a\subseteq A\,[a\in S_a \wedge S_a \text{ is a (n)-singleton}]$
$\leftrightarrow \forall a,b\in A\, j(j(a=b)\to a=b)$, for $n\in\{3,6\}$.

Proof. (i) (→) Assume $j(a=b)$ for $a,b\in A$. Then $a\in S_a$ and $j(b\in S)_a$. When S_a were a (i)-singleton or a (4)-singleton, this would imply $a=b$.
Hence $\forall a,b\in A\,(j(a=b)\to a=b)$.
(→) $\forall a,b\in A\,(j(a=b)\to a=b)$ implies that for all $a\in A$ the subset $\{a\}$ is a (1)-singleton.
(ii) and (iii) are similarly proven.

□

Before we will treat the case

$\exists a\in A\,\forall b\in A\,[(b\in S\to a=b)\wedge(a=b\to b\in S)]$

in full generality, we first proceed with the singleton notions related to

$\exists a\in A\,\forall b\in A\,(b\in S\leftrightarrow a=b)$.

Again we consider only notions implying local inhabitedness.

(11) $j\exists a\in A\,\forall b\in A\,(b\in S\leftrightarrow j(a=b))$
(12) $j\exists a\in A\,j\forall b\in A\,(b\in S\leftrightarrow j(a=b))$
(13) $j\exists a\in A\,\forall b\in A\,j(b\in S\leftrightarrow j(a=b))$
(14) $j\exists a\in A\,\forall b\in A\,(j(b\in S)\leftrightarrow a=b)$
(15) $j\exists a\in A\,j\forall b\in A\,(j(b\in S)\leftrightarrow a=b)$
(16) $j\exists a\in A\,\forall b\in A\,j(j(b\in S)\leftrightarrow a=b)$
(17) $j\exists a\in A\,\forall b\in A\,(b\in S\leftrightarrow a=b)$
(18) $j\exists a\in A\,j\forall b\in A\,(b\in S\leftrightarrow a=b)$
(19) $j\exists a\in A\,\forall b\in A\,j(b\in S\leftrightarrow a=b)$
(20) $j\exists a\in A\,\forall b\in A\,(j(b\in S)\leftrightarrow j(a=b))$

Note that (8)=(17), (9)=(19), (10)=(20),
(11)=(12), (14)=(15), (17)=(18) because $j\exists x\, j\phi \leftrightarrow j\exists x\, \phi$,
(4)⇒(14), (5)⇒(15) and (6)⇒(16)
(11)⇒(14), (12)⇒(15) and (13)⇒(16).
(11)⇒(12)⇒(13)⇒(20)
(14)⇒(15)⇒(16)⇒(20)
(17)⇒(18)⇒(19)⇒(20)

.c3.**3.1.2 Lemma.**;

(i) $\forall a\in A\, \exists S_a \subseteq A\, [a\in S_a \wedge S_a$ is a (n)-singleton$]$
$\leftrightarrow \forall a\in A\, j\, \forall b\in A\, (j(a=b)\to a=b)$, for $n\in\{14,15\}$

(ii) $\forall a\in A\, \exists S_a \subseteq A\, [a\in S_a \wedge S_a$ is a (n)-singleton$]$
$\leftrightarrow \forall a,b\in A\, j(j(a=b)\to a=b)$, for $n\in\{16\}$.

Proof. (ii) (←) $\forall a,b\in A\, j(j(a=b)\to a=b)$ implies that $\{a\}$ is a (16)-singleton.
(→) If for $a\in A$ there is $S_a \subseteq A$ such that both $a\in S_a$ and S_a is a (n)-singleton, then we can conclude successively

$j\exists b\in A\, \forall c\in A\, j(jc\in S_a \leftrightarrow b=c)$,
$j\exists b\in A[jb=a \wedge \forall c\in A\, j(ja=c\leftrightarrow b=c)]$,
$j\forall c\in A\, j(ja=c\leftrightarrow a=c)$.

We have shown that

$\forall a\in A\, j\, \forall c\in A\, j(ja=c\leftrightarrow a=c)$,

which is equivalent to

$\forall a\in A\, \forall b\in A\, j(j(a=b)\to a=b)$.

□

Hence we can discard the (n)-singletons with $14\le n\le 16$.

Finally we only list all variations on $\exists a\in A\, \forall b\in A\, [(b\in S\to a=b)\wedge(a=b\to b\in S)]$ for which either $\{a\}$ or $\{a\}^j$ are singletons. We group them in blocks of five items.

(21) $j\exists a\in A\, \forall b\in A\, [(j(b\in S)\to a=b)\wedge(j(a=b)\to b\in S)]$
(22) $j\exists a\in A\, \forall b\in A\, [(j(b\in S)\to a=b)\wedge j(j(a=b)\to b\in S)]$
(23) $j\exists a\in A\, \forall b\in A\, [(j(b\in S)\to a=b)\wedge(a=b\to b\in S)]$
(24) $j\exists a\in A\, \forall b\in A\, [(j(b\in S)\to a=b)\wedge j(a=b\to b\in S)]$
(25) $j\exists a\in A\, \forall b\in A\, [(j(b\in S)\to a=b)\wedge(a=b\to j(b\in S))]$ (⇔ 14)

Since (14) does not satisfy $\forall a\in A\, \exists S_a \subseteq A\, [a\in S_a \wedge S_a$ is a (n)-singleton$]$ we discard all.

(26) $j\exists a\in A\,\forall b\in A\,[j(j(b\in S)\to a=b)\wedge(j(a=b)\to b\in S)]$
(27) $j\exists a\in A\,\forall b\in A\,[j(j(b\in S)\to a=b)\wedge j(j(a=b)\to b\in S)]$
(28) $j\exists a\in A\,\forall b\in A\,[j(j(b\in S)\to a=b)\wedge(a=b\to b\in S)]$
(29) $j\exists a\in A\,\forall b\in A\,[j(j(b\in S)\to a=b)\wedge j(a=b\to b\in S)]$
(30) $j\exists a\in A\,\forall b\in A\,[j(j(b\in S)\to a=b)\wedge(a=b\to j(b\in S))]$ ($\Leftrightarrow$ 16)

Since (16) does not satisfy $\forall a\in A\,\exists S_a\subseteq A\,[a\in S_a\wedge S_a$ is a (n)-singleton$]$ we discard all.

(31) $j\exists a\in A\,\forall b\in A\,[(b\in S\to a=b)\wedge(j(a=b)\to b\in S)]$ discard
(32) $j\exists a\in A\,\forall b\in A\,[(b\in S\to a=b)\wedge j(j(a=b)\to b\in S)]$ discard
(33) $j\exists a\in A\,\forall b\in A\,[(b\in S\to a=b)\wedge(a=b\to b\in S)]$ ($\Leftrightarrow$ 8)
(34) $j\exists a\in A\,\forall b\in A\,[(b\in S\to a=b)\wedge j(a=b\to b\in S)]$
(35) $j\exists a\in A\,\forall b\in A\,[(b\in S\to a=b)\wedge(a=b\to j(b\in S))]$

(36) $j\exists a\in A\,\forall b\in A\,[j(b\in S\to a=b)\wedge(j(a=b)\to b\in S)]$ discard
(37) $j\exists a\in A\,\forall b\in A\,[j(b\in S\to a=b)\wedge j(j(a=b)\to b\in S)]$ discard
(38) $j\exists a\in A\,\forall b\in A\,[j(b\in S\to a=b)\wedge(a=b\to b\in S)]$
(39) $j\exists a\in A\,\forall b\in A\,[j(b\in S\to a=b)\wedge j(a=b\to b\in S)]$ ($\Leftrightarrow$ 19)
(40) $j\exists a\in A\,\forall b\in A\,[j(b\in S\to a=b)\wedge(a=b\to j(b\in S))]$

(41) $j\exists a\in A\,\forall b\in A\,[(b\in S\to j(a=b))\wedge(j(a=b)\to b\in S)]$ ($\Leftrightarrow$ 11)
(42) $j\exists a\in A\,\forall b\in A\,[(b\in S\to j(a=b))\wedge j(j(a=b)\to b\in S)]$ ($\Leftrightarrow$ 13)
(43) $j\exists a\in A\,\forall b\in A\,[(b\in S\to j(a=b))\wedge(a=b\to b\in S)]$
(44) $j\exists a\in A\,\forall b\in A\,[(b\in S\to j(a=b))\wedge j(a=b\to b\in S)]$
(45) $j\exists a\in A\,\forall b\in A\,[(b\in S\to j(a=b))\wedge(a=b\to j(b\in S))]$ ($\Leftrightarrow$ 20)

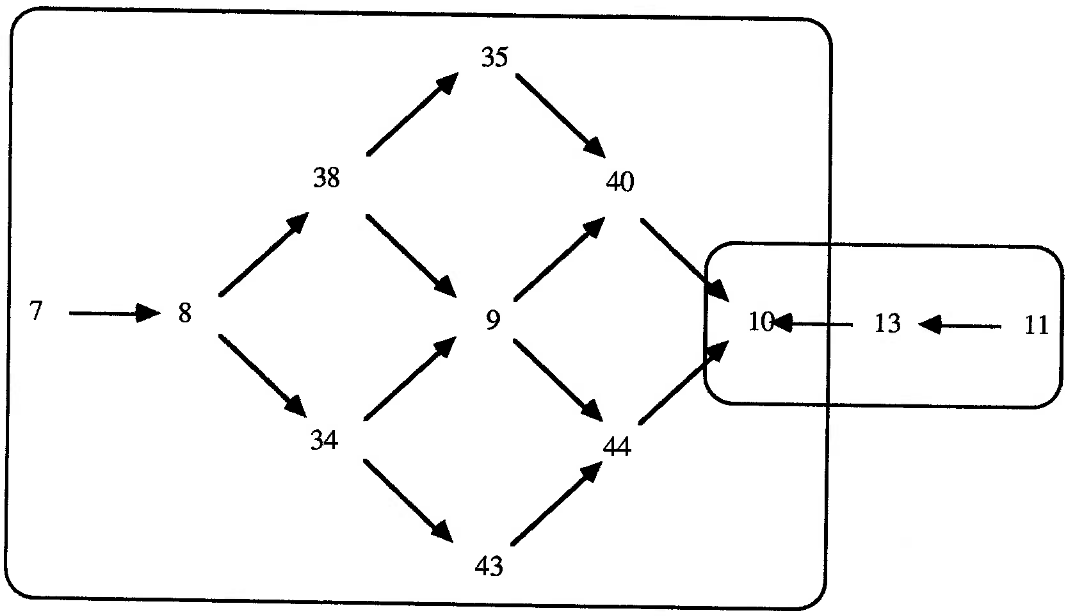
After discarding all notions for which neither $\{a\}$ nor $\{a\}^j$ is a singleton and taking care of the equivalent notions we end up with the following definition:

3.1.3 Definition. Let A be a type. A *(n)-singleton* of A (where n is the number of the line) is a subtype S of A such that

(7) $j\exists a\in A\; a\in S\wedge\forall a,b\in A\,(a\in S\wedge b\in S\to a=b)$
(8) $j\exists a\in A\; a\in S\wedge j\forall a,b\in A\,(a\in S\wedge b\in S\to a=b)$
(9) $j\exists a\in A\; a\in S\wedge\forall a,b\in A\; j(a\in S\wedge b\in S\to a=b)$
(10) $j\exists a\in A\; a\in S\wedge\forall a,b\in A\,(a\in S\wedge b\in S\to j(a=b))$
(11) $j\exists a\in A\,\forall b\in A\,(b\in S\leftrightarrow j(a=b))$
(13) $j\exists a\in A\,\forall b\in A\; j(b\in S\leftrightarrow j(a=b))$
(34) $j\exists a\in A\,\forall b\in A\,[(b\in S\to a=b)\wedge j(a=b\to b\in S)]$
(35) $j\exists a\in A\,\forall b\in A\,[(b\in S\to a=b)\wedge(a=b\to j(b\in S))]$
(38) $j\exists a\in A\,\forall b\in A\,[j(b\in S\to a=b)\wedge(a=b\to b\in S)]$

(40) $j\exists a\in A\ \forall b\in A\ [j(b\in S\to a=b)\wedge(a=b\to j(b\in S))]$
(43) $j\exists a\in A\ \forall b\in A\ [(b\in S\to j(a=b))\wedge(a=b\to b\in S)]$
(44) $j\exists a\in A\ \forall b\in A\ [(b\in S\to j(a=b))\wedge j(a=b\to b\in S)]$

with the following pattern of provable dependency (the two quadrangles correspond to the two classes of singletons distinguished in lemma 3.1.4):



3.1.4 Lemma. Let a be an element of type A.
(i) $\{a\}$ is a (n)-singleton for $n\in\{7,8,38,34,35,9,43,40,44,10\}$.
(ii) $\{a\}^j$ is a (n)-singleton for $n\in\{10,13,11\}$

Proof.
(i), (ii) It suffices to show for an element a of a type A that $\{a\}$ is a (7)-singleton of A and $\{a\}^j$ is a (11)-singleton of A. Which is trivial.

□

3.1.5 Definition.
For a type A we define an equivalence relation depending on the topology:
$a\approx b := j(a=b)$ for $a,b\in A$.

3.1.6 Lemma.
For each type A it holds that $A_{/\approx}$ is separated.

Proof. Suppose for equivalence classes $S,T \in A/_{\approx}$ it holds that $jS=T$. This means that for representatives a and b of respectively S and T we have $ja \approx b$, i.e., $jj(a=b)$ and so $j(a=b)$. But then we get $a \approx b$, and so $S=[a]=[b]=T$.

□

Following [van Dalen] we could define for (n)-singletons a number of other equivalence relations related to $\approx$. After all, for (n)-singletons S,T of A we have

$$S \approx T \leftrightarrow jS=T \leftrightarrow j\forall x \in A\,(x \in S \leftrightarrow x \in T).$$

This formulation invites us to a whole spectrum of other equivalences. We will not pursue this possibility.

Let us finally remark that in the case of a topology that commutes with the implication, like the double negation topology $\neg\neg : \Omega \to \Omega$ and topologies of the form $j^p : \Omega \to \Omega : \omega \mapsto p \to \omega$ there are only three rewritings of $\phi \to \psi$: $j\phi \to \psi$, $\phi \to \psi$ and $\phi \to j\psi$. As a consequence there are less notions of (n)-singletons for such topologies.

3.2 The generalized Grothendieck construction.

We will now describe constructions that produce sheaves from types.
Given a type A, the construction of A^+, i.e., one step of the Grothendieck construction proceeds as follows (cf. [Johnstone 77], page 85):

(i) $\bar{A} := \{S \subseteq A \mid \forall x,y \in A\,(x \in S \wedge y \in S \to x=y)\}$,
(ii) $\hat{A}$ is the closure of the image of A in $\bar{A}$ via $\{\,\} : A \to \bar{A} : a \mapsto \{a\}$,
(iii) $A^+ := \hat{A}/_{\approx}$.

[Johnstone 77] obtains A^+ via an appropriate internal colimit. If we describe $\hat{A}$ internally we get

$$\begin{aligned}\hat{A} &:= \{S \in \bar{A} \mid j\exists a \in A\; S=\{a\}\}\\ &= \{S \subseteq A \mid j\exists a \in A\; a \in S \wedge \forall x,y \in A\,(x \in S \wedge y \in S \to x=y)\}\\ &= \{S \subseteq A \mid S \text{ is a (7)-singleton of } A\}.\end{aligned}$$

We will generalize this construction to arbitrary (n)-singletons:

3.2.1 Definition. *The generalized Grothendieck construction.*

Let A be a type.

(i) For all (n)-singleton notions define:

$$A^{+}{}_{n}:=\{S\subseteq A \mid S \text{ is a (n)-singleton}\}_{/\approx}$$

(ii) For (n)-singleton notions with $n\notin\{11,13\}$ define:

$$\varepsilon_{An}:=A\longrightarrow A^{+}:a\mapsto[\{a\}],$$

(iii) For (n)-singleton notions with $10\leq n\leq 13$ define:

$$\varepsilon_{An}:=A\longrightarrow A^{+}:a\mapsto[\{a\}^{j}],$$

(iv) Finally, for any (n)-singleton notions define:

$$\gamma_{An}:A\longrightarrow A^{++}:=\varepsilon_{A^{+}n}\circ\varepsilon_{An}.$$

If it is clear which notion of singleton we use, we will drop the subscripts n.

We will show that for all notions of (n)-singleton of definition (3.1.3) the generalized Grothendieck construction applied to a type A results in an associated sheaf.

3.2.2 Theorem.; For any notion of (n)-singleton and all types A it holds that:

(i) A^{+} is separated for any n.
(ii) If A is separated, then A^{+} is a sheaf.
(iii) If A is sheaf, then A^{+} is isomorphic to A.
(iv) A^{++} is a sheaf.

Proof.

(i) Apply lemma (3.1.4) to $\hat{A}$. Note that this does not depend on the chosen notion of (n)-singleton.

(ii) Suppose A is separated. In order to show that A^{+} is a sheaf we suppose that for $X\subseteq A^{+}$ we have $j\exists!S\in A^{+}\, S\in X$.
Define $T:=\{x\in A \mid \exists[S]\in X\ \exists S\in[S]\ x\in S\}$.
For each notion of (n)-singleton we claim that :
(a) T is a (n)-singleton of A,
(b) $j[T]\in X$,
(c) $jR_1\in X\wedge jR_2\in X\rightarrow R_1=R_2$.
From this claim follows $\exists!S\in A^{+}\ jS\in X$, which enables us to conclude that A^{+} is a sheaf.

Proof of (a). We can not give a uniform proof for all (n)-singletons that T is a (n)-singleton. It is easy to show separately that T is a (11)-singleton if one started the construction out with (11)-singletons. Idem for n=13. The other cases can be proved in one big sweep. We show for $n\notin\{11,13\}$ that T is a (7)-singleton of A,

so that we can conclude that T is a (n)-singleton. We split this proof in two parts, first we prove the existential clause, then the unicity clause for T to be a (7)-singleton.
In order to show the existential part it suffices to prove that

$\exists! S\in A^+ \; S\in X \rightarrow j\exists a\in A \; a\in T.$

Assume there is a unique $[S]\in A^+$ such that $[S]\in X$. Let the (n)-singleton S be a representative in [S]. Then for T holds $j\exists x\in A \; x\in T$, since $a\in S$ implies $a\in T$ and S is a (n)-singleton implies $j\exists x\in A \; x\in S$.
To prove unicity suppose that $a\in T$ and $b\in T$. Then $[\{a\}]\in X$ and $[\{b\}]\in X$ in case $n\notin\{11,13\}$. Hence by the initial assumption $j\exists! S\in A^+ \; S\in X$ we get $j[\{a\}]=[\{b\}]$, and so $j(a=b)$. Using singletons of the form $\{a\}^j$ the same argument applies for $n\in\{11,13\}$. Hence $j(a=b)$ for arbitrary n. A is separated, therefore $a=b$.
We conclude that $\forall a,b\in A \; (ja\in T \wedge jb\in T \rightarrow a=b)$ for T.

Proof of (b).
In order to prove (b) it is sufficient to prove that $\exists! S\in A^+ \; S\in X \rightarrow [T]\in X$. So, let us assume again that the (n)-singleton S is a representative of the unique [S] in A^+. From the definition of T we get $S\subseteq T$. If $a\in T$, and also $a\in S'\in[S]$ for some singleton S'. But $T\approx\{a\}\approx S'$ for $n\notin\{11,13\}$, and $T\approx\{a\}^j\approx S'$ in case $n\in\{11,13\}$. Both imply $S\approx T$.

Proof of (c). Trivial: if the equivalence classes $[R_1]$ and $[R_2]$ are members of $X\subseteq A^+$, then by our initial assumption $j\exists! S\in A^+ \; S\in X$ we get $j[R_1]=[R_2]$. By (a) we already know that A^+ is separated whence $[R_1]=[R_2]$.

(End proof of (ii))

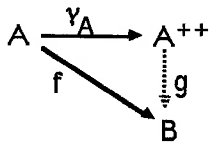
(iii) Let A be a sheaf.
For $n\notin\{11,13\}$ we will prove that $\varepsilon_A: A\rightarrow A^+ : a\mapsto [\{a\}]$ is an isomorphism.
Assume for $a,b\in A$ that $\varepsilon_A(a)=\varepsilon_A(b)$. Then $[\{a\}]=[\{b\}]$, and $j\{a\}=\{b\}$. Hence $j(a=b)$. Since sheaves are separated, we get $a=b$. Thus we see that ε_A is a mono.
Let S be a representative of an element [S] in A^+. S is a (n)-singleton. Hence together with the separatedness of A we get $j\exists! x\in A^+ \; x\in S$. It follows from the sheaf property of A that there is a unique $a\in A$ such that $ja\in S$. Hence $\{a\}\approx S$, and so $\varepsilon_A(a)=[S]$. Thus we have shown that ε_A is epic.
For $n\in\{11,13\}$ one proves similarly that $\varepsilon_A: A\rightarrow A^+ : a\mapsto [\{a\}^j]$ is an isomorphism.

(iv) It follows from the previous results that for each type A and all notions of singleton of definition (1.3.3) that A^{++} is a sheaf.

□

3.2.3 Theorem. For each notion of (n)-singleton and all types A it holds that:

$\gamma_A: A \to A^{++}$ is universal among functions from A into sheaves.
This means that for each sheaf B and function $f: A \to B$ there is a unique function $g: A^{++} \to B$ such that the following diagram commutes:



Proof.
It is sufficient to consider only the case $n \in \{7,10,11\}$. If for these three notions of singleton we have the required universal property, then for a arbitrary type A the three corresponding sheaves $(A^{++})_7$, $(A^{++})_{10}$ and $(A^{++})_{11}$ must be isomorphic. As the sheaves for the other notions are squeezed in between, they are also isomorphic to, say, $(A^{++})_{10}$.

To prove the universality of γ_A it suffices to show the universality of ε_A among morphisms into a sheaf.
Consider a sheaf B and function $f: A \to B$.
For $n \in \{7,10,11\}$ we define $T = \{f(a) \in B \mid \varepsilon_A(a) = S\}$ for any element S in $(A^+)_n$. This T is again a (n)-singleton of B. Therefor, since B is a sheaf, we have a unique element $b_S \in B$ such that $jb \in T$.
Now define $g: A^+ \to B: S \mapsto b_S$. By the construction we get $\forall a \in A\, f(a) = g(\varepsilon_A(a))$. This function g is unique with respect to this property, for suppose we have also $h: A^+ \to B$ such that $f = h \circ \varepsilon_A$. Let S be a representative of an element [S] in A^+. For $a \in S$ we clearly have $\varepsilon_A(a) = [S]$, and hence $h([S]) = h(\varepsilon_A(a)) = g(\varepsilon_A(a)) = g([S])$. Since S is a singleton we have only $j \exists a \in A\; a \in S$. Now we get $jh([S]) = g([S])$. However B is a sheaf, whence separated. We get $h([S]) = g([S])$. Hence $h = g$.

□

3.3 The Lawvere-Tierney construction.

Lawvere-Tierney's construction (cf. [Gray], [Johnstone 77] or [Freyd 72]) takes the closure of the image of A in $\Omega_j{}^A$ via the function $\{\,\}^j: A \to \Omega_j{}^A: a \mapsto \{a\}^j$). Now recall that $\Omega_j{}^A = \{B \subseteq A \mid B \text{ is } j\text{-closed}\}$ (cf. proof of (2.3.6). If we formulate this in type theory we get via lemma (2.3.2):

$$\{S \subseteq A \mid S = S^j \wedge j \exists a \in A\; S = \{a\}^j\}$$

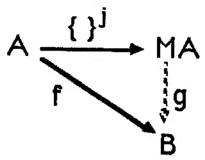
But for $S\subseteq A$ we have that

$$j\exists a\in A\ S=\{a\}^j \leftrightarrow j\exists a\in A\ \forall b\in A\ (b\in S\leftrightarrow j(a=b))$$

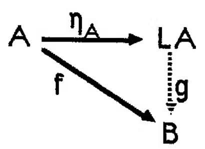
Thus we see that Lawvere-Tierney's construction is based on our notion of (11)-singleton.

3.3.1 Definition. *The Lawvere-Tierney construction*.
For each type A define (cf. exercise (3.4) in [Johnstone 77]):
(i) $MA:=\{S\subseteq A\mid \exists a\in A\ S=\{a\}^j\}$ $(=\{S\subseteq A\mid S=S^j\wedge \exists a\in A\ S=\{a\}^j\})$
(ii) $LA:=\{S\subseteq A\mid S=S^j\wedge S$ is a (11)-singleton$\}$.
(iii) $\eta_A{:=}A\rightarrow LA{:}a\mapsto[\{a\}^j]$.

3.3.2 Theorem. For all types A it holds that:
(i) MA is separated, and $\{\ \}^j{:}A\rightarrow MA$ is universal among functions from A into separated types, i.e., for each separated type B and function $f{:}A\rightarrow B$ there is a unique function $g{:}MA\rightarrow B$ such that the following diagram commutes:



(ii) LA is a sheaf, and $\eta_A{:}A\rightarrow LA$ is universal among functions from A into sheaves, i.e., for each sheaf B and function $f{:}A\rightarrow B$ there is a unique function $g{:}LA\rightarrow B$ such that the following diagram commutes:



Proof.
(i) For $S,T\in MA$ suppose $jS=T$. Then for some $a,b\in A$ we have $S=\{c\in A\mid ja=c\}$ and $T=\{c\in A\mid jb=c\}$. By assumption $j(b\in S)$ and $ja\in T$, i.e., $j(a=b)$. But this implies that $S=T$. Hence MA is separated.

Next, let $f{:}A\rightarrow B$ be a function into a separated type. If $S\in MA$, then there is $a\in A$ such that $S=\{c\in A\mid ja=c\}$. Consider $f(a)$. If for some $b\in A$ we also have

$S=\{c\in A \mid jb=c\}$, then $j(a=b)$. Hence, $jf(a)=f(b)$, and by separatedness of B it follows that $f(a)=f(b)$. Thus we see that $G=\{(S,d)\in MA\times B \mid \exists a\in S\ f(a)=d\}$ is the graph of some function, let us say $g: MA \to A$. Clearly, this $g: MA \to B$ is the only function such that for $a\in A$ we have $f(a)=g(\{a\}j)$.

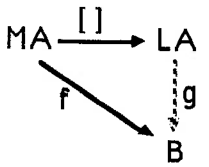
(ii) First we show that LA is separated. So let $S,T \in LA$ such that $jS=T$. Since S and T are j-closed we can argue as follows:

$$\begin{aligned} jS=T &\to j\forall a\in A\,(a\in S \leftrightarrow a\in T) \\ &\to \forall a\in A\,(ja\in S \leftrightarrow ja\in T) \\ &\to \forall a\in A\,(a\in S \leftrightarrow a\in T) \\ &\to S=T. \end{aligned}$$

Hence LA is separated.
Secondly, to prove that LA is a sheaf, suppose for $X\subseteq LA$ that $j\exists! R\in LA\ R\in X$. Define $S:=\{a\in A \mid j\eta_A(a)\in X\}$. We will show S is the unique element in LA such that $jS\in X$. Assume for a moment that that there is a unique R in LA such that $R\in X$. Then there is $a\in A$ such that $R=\{c\in A \mid ja=c\}$. Hence, $jR=\{a\}$. Therefore $j\eta_A(a)\in X$, and $jR=S$. Now we have shown that $\exists! R\in LA\ R\in X$ implies $jS\in X$, and so $j\exists! R\in LA\ R\in X \to \exists R\in LA\ jR\in X$.
Unicity is also simple:

$$\begin{aligned} j\exists! R\in LA\ R\in X &\to j\forall R,S\in LA\ [R,S\in X \to R=S] \\ &\to \forall R,S\in LA\ [R,S\in X \to jR=S] \\ &\to \forall R,S\in LA\ [R,S\in X \to R=S] \end{aligned}$$

Thirdly, to prove the universality of η_A it suffices to show because of the first part of the theorem that for each separated type B and each $f: MA \to B$ there is a unique function $g: LA \to B$ such that the following diagram commutes:



Define $G=\{(S,b)\in LA\times B \mid \exists T\in S\ f(T)=b\}$. Suppose $R,T\in S$. Then $jR=T$, and $jf(R)=f(T)$. Since the sheaf B is separated we get $f(R)=f(T)$. Thus G is the graph of function $g: LA \to B$ such that $g\circ[\ ]=f$. Suppose for $h: LA \to B$ we also have $h\circ[\ ]=f$. Let $S\in MA$ be a representative of $[S]$ in LA. Then $g([S])=f(S)=h([S])$. I.e., $h=g$.

□

3.4 The generalized Lawvere-Tierney construction.

In the PhD thesis of [Rosolini] one finds the following construction for associated sheaves, that according to Rosolini belongs to anonymous folklore:

$$Q(A)=\{S\subseteq A \mid \forall x,y\in S\ [x\in S\wedge y\in S\rightarrow jx=y]\wedge j\exists x\in A\ x\in S\}_{/\approx}.$$

We recognize it as the single step of the Grothendieck construction performed with (10)-singletons. At first sight it may seem unrelated to the Lawvere-Tierney construction. Observe however that $LA=(A^{+})_{11}$, because for (11)-singletons it holds that $jS=S^{j}$, and so:

$$\{S\subseteq A \mid S=S^{j}\wedge S \text{ is a (11)-singleton}\}=\{S\subseteq A \mid S \text{ is a (11)-singleton}\}_{/\approx}.$$

If we then prove as above that LA is a sheaf, then it is a property of the Grothendieck construction that $(A^{+})_{xi}$ is isomorphic to $(A^{++})_{11}$. Therefore LA is an associated sheaf to A as it inherits the universal properties of $(A^{++})_{11}$.

So the Lawvere-Tierney-construction can be described as a single step of the Grothendieck construction performed with (11)-singletons.

Thus we are led to the following definition:

3.4.1 Definition. *The generalized Lawvere-Tierney construction.*
For (n)-singletons with $n\in\{10,11,13\}$ define for each type A:

$$R_n(A)=\{S\subseteq A \mid S \text{ is a (n)-singleton}\}_{/\approx}.$$

3.4.2 Theorem. For all (n)-singleton notions with $n\in\{10,11,13\}$ and all types A it holds that $R_n(A)$ is a sheaf isomorphic to $L(A)$.

Proof. One can prove this in different ways. One is to show that R_x is a sheaf. Another is to observe that $L(A)=R_{xi}(A)\subseteq R_{xiii}(A)\subseteq R_x(A)$. Now order to show that for each $n\in\{10,11,13\}$ $R_n(A)$ are sheaves isomorphic to $L(A)$, it suffices to prove that $R_x(A)$ is isomorphic to $L(A)$:
Define for every type A:

$$rl: R_x(A)\longrightarrow L(A): [S]\mapsto \{a\in A \mid \{a\}^{j}\approx S\}$$
$$lr: L(A)\longrightarrow R_x(A): S\mapsto [S]$$

It is straightforward to check that rl and lr are well-defined and each others inverse.

□

3.5 The Barr-Wells variant.

[Barr & Wells] construct associated sheaves in a one step process:
(i) given a type A, construct $B := A_{/\approx}$ (where $a \approx b \leftrightarrow j(a=b)$)
(ii) let C be the closure of B in P(B) via $\{\,\}: B \to P(B)$
(iii) finally, $BW(A) := C_{/\approx}$ (where $S \approx T \leftrightarrow jS=T$).

If we reformulate this in type theory we get the following:

$$\begin{aligned} BW(A) &:= \{S \subseteq A_{/\approx} \mid j\exists T \in A_{/\approx}\, \{T\}=S\}_{/\approx} \\ &= \{S \subseteq A_{/\approx} \mid j\exists! T \in A_{/\approx}\, T \in S\}_{/\approx} \\ &= \{S \subseteq A_{/\approx} \mid S \text{ is (8)-singleton of } A_{/\approx}\}_{/\approx}. \end{aligned}$$

3.5.1 Definition. *The generalized Barr-Wells construction.*
For any notion of (n)-singletons define for each type A:
(i) $BW_n(A) := \{S \subseteq A_{/\approx} \mid S \text{ is (n)-singleton of } A_{/\approx}\}_{/\approx}$,
(ii) $\gamma_{nA} := A \to BW_n(A): a \mapsto [\{[\{a\}]\}]$.

This definition is only seemingly generalized: on a separated type like $A_{/\approx}$ the notions of (n)-singleton are all identical.

3.5.2 Lemma. BW(A) and L(A) are isomorphic for any type A.

Proof. For each type A define:

$$bwr: BW(A) \to L(A): [S] \mapsto \{a \in A \mid [a] \in S\}$$
$$rbw: L(A) \to BW(A): [S] \mapsto [\{[a] \in A_{/\approx} \mid a \in S\}]$$

It is straightforward to show that these functions are each others inverses.

□

We will denote with L any of the constructions that do give associated sheaves.

3.6 Preservation properties of L

We will end this chapter with a type theoretical proof that the associated sheaf construction preserves (external) finite limits and exponents.

3.6.1 Theorem.
(i) All subobjects of $\mathbb{1}$ are sheaves.

(ii) For each sheaf A there is exactly one function $A \to \mathbb{1}$.
(iii) If P is the pullback of $f: A \to B$ and $g: C \to B$, then LP is the pullback of $Lf: LA \to LB$ and $Lg: LC \to LB$.

Proof.
(i) is straightforward and (ii) follows from (i).

(iii) The pullback P of $f: A \to B$ and $g: C \to B$ is $\{\langle a,c\rangle \in A \times C \mid f(a)=g(c)\}$ (up to isomorphism). Similarly, the type $Q := \{\langle S,T\rangle \in LA \times LC \mid (Lf)(S)=(Lg)(T)\}$ is the pullback of $Lf: LA \to LB$ and $Lg: LC \to LB$. One easily checks that Q is isomorphic with LP.

□

Chapter 4

Algebraic Theories and Subalgebra Classifiers

In this chapter we will work with internal finitary algebraic theories. Given a topos $\mathbb{E}$ one can consider the category $\mathbb{E}_{\mathbb{T}}$ of $\mathbb{T}$-algebras in $\mathbb{E}$. In the case $\mathbb{E} = \mathcal{S}ets$ Johnstone has syntactically characterized the (possibly infinitary) algebraic theories $\mathbb{T}$ such that the variety of $\mathbb{T}$-algebras $\mathcal{S}ets_{\mathbb{T}}$ contains a subobject classifier (cf. [Johnstone 85] and [Johnstone a]). For $\mathbb{E} = \mathcal{S}h(\mathbb{H})$ of sheaves over a locale $\mathbb{H}$ [Borceux and van den Bossche] constructed for commutative finitary algebras $\mathbb{T}$ an object $\Omega_{\mathbb{T}}$ in $\mathcal{S}h(\mathbb{H})$ that acts as an subobject classifier for $\mathcal{S}h(\mathbb{H})_{\mathbb{T}}$ with respect to a certain class of characteristic functions, although $\Omega_{\mathbb{T}}$ itself does not need to be a $\mathbb{T}$-algebra.

Given any algebraic theory $\mathbb{T}$ we will define a class of characteristic functions with codomain $\Omega_{\mathbb{T}}$ such that $\mathbb{1} \rightarrow \Omega_{\mathbb{T}}$ acts as a *subobject classifier* for $\mathbb{T}$-algebras even if $\Omega_{\mathbb{T}}$ and $\mathbb{1}$ are no $\mathbb{T}$-algebras themselves. A better name for such a phenomenon is *subalgebra classifier*. For commutative theories we show that our notion of characteristic morphism and the definition of characteristic morphism as used by Borceux and van den Bossche come down to the same.

We will present this within the constructive setting of type theory. Hence our results apply to any arbitrary elementary topos $\mathbb{E}$.

4.1 Internal finitary algebras

Let $\mathbb{T}$ be some (finitely presented) finitary algebraic theory, that is the language corresponding with the algebraic theory $\mathbb{T}$ has one type, a (finite) set of function symbols of finite arity, and a (finite) set of equations of $\mathbb{T}$. Given a topos $\mathbb{E}$ we define $\mathbb{E}_{\mathbb{T}}$ to be the subcategory of models in $\mathbb{E}$ of $\mathbb{T}$. It is an old result of [Lesaffre] that there exist a free functor $\mathbb{E} \rightarrow \mathbb{E}_{\mathbb{T}}$ (cf. [Johnstone 77]). With respect to toposes this is the external way of looking at algebras.

One can also approach the notion of algebraic theory internally in the style of universal algebra (cf. e.g. [Jacobson]). Suppose we work inside a type theory **H** containing the axioms of Peano.

4.1.1 Definition The data in **H** that define an internal algebra $\mathbb{T}$ consist of two things:

(i) A type F of function symbols, together with a morphism $\delta : F \rightarrow \mathbb{N}$ that assigns an arity to each function in F. Starting from this type, we can construct an type **LANG**(F,δ) containing all well-formed terms and formulas of the first order language of $\mathbb{T}$. Details of this construction resemble the construction of free monoids (1.3.8) and require the presence of a natural number object.

(ii) A subtype **AX** of **LANG**(F,δ) containing the equational axioms the algebra $\mathbb{T}$ has to satisfy.

With this information it is possible to state internally that a $\mathbb{T}$-algebra, is a pair $\langle A, F_A \rangle$, where A is an type, and F_A a type containing the algebraic operations on A.

4.1.2 Free algebras

Let $\mathbb{T}$ be some finitary algebraic theory. We will construct the free algebra on an arbitrary type X in two steps. First we construct the type WF_X of well formed words of elements of X and the operation symbols of the algebra (cf. for instance [Jacobson]). Then we take quotients dictated by the equational axioms of the algebra.

The first part of this task is unproblematic, though somewhat tedious to write out in full detail. The second part is of a bit more interest. With each element α of **Ax** there corresponds a congruence relation R_α on WF_X. If we are able to define the smallest congruence $R \subseteq WF_X \times WF_X$ containing all these R_α, then we can define the free universal $\mathbb{T}$-algebra generated by X to be the quotient $F_X = (WF_X)/R$. This smallest congruence R can be constructed by similar methods as in (1.6.5):

(i) define $S_0 := \bigcup\{R_\alpha \mid \alpha \in \mathbf{Ax}\}$, this is a reflexive and symmetric relation.

(ii) for $S \subseteq WF_X \times WF_X$, define
$S^+ := \{(v,w) \in WF_X \times WF_X \mid \exists u \in WF_X (v,u) \in S \wedge (u,w) \in S\}$

(iii) $R := \bigcup \bigcap \{\Sigma \subseteq P(WF_X \times WF_X) \mid S_0 \in \Sigma \wedge \forall S \in \Sigma\, [S \text{ is a relation on } FW_X] \wedge \forall S \in \Sigma\, S^+ \in \Sigma\}$.

It is straightforward to see that FW_X/R has the universal property which identifies it up to isomorphism as a free universal algebra on X.

Note that in the above construction of free algebra we did not use the condition that the algebraic theory is finitely presented, so for free we get a generalization of the result of [Lesaffre], provided that we can internalize an external finitary algebraic theory in a topos. This is not difficult, as long as the topos has sufficiently many coproducts of $\mathbb{1}$: from the externally given object and morphisms plus the knowledge of the domains of the functions we can construct an internal pair $\langle F,\delta\rangle$ carrying essentially the same information provided that the topos $\mathbb{E}$ one works in has coproducts of $\mathbb{1}$ of sufficient cardinality.

4.1.4 Notations.

(i) $A\leq B$ denotes that A is a subalgebra of the algebra B.

(ii) Let Fn denote the *free algebra generated by n elements*, e.g., $\{*_1,\ldots,*_n\}$.

(iii) The *free algebra generated by a subset* $\{*\in\mathbb{1}\mid\omega\}$ of $\mathbb{1}$, for $\omega\in\Omega$, will be denoted by $F(\mathbb{1}\restriction\omega)$, and the free algebra generated by the empty set, $F(\mathbb{1}\restriction\perp)$ denoted as $F\varnothing$.

(iv) With an element a of an algebra A corresponds an *unary* algebraic operation

$F\mathbb{1}\longrightarrow A: v\mapsto v[*:=a]$ (in short $v(a)$)

Similarly with a word w in Fn corresponds an *unary* operation

$F\mathbb{1}\longrightarrow Fn: v\mapsto v(w)$.

We will denote word and corresponding operation by the same letter w.
Hence for $w\in Fn$ and $R\subseteq Fn$ we can use the notation $w^{-1}(R)$:

$w^{-1}(R):=\{v\in F\mathbb{1}\mid v(w)\in R\}$.

(v) There are several equivalent ways of defining the join of an indexed set of subalgebras $(A_i)_{i\in I}$ of a given algebra C. The first one is a formulation that does not make use of natural numbers:

(a) $\Sigma^1_{i\in I}A_i:=\bigcap\{B\subseteq C\mid B \text{ is an algebra} \wedge \forall i\in I\ A_i\subseteq B\}$

(b) $\Sigma^2_{i\in I}A_i:=\{w\in C\mid \exists n\in\mathbb{N}\ \exists v\in Fn\ \exists a_1,\ldots,a_n\in\bigcup_{i\in I}A_i\ w=v[*_1:=a_1,\ldots,*_n:=a_n]\}$

Clearly $\Sigma^1_{i\in I}A_i\subseteq\Sigma^2_{i\in I}A_i$, since $\Sigma^2_{i\in I}A_i$ contains all the A_i, and $\Sigma^2_{i\in I}A_i$ is an subalgebra of C. On the other hand if we have $w\in\Sigma^2_{i\in I}A_i$ then w is of the form $v[*_1:=a_1,\ldots,*_n:=a_n]$ for a finite number of elements $a_1,\ldots,a_n$ in $\bigcup_{i\in I}A_i$. (Note that we make use of the fact that we work with finitary algebras.) Now w is an element of any B containing all A_i. Hence $w\in\Sigma^1_{i\in I}A_i$. It follows that $\Sigma^1_{i\in I}A_i=\Sigma^2_{i\in I}A_i$. From now on we will use the notation $\Sigma_{i\in I}A_i$ for the join of the algebras A_i.

(vi) Consider the canonical injections $\sigma_i: F\mathbb{1} \to Fn: w \mapsto w[* := *_i]$. Let $A_1,...,A_n$ be subalgebras of $F\mathbb{1}$. Define $A_i*_i := \{w(*_i) \in F\{*_i\} \mid w \in A_i\}$ for $1 \leq i \leq n$. Then we denote the join $\Sigma_{1\leq i\leq n}A_i*_i$ in Fn by $A_1 \oplus ... \oplus A_n$.

(vii) We will sometimes make a distinction between *operators*, the basic functions of the algebraic theory, and *operations*, the compound functions, that can be build from the operators.

4.2 A generalized type of truth values

We will need a type of *algebraic* truth values to define a notion of topology in this algebraic context. Recall that Ω is isomorphic with the type of subtypes of $\mathbb{1}$. We replace $\mathbb{1}$ by the free algebra $F\mathbb{1}$ generated by $\mathbb{1}$ and consider subalgebras instead of subtypes.

4.2.1 Definition.
$\Omega_{\mathbb{T}} := \{A \subseteq F\mathbb{1} \mid A \text{ is a subalgebra of } F\mathbb{1}\}$

In order to compare Ω with $\Omega_{\mathbb{T}}$ we define $e: \Omega \to \Omega_{\mathbb{T}}: \omega \mapsto F(\mathbb{1}\restriction\omega)$. It is tempting to think that this is an embedding. The degenerate algebra that consists of one constant c and one axiom $x=c$ provides a counterexample, because then we have $F(\mathbb{1}\restriction\omega) = F(\mathbb{1}) = \mathbb{1}$ for all $\omega \in \Omega$.

It is natural to ask whether $\Omega_{\mathbb{T}}$ has the following three desirable properties.
(i) Is $\Omega_{\mathbb{T}}$ a complete Heyting algebra?
(ii) Is $\Omega_{\mathbb{T}}$ an ($\mathbb{T}$-)algebra?
(iii) Is $\mathbb{1} \to \Omega_{\mathbb{T}}: * \mapsto F\mathbb{1}$ a subalgebra classifier for algebras, even if $\Omega_{\mathbb{T}}$ itself is perhaps not an algebra (remember [Borceux and van den Bossche])?

[Johnstone a] provides us with the tools to tackle the last of these questions. He imposes also the following structure on $\Omega_{\mathbb{T}}$:

4.2.2 Definition. For an operator p of $\mathbb{T}$ define:
$p: \Omega_{\mathbb{T}} := \{w \in F\mathbb{1} \mid w(p) \in F0\} = F\mathbb{1}$, if p is a constant

$p:(\Omega_{\mathbb{T}})^n \to \Omega_{\mathbb{T}} := (A_1,\ldots,A_n) \mapsto \{w \mid w(p(*_1,\ldots,*_n)) \in \oplus_{1\leq i\leq n} A_i\}$ $(= p^{-1}(\oplus_{1\leq i\leq n} A_i)$, cf. (4.1.4)), if p is n-ary, $n\geq 1$.

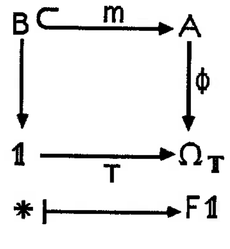
Even if we do not know that $\Omega_{\mathbb{T}}$ is a $\mathbb{T}$-algebra we can still check that $\Omega_{\mathbb{T}}$ acts as a subalgebra classifier with respect to the right class of characteristic morphisms. [Borceux and van den Bossche] have defined another class of characteristic morphisms, with respect to which $\Omega_{\mathbb{T}}$ acts as a subalgebra classifier when the algebraic theory $\mathbb{T}$ satisfies an extra commutativity condition. With help of the extra structure on $\Omega_{\mathbb{T}}$ provided by [Johnstone a] we will define a class of characteristic morphisms such that $\Omega_{\mathbb{T}}$ behaves as a subobject classifier for *any* algebraic theory $\mathbb{T}$.

4.2.3 Definition. Let A be an algebra.
A function $\phi: A \to \Omega_{\mathbb{T}}$ is called *characteristic* if

(i) for any $n \in \mathbb{N}$, all n-ary algebraic operators p and all $a_1,\ldots,a_n \in A$
$p(\phi(a_1),\ldots,\phi(a_n)) \subseteq \phi(p(a_1,\ldots,a_n))$

(ii) $* \in \phi(u(a)) \leftrightarrow u(*) \in \phi(a)$ for all unary operators u and all $a \in A$,

4.2.4 Theorem.
(i) There is a one-one correspondence between subalgebras $m: B \rightarrowtail A$ and characteristic functions $A \to \Omega_{\mathbb{T}}$.
(ii) $\Omega_{\mathbb{T}}$ together with $T: \mathbb{1} \to F\mathbb{1}: * \mapsto \Omega_{\mathbb{T}}$ *acts as subalgebra classifier with respect to characteristic functions* into $\Omega_{\mathbb{T}}$. That is, for each $\mathbb{T}$-monomorphism $m: B \rightarrowtail A$ there is one and only one characteristic function $\phi: A \to \Omega_{\mathbb{T}}$ such that the following diagram is a pullback:



Moreover, the pullback preserves the correspondence of (i) between subalgebras and characteristic functions.

Proof.
(i) Let $m: B \rightarrowtail A$ be a monomorphism.
Define $\phi_B: A \to \Omega_{\mathbb{T}}$ by $a \mapsto \{w \in F\mathbb{1} \mid \exists b \in B\ w(a) = m(b)\}$. ϕ_B is well-defined, as

each $\phi_B(a)$ is a subalgebra of $F\mathbb{1}$. We will show that this ϕ_B is a characteristic function.

First, for p an n-ary operator and for $a_1,...,a_n \in A$ suppose that

$$w \in p(\phi_B(a_1),...,\phi_B(a_n)).$$

That is

$$w(p(*_1,...,*_n)) \in \oplus_{1 \le i \le n} \phi_B(a_i). \qquad \text{(recall definition of } \oplus \text{ in 4.1.4.iv)}$$

Hence $w(p(*_1,...,*_n))$ can be written as $u(v_1(*_{\alpha(1)}),...,v_n(*_{\alpha(n)}))$, where $u \in Fn$, α is a permutation on $\{1,...,n\}$ and $v_i \in \phi_B(a_i)$ for $1 \le i \le n$. For each a_i there is a $b_i \in B$ such that $m(b_i)=v_i(a_i)$. If we substitute a_i for $*_i$, then we get

$$\begin{aligned} w(p(a_1,...,a_n)) &= u(v_1(a_{\alpha(1)}),...,v_n(a_{\alpha(n)})) \\ &= u(m(b_{\alpha(1)}),...,m(b_{\alpha(n)})) \\ &= m(u(b_{\alpha(1)},...,b_{\alpha(n)})). \end{aligned}$$

Since $u(b_{\alpha(1)},...,b_{\alpha(n)}) \in B$ we get $w \in \phi_B(p(a_1,...,a_n))$.

Secondly, for all unary operators u and all $a \in A$,

$$\begin{aligned} * \in \phi_B(u(a)) &\leftrightarrow \exists b \in B\ m(b)=u(a) \\ &\leftrightarrow u(*) \in \phi_B(a). \end{aligned}$$

Thus we see that $\phi_B : A \rightarrow \Omega_{\mathbf{T}}$ is a characteristic function.

Next let $\phi : A \rightarrow \Omega_{\mathbf{T}}$ be a characteristic function.
Define $B_\phi = \{b \in A \mid \phi(b)=F(\mathbb{1})\}$. We will show that B_ϕ is a subalgebra of A.
It is trivial to see that B_ϕ contains all constants.
For $b_1,...,b_n \in B_\phi$ and n-ary algebraic operator p on A it holds that

$$\begin{aligned} F\mathbb{1} &= p(F\mathbb{1},...,F\mathbb{1}) \\ &= p(\phi(b_1),...,\phi(b_n)) \\ &\subseteq \phi(p(b_1,...,b_n)) \\ &\subseteq F\mathbb{1}. \end{aligned}$$

It follows that $\phi(p(b_1,...,b_n))=F\mathbb{1}$. Thus we see that $p(b_1,...,b_n) \in B_\phi$. Hence B_ϕ is an algebra.

We check that $B \mapsto \phi_B$ and $\phi \mapsto B_\phi$ are each others inverse by the following calculations:

$$\begin{aligned} \phi(B_\phi)(a) &= \{w \in F(\mathbb{1}) \mid \exists b \in A\ \ \phi(b)=F(\mathbb{1}) \wedge w(a)=b\} \\ &= \{w \in F(\mathbb{1}) \mid \phi(w(a))=F(\mathbb{1})\} \\ &= \{w \in F(\mathbb{1}) \mid * \in \phi(w(a))\} \\ &= \{w \in F(\mathbb{1}) \mid w \in \phi(a)\} \qquad \text{(4.2.3.ii)} \\ &= \phi(a). \end{aligned}$$

And similar:
$$\begin{aligned} B(\phi_B) &= \{a \in A \mid \phi_B(a)=F(\mathbb{1})\} \\ &= \{a \in A \mid \exists b \in B\ \ a=m(b)\} \end{aligned}$$

$\approx B$

(ii) Existence and the pullback property follow from the relationship of $m{:}B \rightarrowtail A$ and ϕ_B. The uniqueness of ϕ among the characteristic morphisms follows from the properties of characteristic functions:

Suppose we have a characteristic morphism $f{:}A \to \Omega_T$ that classifies $B \rightarrowtail A$. Then

$\forall a \in A[\phi_B(b)=F\mathbb{1} \leftrightarrow f(b)=F\mathbb{1}]$

For $w \in F\mathbb{1}$ we can argue

$$\begin{aligned} w\in\phi_B(b) &\leftrightarrow *\in\phi_B(w(b)) \\ &\leftrightarrow \phi_B(w(b))=F\mathbb{1} \\ &\leftrightarrow f(w(b))=F\mathbb{1} \\ &\leftrightarrow *\in f(w(b)) \\ &\leftrightarrow w\in f(b). \end{aligned}$$

Hence, we see that $f=\phi_B$.

□

In [Borceux and van den Bossche] the foregoing theorem is proved for a restricted class of algebras and a different class of characteristic functions. However, if we put the restriction on the algebras, we can show that our characteristic functions satisfy the conditions of Borceux and van den Bossche.

4.2.5 Theorem. For algebraic theories such that unary operations commute with operators of arbitrary arity, a function $\phi{:}A \to \Omega_T$ is characteristic if and only if $\phi{:}A \to \Omega_T$ satisfies the conditions of [Borceux and van den Bossche]:

(i) $*\in\phi(u(a)) \leftrightarrow u(*)\in\phi(a)$ for all unary operators u and all $a \in A$

(ii) for all $n \in \mathbb{N}$, all n-ary algebraic operators p and all $a_1,\dots,a_n \in A$
$\phi(a_1)\cap\dots\cap\phi(a_n) \subseteq \phi p(a_1,\dots,a_n)$.

Proof. First we make two claims.

Claims.

(a) for $A_1,\dots,A_n \in \Omega_T$ and p an n-ary operator we have:
$A_1\cap\dots\cap A_n \subseteq \{w \in F\mathbb{1} \mid p(w(*_1),\dots,w(*_n)) \subseteq \oplus_{1\le i\le n}A_i\}$.

(b) for a function $\phi{:}A \to \Omega_T$ satisfying conditions (i) and (ii) of the theorem, we have

$$\phi(a_1)\cap\dots\cap\phi(a_n) \subseteq \phi u(a_1,\dots,a_n)$$

for all $n \in \mathbb{N}$, all $u \in Fn$ and all $a_1,\dots,a_n \in A$.

The proof of the claims is not difficult: (i) is trivial and (ii) follows by an easy induction on the structure of u.

Now, if $\phi: A \rightarrow \Omega_{\mathbb{T}}$ is a characteristic function, then, assuming that unary operations commute with n-ary operators, $\phi: A \rightarrow \Omega_{\mathbb{T}}$ satisfies the conditions (i) and (ii) of the theorem. We show (ii).
For all $a_1,\ldots,a_n \in A$ and p an n-ary operator we have

$$\begin{aligned} \phi(a_1)\cap\ldots\cap\phi(a_n) &\subseteq \{w\in F\mathbb{1} \mid p(w(*_1),\ldots,w(*_n))\subseteq\oplus_{1\le i\le n}\phi(a_i)\} &&\text{(claim a)}\\ &= \{w\in F\mathbb{1} \mid w(p(*_1,\ldots,*_n))\subseteq\oplus_{1\le i\le n}\phi(a_i)\}\\ &= p(\phi(a_1),\ldots,\phi(a_n))\\ &\subseteq \phi(p(a_1,\ldots,a_n)). \end{aligned}$$

Next, we assume that $\phi: A \rightarrow \Omega_{\mathbb{T}}$ satisfies conditions (i) and (ii) of the theorem. Suppose we have $w\in p(\phi(a_1),\ldots,\phi(a_n))$ for $a_1,\ldots,a_n \in A$ and p an n-ary operator. We will show that $w\in\phi(p(a_1,\ldots,a_n))$.

$$\begin{aligned} w\in p(\phi(a_1),\ldots,\phi(a_n)) &\rightarrow wp(*_1,\ldots,*_n)\in\oplus_{1\le i\le n}\phi(a_i)\\ &\rightarrow p(w(*_1),\ldots,w(*_n))\in\oplus_{1\le i\le n}\phi(a_i) \end{aligned}$$

Hence $p(w(*_1),\ldots,w(*_n))$ is of the form $u(v_1(*_{\alpha(1)}),\ldots,v_n(*_{\alpha(n)}))$ where $u\in Fn$, α is some permutation on $\{1,\ldots,n\}$ and $v_i\in\phi(a_i)$ for all $1\le i\le n$.
By condition (ii) we obtain that $*\in\phi(v_i(a_{\alpha(i)}))$ for all $1\le i\le n$. It follows that $*\in\bigcap_{1\le i\le n}\phi(v_i(a_{\alpha(i)}))$. Applying the condition (i) together with claim (b) we get

$$*\in\phi(u(v_1(a_{\alpha(1)}),\ldots,v_n(a_{\alpha(n)})))$$

However:

$$\begin{aligned} *\in\phi(u(v_1(a_{\alpha(1)}),\ldots,v_n(a_{\alpha(n)}))) &\rightarrow *\in\phi(p(w(a_1),\ldots,w(a_n)))\\ &\rightarrow *\in\phi(w(p(a_1,\ldots,a_n)))\\ &\rightarrow w\in\phi(p(a_1,\ldots,a_n)). \end{aligned}$$

Thus we can conclude that $w\in\phi(p(a_1,\ldots,a_n))$.
All together we have shown that ϕ is characteristic function.

□

4.2.6 Definition.
An algebraic theory $\mathbb{T}$ is called *sufficiently unary* if for any n-ary algebraic operation $w\in Fn$ there is an algebraic operation q of, say, arity m, together with two m-tuples of unary operators t_j and operations v_j $(1\le j\le m)$ and a function $\alpha:\{1,\ldots,m\}\rightarrow\{1,\ldots,n\}$ such that the following equations hold:

$$q(v_1w(x_1,\ldots,x_n),\ldots,v_mw(x_1,\ldots,x_n))=w(x_1,\ldots,x_n)$$
$$v_jw(x_1,\ldots,x_n)=t_j(x_{\alpha(j)}) \quad (1\le j\le m).$$

The notion of sufficiently unary enables us to generalize definition (4.2.2) to arbitrary n-ary operators. We have borrowed the name sufficiently unary from

[Johnstone 85], where it was used for a related syntactic condition.

4.2.7 Lemma. (implicit in [Johnstone a])
For a sufficiently unary algebraic theory $\mathbb{T}$ we have that

$$\forall n\in\mathbb{N}\ [n\geq 1\rightarrow \forall w\in Fn\ \forall A_1,\dots,A_n\in\Omega_{\mathbb{T}}\ \forall u\in F\mathbb{1}\ (u\in w(A_1,\dots,A_n)\leftrightarrow u(w(*_1,\dots,*_n))\in\oplus_{1\leq i\leq n}A_i)].$$

Proof.
For $w\in Fn$, $A_1,\dots,A_n\in\Omega_{\mathbb{T}}$ and $u\in F\mathbb{1}$, we will show by induction to the structure of the word w that

$$u\in w(A_1,\dots,A_n)\leftrightarrow u(w(*_1,\dots,*_n))\in\oplus_{1\leq i\leq n}A_i$$

for any $w\in Fn$.
If w is a algebraic operation of $\mathbb{T}$ then by definition (4.2.3) $u\in w(A_1,\dots,A_n)$ if and only if $u(w(*_1,\dots,*_n))\in\oplus_{1\leq i\leq n}A_i$.
So we need only to consider the case $w(*_1,\dots,*_n)=p(r_1(*_1,\dots,*_n),\dots,r_k(*_1,\dots,*_n))$ with induction hypotheses

$$\forall u\in F\mathbb{1}\ [u\in r_j(A_1,\dots,A_n)\leftrightarrow u(r_j(*_1,\dots,*_n))\in\oplus_{1\leq i\leq n}A_i \text{ for } (1\leq j\leq k).$$

($\rightarrow$) Assume first $u\in w(A_1,\dots,A_n)$. We can rewrite this assumption as:

$$u\in p(r_1(A_1,\dots,A_n),\dots,r_k(A_1,\dots,A_n))$$

and further

$$u(p(*_1,\dots,*_k))\in\oplus_{1\leq i\leq k}r_i(A_1,\dots,A_n).$$

Hence, there are $v\in Fk$, a permutation α on $\{1,\dots,k\}$ and $v_i\in r_i(A_1,\dots,A_n)$ for $1\leq i\leq k$ such that

$$u(p(*_1,\dots,*_k))=v(v_1(*_{\alpha(1)}),\dots,v_k(*_{\alpha(k)})).$$

By induction hypothesis $v_i\in r_i(A_1,\dots,A_n)$ implies $v_i(r_i(*_1,\dots,*_k))\in\oplus_{1\leq i\leq n}A_i$.
And so we get

$$u(p(r_1(**),\dots,r_k(**)))=v(v_1(r_{\alpha(1)}(**)),\dots,v_k(r_{\alpha(k)}(**))\in\oplus_{1\leq i\leq n}A_i,$$

where $**$ denotes the sequence $*_1,\dots,*_n$.
Therefore:

$$u(w(*_1,\dots,*_n))=u(p(r_{\alpha(1)}(*_1,\dots,*_n),\dots,r_{\alpha(k)}(*_1,\dots,*_n)))\in\oplus_{1\leq i\leq n}A_i.$$

($\leftarrow$) Secondly assume $u(w(*_1,\dots,*_n))\in\oplus_{1\leq i\leq n}A_i$.
Hence, by rewriting we get

$$u(p(r_1(*_1,\dots,*_n),\dots,r_k(*_1,\dots,*_n)))\in\oplus_{1\leq i\leq n}A_i.$$

Now it follows from the *sufficient unarity* of $\mathbb{T}$ applied to $u(p(y_i,\dots,y_k))$ that there is an algebraic operation q of, say, arity m, together with two m-tuples of unary algebraic operators t_j and algebraic operations v_j $(1\leq j\leq m)$ and a function $\alpha:\{1,\dots,m\}\rightarrow\{1,\dots,k\}$ such that the following equations hold:

$$up(y_1,\dots,y_k)=q(v_1up((y_1,\dots,y_k),\dots,v_mup(y_1,\dots,y_k))$$
$$t_jy_{\alpha(j)}=v_jup(y_1,\dots,y_k)\quad(1\leq j\leq m).$$

Substituting $r_j(*_1,\dots,*_n)$ for y_j in these equations we get:

$$t_j r_{\alpha(j)}(*_1,\dots,*_n)=v_j up(r_1(*_1,\dots,*_n),\dots,r_k(*_1,\dots,*_n)).$$

Since all $r_{\alpha(j)}(*_1,\dots,*_n)$ belong to $\oplus_{1\le i\le n}A_j$ we get by induction hypothesis

$$t_j r_{\alpha(j)}(*_1,\dots,*_n)\in\oplus_{1\le i\le n}A_i \text{ for } 1\le j\le m.$$

And so we get

$$\begin{aligned} u(p(r_1(**),\dots,r_k(**))) &= q(v_1up(**),\dots,v_mup(**_k)) \\ &= q(t_1r_{\alpha(1)}(**),\dots,t_mr_{\alpha(m)}(**)) \end{aligned}$$

where $**$ denotes the sequence $*_1,\dots,*_n$.

Since the latter term is an element of $\oplus_{1\le j\le m}r_j(A_1,\dots,A_n)$ we get

$$u(p(r_1(*_1,\dots,*_n),\dots,r_k(*_1,\dots,*_n)))\in\oplus_{1\le j\le m}r_j(A_1,\dots,A_n)$$

Therefore by definition (4.2.2)

$$u\in p(r_1(A_1,\dots,A_n),\dots,r_k(A_1,\dots,A_n))$$

or

$$u\in w(A_1,\dots,A_n).$$

□

The converse can be proved without any restriction on the algebraic theory $\mathbb{T}$.

4.2.8 Lemma.

For any function $\phi: A\to\Omega_{\mathbb{T}}$ of the form $a\mapsto\{w\in F\mathbb{1}\mid w(a)\in B\}$ for some subalgebra $B\subseteq A$, it holds that

$$p(\phi(a_1),\dots,\phi(a_n))\subseteq\phi(p(a_1,\dots,a_n))$$

for any $a_1,\dots a_n\in A$ and algebraic operator p.

Proof.

Suppose we have

$$w\in p(\phi(a_1),\dots,\phi(a_n))$$

then

$$w(p(*_1,\dots,*_n))\in\oplus_{1\le i\le n}\phi(a_i).$$

Hence $w(p(*_1,\dots,*_n))$ can be written as $u(v_1(*_{\alpha(1)}),\dots,v_n(*_{\alpha(n)}))$, where $u\in Fn$, α is a permutation on $\{1,\dots,n\}$ and $v_i\in\phi(a_{\alpha(i)})$ for $1\le i\le n$. For each a_i it follows that $v_i(a_{\alpha(i)})\in B$. If we substitute a_i for $*_i$, then we get

$$w(p(a_1,\dots,a_n)) = u(v_1(a_{\alpha(1)}),\dots,v_n(a_{\alpha(n)}))$$

Since $u(v_1(a_{\alpha(1)}),\dots,v_n(a_{\alpha(n)}))\in B$ we get $w\in\phi(p(a_1,\dots,a_n))$.

□

It seems natural to require for algebraic theories the property:

$$w(*)\in A\leftrightarrow w(*_1)\in A\oplus B.$$

In general this can not be deduced. We need the following restriction on algebraic theories:

4.2.11 Definition.; (cf. [Johnstone a])
An algebraic theory $\mathbb{T}$ is *strongly-nonconstant* if, whenever we have an equation of the form

$$w(*_1,\dots,*_m)=v(*_1,\dots,*_n),$$

where $n,m\in\mathbb{N}$, $w\in Fm$ and $v\in F(n)$, then $m=n$,
that is, the left hand side and the right hand side of equations contain the same generators.

4.2.12 Lemma. If $\mathbb{T}$ is an strongly-nonconstant theory, then for all $n,m\in\mathbb{N}$, $w\in Fn$, $A_1,\dots,A_{n+m}\in\Omega_{\mathbb{T}}$ and all $u\in F\mathbb{1}$ it holds that:

$$w(*_1,\dots *_m)\in\oplus_{1\le i\le m}A_i \leftrightarrow w(*_1,\dots *_m)\in\oplus_{1\le i\le m+n}A_i.$$

Proof. ($\rightarrow$) is trivial.
So, assume for $n,m\in\mathbb{N}$, $w\in Fn$, $A_1,\dots,A_{n+m}\in\Omega_{\mathbb{T}}$ and $u\in F\mathbb{1}$ that

$$w(*_1,\dots *_m)\in\oplus_{1\le i\le m+n}A_i.$$

Then by definition of the join $\oplus$ we can find $v\in F(m+n)$, $v_i\in F\mathbb{1}$ for $1\le i\le m+n$ such that

$$w(*_1,\dots *_m)=v(v_1(*_1),\dots,v_{m+n}(*_{m+n})).$$

The assumption strongly constant now implies that $m=n$. Hence we get:

$$w(*_1,\dots *_m)\in\oplus_{1\le i\le m}A_i$$

□

4.2.13 Lemma. If $\mathbb{T}$ is a strictly unary and strongly-nonconstant algebraic theory then $\Omega_{\mathbb{T}}$ with the operations of (4.2.2) is a $\mathbb{T}$-algebra.

Proof.
To show that the definition (4.2.2) does turn $\Omega_{\mathbb{T}}$ into an algebra, we have to check that all equations of $\mathbb{T}$ are satisfied for $\Omega_{\mathbb{T}}$.
Let $w(x_1,\dots,x_k,y_1,\dots,y_m)=v(y_1,\dots,y_m,z_1,\dots,z_n)$ be such an equation, in which we have listed all variables at both sides.
Then

$$\begin{aligned}
u\in w(A_1,\dots A_{k+m}) &\leftrightarrow u(w(*_1,\dots,*_{k+m}))\in\Sigma_{1\le i\le k+m}A_i && (4.2.7)\\
&\leftrightarrow u(w(*_1,\dots,*_{k+m}))\in\Sigma_{1\le i\le k+m+n}A_i && (4.2.10)\\
&\leftrightarrow u(v(*_{k+1},\dots,*_{k+m+n}))\in\Sigma_{1\le i\le k+m+n}A_i && \text{(equation)}\\
&\leftrightarrow u(v(*_1,\dots,*_{m+n}))\in\Sigma_{1\le i\le m+n}A_{k+i} && (4.2.10)\\
&\leftrightarrow u\in v(A_1,\dots,A_n), && (4.2.7)
\end{aligned}$$

and hence $w(A_1,\dots,A_n)=v(A_1,\dots,A_n)$, where the A_i range over $\Omega_{\mathbb{T}}$.

We will prove the following version of theorem (1.3) in [Johnstone a]:

4.2.14 Theorem.
For a finitary algebraic theory $\mathbb{T}$, the following are equivalent:

(i) for all maps $\phi: A \longrightarrow \Omega_{\mathbb{T}}$ of the form $a \mapsto \{w \in F\mathbb{1} \mid w(a) \in B\}$ for some subalgebra $B \subseteq A$, it holds for any $n \in \mathbb{N}$, $p \in Fn$ and all $a_1,\dots,a_n \in A$ that:
$\phi(p(a_1,\dots,a_n)) \subseteq p(\phi(a_1),\dots,\phi(a_n))$.

(ii) the type of subalgebras of any algebra is a complete Heyting algebra,

(iii) $\mathbb{T}$ is a sufficiently unary algebraic theory.

If $\mathbb{T}$ is a strongly-nonconstant, finitary algebraic theory then the foregoing statements are equivalent to:

(iv) $\top: \mathbb{1} \longrightarrow \Omega_{\mathbb{T}}: * \mapsto F\mathbb{1}$ is a morphism of $\mathbb{T}$-algebras and a subobject classifier for $\mathbb{T}$-algebras.

Proof.
(i)⇒(ii) Consider the type of subalgebras of a type A defined as
$\mathrm{Sub}(A) := \{B \subseteq A \mid B \text{ is a subalgebra of } A\}$.
With the definition of join we already have that $\langle \mathrm{Sub}(A), \subseteq, A, \Sigma \rangle$ is an complete join-semilattice. It will follow that $\langle \mathrm{Sub}(A), \subseteq, F0, \cap, A, \Sigma \rangle$ is an complete Heyting algebra if we can show the infinite distributive schema $B \cap \Sigma_{i \in I} A_i = \Sigma_{i \in I}(B \cap A_i)$ holds for subalgebras of A, or, equivalently, if we can define $\Rightarrow_A: \mathrm{Sub}(A) \longrightarrow \mathrm{Sub}(A)$ with the property $D \subseteq B \Rightarrow C$ if and only if $D \cap B \subseteq C$ for all $B, C, D \in \mathrm{Sub}(A)$.
Define for $B, C \in \mathrm{Sub}(A)$:
$B \Rightarrow_A C := \{a \in A \mid \phi_B(a) \subseteq \phi_C(a)\}$.
This can be rewritten as (see proof of (4.2.4)):
$B \Rightarrow_A C := \{a \in A \mid \forall w \in F\mathbb{1}\, [w(a) \in B \rightarrow w(a) \in C]\}$

In order to show that $B \Rightarrow_A C$ is a subalgebra of A, let p be an algebraic operation of $\mathbb{T}$, and suppose $a_1,\dots,a_n \in B \Rightarrow_A C$. Then for each a_i we have
$\phi_B(a_i) \subseteq \phi_C(a_i)$
Hence for $w \in F\mathbb{1}$ we can argue

$$\begin{aligned} w \in \phi_B(p(a_1,\dots,a_n)) &\rightarrow w \in p(\phi_B(a_1),\dots,\phi_B(a_n)) \\ &\rightarrow w \in p(\phi_C(a_1),\dots,\phi_C(a_n)) \\ &\rightarrow w \in \phi_C(p(a_1,\dots,a_n)). \qquad (4.2.8) \end{aligned}$$

Thus we see that $p(a_1,\dots,a_n) \in B \Rightarrow_A C$.

Next assume for $B,C,D\in Sub(A)$ we have that $D\subseteq B\Rightarrow C$. Now for $a\in D$ it holds that

$$a\in B\Rightarrow_A C \;\leftrightarrow\; \forall w\in F\mathbb{1}\,[w(a)\in B\rightarrow w(a)\in C]$$
$$\rightarrow\; a\in B\rightarrow a\in C$$

So we get $D\cap B\subseteq C$.

If, on the other hand, we have $D\cap B\subseteq C$ for $B,C,D\in Sub(A)$, suppose $a\in D$ and $wa\in B$ for $w\in F\mathbb{1}$. Then $wa\in D$, and thus $wa\in D\cap B$. And so $wa\in C$. Hence $a\in B\Rightarrow_A C$.

(ii)$\Rightarrow$(iii) We copy Johnstone's elegant little proof:
Given an operation $p\in Fn$, let $A\subseteq Fn$ be the subalgebra generated by p, and let B_i $(1\leq i\leq n)$ be the subalgebra generated by $*_i$.
Then $p\in A\cap\Sigma_{i=1}^{n}B_i$, and by hypothesis we have $p\in\Sigma_{i=1}^{n}(A\cap B_i)$.

That is p may be obtained by applying some term (q, say) to a family of terms in $\bigcup_{i=1}^{n}(A\cap B_i)$.

But $A\cap B_i$ consists of terms $up(*_1,\ldots,*_n)$ of the form $v(*_i)$, where u,v are unary. Hence (iii) holds.

(iv)$\Rightarrow$(ii) Trivial,

Finally, we assume that $\mathbb{T}$ is a strongly-nonconstant algebraic theory in order to prove the remaining implication.

(iii)$\Rightarrow$(iv) By (4.2.11) we get that $\Omega_{\mathbb{T}}$ is a $\mathbb{T}$-algebra. It is trivial to see that $\mathbb{1}\approx\{F\mathbb{1}\}$ is a subalgebra of $\Omega_{\mathbb{T}}$ and $T:\mathbb{1}\rightarrow\Omega_{\mathbb{T}}:*\mapsto F\mathbb{1}$ is a morphism of $\mathbb{T}$-algebras.

The subobject classifying properties follow from (4.2.6).

Clearly, if a function $\phi:A\rightarrow\Omega_{\mathbb{T}}$ is of the form $a\mapsto\{w\in F\mathbb{1}\mid w(a)\in B\}$ for some subalgebra $B\subseteq A$, that is, ϕ is a characteristic function, then by lemma (4.2.10) ϕ satisfies

$$p(\phi(a_1),\ldots,\phi(a_n))\subseteq\phi(p(a_1,\ldots,a_n))$$

for any $n\in\mathbb{N}$, $p\in Fn$ and all $a_1,\ldots,a_n\in A$. Hence ϕ is morphism, if we can also prove

$$\phi(p(a_1,\ldots,a_n))\subseteq p(\phi(a_1),\ldots,\phi(a_n)).$$

As in proof of proposition (2.4) of [Johnstone a]. Suppose $w\in\phi(p(a_1,\ldots,a_n))$, then $wp(a_1,\ldots,a_n)\in B$.
Now consider $wp(*_1,\ldots,*_n)$. Since $\mathbb{T}$ is supposed to be strictly unary, there is an algebraic operation q of, say, arity m, together with two m-tuples of unary operators t_j and operations v_j $(1\leq j\leq m)$ and a function $\alpha:\{1,\ldots,m\}\rightarrow\{1,\ldots,n\}$ such that the following equations hold:

$$q(v_1wp(*_1,\ldots,*_n),\ldots,v_mwp(*_1,\ldots,*_n))=wp(*_1,\ldots,*_n)$$

$$v_j w p(*_1,\ldots,*_n)=t_j(*_{\alpha(j)}) \quad (1\leq j\leq m).$$

We see that

$$v_j w p(a_1,\ldots,a_n)\in B,$$

and therefore by (4.2.7) that

$$t_j\in\phi(a_{\alpha(j)})).$$

Putting everything together we get

$$wp(*_1,\ldots,*_n)=q(t_1(*_{\alpha(1)}),\ldots,t_m(*_{\alpha(m)}))\in\oplus_{1\leq i\leq n}\phi(a_i)).$$

Hence we can conclude

$$w\in p(\phi(a_1),\ldots,\phi(a_n)).$$

Thus we have shown that $\phi(p(a_1,\ldots,a_n))\subseteq p(\phi(a_1),\ldots,\phi(a_n))$.

□

Chapter 5

Topologies, Sheaves and Localizations for some Algebraic Theories

Borceux, Kelly, Veit and Van den Bossche have studied the relationships between localizations, universal closure operations, Gabriel-Grothendieck topologies and Lawvere-Tierney topologies in all sorts of categories (cf. [Borceux], [Borceux and Veit (86 and 88)], [Borceux and Kelly] and [Borceux and Van den Bossche]). Roughly summarizing, typical results obtained are:

(i) universal closure operations are in 1-1 correspondence with both Gabriel-Grothendieck topologies and Lawvere-Tierney topologies
(ii) each localization induces a universal closure operation, and the 1-1 correspondence exists only if the category satisfies some condition
(iii) Lawvere-Tierney topologies, Gabriel-Grothendieck topologies and localizations each form a locale.

The techniques used provide a uniform approach to notions of localization in seemingly unrelated areas as topos theory and ring or module theory.

Let $\mathbb{T}$ be some finitary algebraic theory in a topos $\mathbf{E}$. We will reason internally in some type theory with a natural number object. We will start with the correspondence between universal closure operations and topologies in the style of Lawvere-Tierney and Grothendieck.

5.1 Universal closure operations and topologies on $\Omega_{\mathbb{T}}$

We start with the definition of a closure operation. When described in type theory, it is a kind of meta process that given an algebra returns a closure operation on the type of subalgebras of the given algebra. Taking the closure of $\{F\mathbb{1}\} \hookrightarrow \Omega_{\mathbb{T}}$ is the standard construction to produce the corresponding Grothendieck topology on $\Omega_{\mathbb{T}}$. Given something like a Grothendieck topology $J \subseteq \Omega_{\mathbb{T}}$, we suggest

$$\Omega_{\mathbb{T}} \to \Omega_{\mathbb{T}} : R \mapsto \{w \in F\mathbb{1} \mid w^{-1}(R) \in J\}$$

as the generalization of the earlier correspondence in chapter 2 between Grothendieck topologies and Lawvere-Tierney topologies. This makes sense because we know that $\Omega_{\mathbf{T}}$ is a subalgebra classifier (cf. 4.2.4). With the notion of Lawvere-Tierney topology, that we get by this analogy on $\Omega_{\mathbf{T}}$, we can complete the circle: given a topology $j:\Omega_{\mathbf{T}}\rightarrow\Omega_{\mathbf{T}}$ we define a closure operation on algebras by mapping a subalgebra $B\hookrightarrow A$ to the subalgebra classified $j\circ\phi_B:A\rightarrow\Omega_{\mathbf{T}}$ (c.f. notations of theorem 4.2.4). We will show that there is a bijection between the resulting notions.

Let us start with universal closure operations. We take the usual definition, as one defines them on any category with pullbacks, cf. [Johnstone 77].

5.1.1 Definition. A *universal closure operation* on $\mathbb{T}$-algebras is defined by specifying for each algebra A a closure operation $(\ )^{cl}$ on the type Subalg(A) of subalgebras of A in such as way that the closure operation commutes with pullback along morphisms.

More specifically, for each algebra A we have a function

$(\ )^{cl}: \mathrm{Subalg}(A)\rightarrow\mathrm{Subalg}(A)$,

such that for any R and S subalgebras of A the following holds:

(i) $R\leq R^{cl}$

(ii) $R\leq S\rightarrow R^{cl}\leq S^{cl}$

(iii) $(R^{cl})^{cl}=R^{cl}$

(iv) $(f^{-1}(R))^{cl}=f^{-1}(R^{cl})$ for any $\mathbb{T}$-morphism $B\rightarrow A$.

We proceed immediately with the definition of Gabriel-Grothendieck topologies, as Borceux calls them.

5.1.2 Definition. (cf. [Borceux and Veit 86])

A *Gabriel-Grothendieck topology* is a subset $J\subseteq\Omega_{\mathbf{T}}$,

(a) $F\mathbb{1}\in J$

(b) $\forall R\in J\ \forall S\in\Omega_{\mathbf{T}}\ [(\forall w\in R\ w^{-1}(S)\in J)\rightarrow S\in J]$

(c) $\forall S_1...S_n\in J\ \forall w\in Fn\ w^{-1}(S_1\oplus...\oplus S_n)\in J$

The following conditions can be derived from (a,b,c):

(d) $\forall R\in J\ \forall R\hookrightarrow S\ S\in J$

(e) $\forall R,S\in\Omega_{\mathbf{T}}\ [(R\cap S)\in J\leftrightarrow(R\in J\wedge S\in J)]$

Observe how this definition generalizes the previous notion of Grothendieck topology (cf. 2.2.7). The new definition collapses to the old one in the restricted case of a degenerate algebra, i.e., an algebra without constants and axioms simply because then $F\mathbb{1}=\mathbb{1}$ and $\Omega_{\mathbf{T}}=\Omega$. A subalgebra $R\subseteq F\mathbb{1}$ corresponds with subobject $\mathbb{1}\upharpoonright\omega$ of $\mathbb{1}$ and $w\in R$ becomes $\omega=\top$.
One should also note a formal resemblance. This observation allows us to see that we get an equivalent definition if we replace (b) by (g), defined as:

(g) $\forall R\in J\ \forall S\in\Omega_{\mathbf{T}}\ [(\forall w\in R\ w^{-1}(R\cap S)\in J)\rightarrow S\in J]$.

For clearly (d,e)$\Rightarrow$(g), and as also the combination (a,b,c,g)$\Rightarrow$(e,f) and (f,g)$\Rightarrow$(d) we have (a,b,c,g)$\Rightarrow$(d).

The above definition of Gabriel-Grothendieck topology is taken from [Borceux and Veit 86], who have stated it in the context of the topos *Sets*.

The following definition of Lawvere-Tierney topology is inevitable if one wants to establish an 1-1 correspondence between Lawvere-Tierney topologies and Gabriel-Grothendieck topologies. Recall the notation (4.1.4.iv): if $w\in Fn$, then we denote by w also the function $F\mathbb{1}\longrightarrow Fn$: $v\longrightarrow v(w)$.

5.1.3 Definition.
A morphism $j:\Omega_{\mathbf{T}}\longrightarrow\Omega_{\mathbf{T}}$ is called a *(generalized) Lawvere-Tierney topology* if

(a) $jF\mathbb{1}=F\mathbb{1}$

(b) $\forall R\in\Omega_{\mathbf{T}}\ jjR=jR$

(c) $\forall R,S\in\Omega_{\mathbf{T}}\ j(R\cap S)=jR\cap jS$

(d) $\forall S_1\ldots S_n\in\Omega_{\mathbf{T}}\forall w\in Fn\ [(jS_1=F\mathbb{1}\wedge\ldots\wedge jS_n=F\mathbb{1})\rightarrow jw^{-1}(S_1\oplus\ldots\oplus S_n)=F\mathbb{1}]$

or equivalently to (d)

(d') $\forall S_1\ldots S_n\in\Omega_{\mathbf{T}}\forall w\in Fn\ [j(S_1\cap\ldots\cap S_n)=F\mathbb{1}\rightarrow jw^{-1}(S_1\oplus\ldots\oplus S_n)=F\mathbb{1}]$.

The following can be derived (use $R\subseteq S\leftrightarrow R\cap S=R$):

(e) $\forall R,S\in\Omega_{\mathbf{T}}\ [R\subseteq S\rightarrow jR\subseteq jS]$

The following condition (f) seems too strong: it implies (d), but we could not establish the converse, that is (a,b,c,d)$\rightarrow$(a,b,c,f):

(f) $\forall S_1\ldots S_n\in\Omega_{\mathbf{T}}\forall w\in Fn: jw^{-1}(S_1\oplus\ldots\oplus S_n)=w^{-1}(jS_1\oplus\ldots\oplus jS_n)$.

5.1.4 Examples.
(i) Of course we have the trivial Lawvere-Tierney topologies: the identity $k_{id}:\Omega_{\mathbf{T}}\longrightarrow\Omega_{\mathbf{T}}$ and the constant topology $k_C:\Omega_{\mathbf{T}}\longrightarrow\Omega_{\mathbf{T}}:R\mapsto F\mathbb{1}$ like in the case for topologies on Ω.

(ii) There are natural connections between Lawvere-Tierney topologies on $\Omega_{\mathbf{T}}$ and Lawvere-Tierney topologies on Ω:

5.1.4.1 Lemma. Let a topology $j:\Omega \to \Omega$ be given. Then the following are Lawvere-Tierney topologies on $\Omega_{\mathbf{T}}$:

(i) $k_j = \Omega_{\mathbf{T}} \to \Omega_{\mathbf{T}}: R \mapsto \{w \in F\mathbb{1} \mid jw \in R\}$,

(ii) $k_{j_2} = \Omega_{\mathbf{T}} \to \Omega_{\mathbf{T}}: R \mapsto \{w \in F\mathbb{1} \mid j* \in R\} \cup F\emptyset$.

Proof. (i) is easy.

(ii) This $k_{j_2}:\Omega_{\mathbf{T}} \to \Omega_{\mathbf{T}}$ is indeed a Lawvere-Tierney topology. The first and third property, preservation of $\top$ and $\wedge$, are trivial, the second follows by a calculation:

$$\begin{aligned} k_{j_2}k_{j_2}(R) &= k_{j_2}(\{w \in F\mathbb{1} \mid j* \in R\} \cup F\emptyset) \\ &= \{w \in F\mathbb{1} \mid j* \in \{w \in F\mathbb{1} \mid j* \in R\} \cup F\emptyset\} \cup F\emptyset \\ &= \{w \in F\mathbb{1} \mid jj* \in R\} \cup F\emptyset \\ &= \{w \in F\mathbb{1} \mid j* \in R\} \cup F\emptyset \\ &= k_{j_2}(R). \end{aligned}$$

For $S_1 \ldots S_n \in \Omega_{\mathbf{T}}$ and $w \in Fn$ suppose $k_{j_2}S_1 = F\mathbb{1} \wedge \ldots \wedge k_{j_3}S_n = F\mathbb{1}$. Note that $k_{j_2}S = F\mathbb{1}$ is equivalent to $* \in \{w \in F\mathbb{1} \mid j* \in S\} \cup F\emptyset$. Similarly in order to prove

$$k_{j_2}w^{-1}(S_1 \oplus \ldots \oplus S_n) = F\mathbb{1}$$

it suffices to prove that

$$(j* \in w^{-1}(S_1 \oplus \ldots \oplus S_n)) \vee * \in F\emptyset$$

If from the assumption follows $* \in F\emptyset$, then we are done. So let us consider the case that $j* \in S_i$ for $1 \le i \le n$. If we assume the stronger asssumption that $* \in S_i$ for $1 \le i \le n$. Then

$$S_1 \oplus \ldots \oplus S_n = Fn.$$

And so

$$w^{-1}(S_1 \oplus \ldots \oplus S_n) = F\mathbb{1}.$$

Now if we apply the j-operator to the last, too strong assumption, we weaken the last assumption to the first, and so we derive the desired localized conclusion from the our earlier weaker assumption.

□

5.1.4.2 Lemma. Let a Lawvere-Tierney topology $k:\Omega_{\mathbf{T}} \to \Omega_{\mathbf{T}}$ be given. Then $j_k = \Omega \to \Omega: \omega \mapsto * \in k[\{w \in F(\mathbb{1}) \mid \omega\} \cup F\emptyset]$ is a topology on Ω.

Proof. The only interesting case is to show that $j_k j_k \omega \to j_k \omega$, for all $\omega \in \Omega$. So let $\omega \in \Omega$. It suffices to show that

$$k[\{w \in F(\mathbb{1}) \mid j_k\omega\} \cup F\emptyset] \subseteq k[\{w \in F(\mathbb{1}) \mid \omega\} \cup F\emptyset],$$

i.e., by $k \circ k = k$ it is sufficient to show that

$$\{w \in F(\mathbb{1}) \mid * \in k[\{w \in F(\mathbb{1}) \mid \omega\} \cup F\emptyset]\} \cup F\emptyset \subseteq k[\{w \in F(\mathbb{1}) \mid \omega\} \cup F\emptyset],$$

i.e., it is sufficient to show that

if $*\in k[\{w\in F(\mathbb{1}) \mid \omega\}\cup F\emptyset]$ then $w\in k[\{w\in F(\mathbb{1}) \mid \omega\}\cup F\emptyset]$.
Which is trivial.

□

5.1.4.3 Lemma. For a topology $k:\Omega_{\mathbb{T}}\to\Omega_{\mathbb{T}}$ there there is a topology $j:\Omega\to\Omega$ such that $k=k_j$ if and only if $k=k_{j_k}$.

Proof. (⇐) Trivial.
(⇒) If $k=k_j$ for some $j:\Omega_{\mathbb{T}}\to\Omega_{\mathbb{T}}$. Then for $R\in\Omega_{\mathbb{T}}$.
Then for $R\in\Omega_{\mathbb{T}}$ we have

$$\begin{aligned}
k_{j_k}(R) &= \{w\in F(\mathbb{1}) \mid j_k(w\in R)\}\\
&= \{w\in F(\mathbb{1}) \mid *\in k[\{v\in F(\mathbb{1}) \mid w\in R\}\cup F\emptyset]\}\\
&= \{w\in F(\mathbb{1}) \mid *\in k_j[\{v\in F(\mathbb{1}) \mid w\in R\}\cup F\emptyset]\}\\
&= \{w\in F(\mathbb{1}) \mid j*\in[\{v\in F(\mathbb{1}) \mid w\in R\}\cup F\emptyset]\}\\
&= \{w\in F(\mathbb{1}) \mid j(w\in R\vee *\in F\emptyset)\}\\
&= \{w\in F(\mathbb{1}) \mid jw\in R\}\\
&= k_j(R)\\
&= k(R).
\end{aligned}$$

□

(iii) [Borceux and Van den Bossche 84] have given in the context of classical logic an example of a Lawvere-Tierney topology for the theory of *abelian groups*. We adopt it to our constructive setting. Then $F(\mathbb{1})=\mathbb{Z}$ and $\Omega_{\mathbb{T}}$ is the set of subgroups of $\langle\mathbb{Z},+\rangle$. Subsets of the form $\{0\}\cup\{nm \mid m\in\mathbb{Z}\wedge\omega\}$ (or in short notation: $n\mathbb{Z}\restriction\omega$) are subgroup of $\mathbb{Z}$ for each $n\in\mathbb{N}$ and $\omega\in\Omega$. Each subgroup $A\subseteq\mathbb{Z}$ is generated by such subgroups, as $A=\langle n\mathbb{Z}\restriction n\in A\rangle_{n\in\mathbb{N}}$. As a consequence $\Omega_{\mathbb{T}}$ is far from isomorphic to $\mathbb{N}$ as it is in the classical case.
An algebraic Lawvere-Tierney topology in this setting is the following:

$$k_p:\Omega_{\mathbb{T}}\to\Omega_{\mathbb{T}}:R\mapsto\{m\in\mathbb{Z} \mid \exists n\in\mathbb{N}\ \ p^n m\in R\}$$

for each prime $p\in\mathbb{N}$, and where $p^n m$ is interpreted as a sum of p^n terms m.
It is straightforward to check that this is a topology on $\Omega_{\mathbb{T}}$.

We will now proceed to establish the bijection between the three afore mentioned notions: universal closure operations, Gabriel-Grothendieck topologies and Lawvere-Tierney topologies.

5.1.5 Lemma.

(i) If we are given a closure operation $(\ )^{cl}$ on algebras, then $J_{cl}=\{R\leq F\mathbb{1} \mid R^{cl}=F\mathbb{1}\}$ is a Gabriel-Grothendieck topology.

(ii) If $J\subseteq\Omega_T$ is a Gabriel-Grothendieck topology, then if for any algebra A we define for $R\leq A$ that

$$R^{cl_J}=\{b\in A\mid \phi_R(b)\in J\},$$

we have defined a universal closure operation.

(iii) There is a one-one correspondence between universal closure operations and Gabriel-Grothendieck topologies.

Proof.

(i) We will prove the three properties of Gabriel-Grothendieck topologies for J_{cl}. We argue for subalgebras of $F\mathbb{1}$:

(a) $F\mathbb{1}\leq(F\mathbb{1})^{cl}\leq F\mathbb{1}$ by (5.1.1.i), hence $F\mathbb{1}\in J_{cl}$,

(b) For $R:=J_{cl}$ and $S\leq F\mathbb{1}$, assume that for all $w\in R$ we have $w^{-1}(S)\in J_{cl}$. Now we can argue as follows:

$$\begin{aligned} w^{-1}(S)\in J_{cl} &\rightarrow (w^{-1}(S))^{cl}=F\mathbb{1}\\ &\rightarrow w^{-1}(S^{cl})=F\mathbb{1} \qquad\qquad (5.1.1.iv)\\ &\rightarrow w\in S^{cl}\end{aligned}$$

Hence we get from the assumption that we get $R\leq S^{cl}$. And so $F\mathbb{1}=R^{cl}\leq S^{cl\,cl}=S^{cl}\leq F\mathbb{1}$ by (5.1.1.ii,iii), that is $S:=J^{cl}$.

(b) For $S_1,\ldots,S_n\in J^{cl}$ observe that in order to show that

$$\forall w\in Fn\ w^{-1}(S_1\oplus\ldots\oplus S_n)\in J^{cl}$$

it suffices to show that

$$Fn=(S_1\oplus\ldots\oplus S_n)^{cl},$$

because

$$\begin{aligned}\forall w\in Fn\ w^{-1}(S_1\oplus\ldots\oplus S_n)\in J^{cl} &\leftrightarrow \forall w\in Fn\ (w^{-1}(S_1\oplus\ldots\oplus S_n))^{cl}=F\mathbb{1}\\ &\leftrightarrow \forall w\in Fn\ w^{-1}(S_1\oplus\ldots\oplus S_n)^{cl}=F\mathbb{1}\\ &\leftrightarrow \forall w\in Fn\ w\in(S_1\oplus\ldots\oplus S_n)^{cl}\\ &\leftrightarrow Fn=(S_1\oplus\ldots\oplus S_n)^{cl}\end{aligned}$$

So, provided we know that $(S_1{}^{cl}\oplus\ldots\oplus S_n{}^{cl})\leq(S_1\oplus\ldots\oplus S_n)^{cl}$, we are done, for then we can argue that

$$\begin{aligned}S_1,\ldots,S_n\in J^{cl} &\rightarrow S_1{}^{cl}=F\mathbb{1}\wedge\ldots\wedge S_n{}^{cl}=F\mathbb{1}\\ &\rightarrow Fn=S_1{}^{cl}\oplus\ldots\oplus S_n{}^{cl}\leq(S_1\oplus\ldots\oplus S_n)^{cl}\leq Fn\\ &\rightarrow Fn=(S_1\oplus\ldots\oplus S_n)^{cl}.\end{aligned}$$

This proviso is easy, as it suffices to show that $S^{cl}\oplus R^{cl}\leq(S\oplus R)^{cl}$. But any $w\in S^{cl}\oplus R^{cl}$ is of the form $v(s,r)$ with $v\in F2$, $s\in S^{cl}$ and $r\in R^{cl}$. Because $S\leq S\oplus R$ we get $S^{cl}\leq(S\oplus R)^{cl}$, and similar for R. Thus we see that $s,r\in(S\oplus R)^{cl}$ and hence $w\in(S\oplus R)^{cl}$.

(ii) Let $J\subseteq\Omega_T$ be a Gabriel-Grothendieck topology. Consider the subalgebras R,S of an arbitrary algebra A.

(a) $R\leq R^{cl_J}$, since for $a\in R$ we have $\phi_R(a)=F\mathbb{1}\in J$.

(b) Assume $R\leq S$. Then for all $a\in A$ we have $\phi_R(a)\leq\phi_S(a)$. Hence if $b\in R^{cl_J}$ then $\phi_R(b)\in J$ and so by (5.1.2.e) we get $\phi_S(b)\in$, i.e., $b\in S^{cl_J}$. Therefor $R^{cl_J}\leq S^{cl_J}$.

(c) We need only to show that $(R^{cl_J})^{cl_J} \leq R^{cl_J}$, for which it suffices to prove that

$$\forall a \in A\ [\phi_{R^{cl_J}}(a) \in J \rightarrow \phi_R(a) \in J].$$

This follows from (5.1.2.b), if we can show that:

$$\forall w \in \phi_{R^{cl_J}}(a)\ w^{-1}(\phi_R(a)) \in J.$$

Which is trivial, since

$$\begin{aligned} w \in \phi_{R^{cl_J}}(a) &\rightarrow w(a) \in R^{cl_J} \\ &\rightarrow \phi_R(w(a)) \in J \\ &\rightarrow w^{-1}(\phi_R(a)) \in J. \end{aligned}$$

(d) For $f: B \longrightarrow A$ and $R \leq A$ we argue

$$\begin{aligned} ((f^{-1})(R))^{cl_J} &= \{b \in B \mid \phi_{f^{-1}(R)}(b) \in J\} \\ &= \{b \in B \mid \{w \in F\mathbb{1} \mid w(b) \in f^{-1}(R)\} \in J\} \\ &= \{b \in B \mid \{w \in F\mathbb{1} \mid f(w(b)) \in R\} \in J\} \\ &= \{b \in B \mid \phi_R(f(b)) \in J\} \\ &= \{b \in B \mid f(b) \in R^{cl_J}\} \\ &= f^{-1}(R^{cl_J}) \end{aligned}$$

(iii) If we start with a Gabriel-Grothendieck topology J we get

$$\begin{aligned} J_{cl_J} &= \{R \leq F\mathbb{1} \mid R^{cl_J} = F\mathbb{1}\} \\ &= \{R \leq F\mathbb{1} \mid \{w \in F\mathbb{1} \mid \phi_R(w) \in J\} = F\mathbb{1}\} \\ &= \{R \leq F\mathbb{1} \mid \phi_R(*) \in J\} \\ &= \{R \leq F\mathbb{1} \mid \{w \in F\mathbb{1} \mid w \in R\} \in J\} \\ &= \{R \leq F\mathbb{1} \mid R \in J\} \\ &= J. \end{aligned}$$

If, on the other hand, we begin with a universal closure operation, we get for $R \leq A$:

$$\begin{aligned} R^{cl(J_{cl})} &= \{a \in A \mid \phi_R(a) \in J_{cl}\} \\ &= \{a \in A \mid (\phi_R(a))^{cl} = F\mathbb{1}\} \\ &= \{a \in A \mid (a^{-1}(R))^{cl} = F\mathbb{1}\} && (4.1.4.iii) \\ &= \{a \in A \mid a^{-1}(R^{cl}) = F\mathbb{1}\} \\ &= \{a \in A \mid a \in R^{cl}\} \\ &= R^{cl}. \end{aligned}$$

We made use of the little trick of (4.1.4) that an element $a \in A$ defines a morphism denoted with the same letter $a: F\mathbb{1} \longrightarrow A: w \mapsto w(a)$, and the trivial observation that

$$\phi_R(a) = \{w \in F\mathbb{1} \mid w(a) \in R\} = a^{-1}(R).$$

□

5.1.6 Definition.

(i) $\text{GG-TOP} = \{J \subseteq \Omega_{\mathbf{T}} \mid J \text{ is a GG-topology}\}$

(ii) $\text{LT-TOP} = \{j: \Omega_{\mathbf{T}} \longrightarrow \Omega_{\mathbf{T}} \mid j \text{ is LT-topology}\}.$

Note that there is not something like a type containing universal closure operations. In the setting of a topos universal closure operations correspond to the global elements of the above two types of topologies. The notion universal closure operation is an essentially external notion, that we can treat via meta type theoretic terms only.

5.1.7 Theorem.
There exists an bijection between the types GG-TOP and LT-TOP.

Proof. We employ similar constructions as in (2.2.8) and (5.1.4). Define

$J_{()}:\text{LT-TOP}\to\text{GG-TOP}: j\mapsto\{R\in\Omega_{\mathbf{T}}\mid jR=F\mathbb{1}\}$

$j_{()}:\text{GG-TOP}\to\text{LT-TOP}: J\mapsto[\Omega_{\mathbf{T}}\to\Omega_{\mathbf{T}}: R\to\{w\in F\mathbb{1}\mid w^{-1}R\in J\}]$,

We have to check that
(a) $J_{()}$ and $j_{()}$ are well-defined,
(b) $J_{()}$ and $j_{()}$ are each others inverse.

Ad (a).
First for $j\in\text{LT-TOP}$ we show that J_j satisfies the properties of a Gabriel-Grothendieck topology.
(a) $jF\mathbb{1}=F\mathbb{1} \leftrightarrow F\mathbb{1}\in\{R\in\Omega_{\mathbf{T}}\mid jR=F\mathbb{1}\}$
$\leftrightarrow F\mathbb{1}\in J_j$
(b) For $S_1...S_n\in J_j$ and $w\in Fn$ we have $jS_i=F\mathbb{1}$ for $1\leq i\leq n$, hence we get
$jw^{-1}(S_1\oplus...\oplus S_n)=F\mathbb{1}$
by (5.1.3.c), i.e.,
$w^{-1}(S_1\oplus...\oplus S_n)\in J_j$
and therefor
$w^{-1}(S_1\oplus...\oplus S_n)\in J_j$.
(c) For $R\in J_j, S\in\Omega_{\mathbf{T}}$ assume that $\forall w\in R\ w^{-1}(S)\in J_j$.

Note: $\forall w\in R\ w^{-1}(S)\in J_j \leftrightarrow \forall w\in R\ jw^{-1}(S)=F\mathbb{1}$
$\leftrightarrow \forall w\in R\ w^{-1}(jS)=F\mathbb{1}$
$\leftrightarrow \forall w\in R\ *\in w^{-1}(jS)$
$\leftrightarrow \forall w\in R\ w\in jS$
$\leftrightarrow R\subseteq jS$
$\rightarrow jR\subseteq jjS$
$\leftrightarrow jR\subseteq jS$

From $jR=F\mathbb{1}$ and $jR\subseteq jS$ follows $jS=F\mathbb{1}$ hence $S\in J_j$. That is we have shown (5.1.2.c).

All together we may conclude that J_j is a Gabriel-Grothendieck topology.

Secondly, for $J\in\text{GG-TOP}$ we show that j_J is a Lawvere-Tierney topology.

(a') $R \subseteq \{w\in F\mathbb{1} \mid w^{-1}R\in J\}$ (for $w^{-1}R=F\mathbb{1}$ if $w\in R$)
$= \{w\in F\mathbb{1} \mid w^{-1}R\in J\}$
$= j_J(R)$

(b') We need only to show that $j_J(j_J(R)\leq j_J(R)$, for which it suffices to prove that
$\forall w\in F\mathbb{1}\,[w^{-1}(j_J(R))\in J \rightarrow w^{-1}(R)\in J]$.

This follows from (5.1.2.b), if we can show that:
$\forall v\in w^{-1}(j_J(R))\; v^{-1}(w^{-1}(R))\in J$.

Which is trivial, since

$$\begin{aligned} v\in w^{-1}(j_J(R)) &\rightarrow v(w)\in j_J(R) \\ &\rightarrow (v(w))^{-1}(R)\in J \\ &\rightarrow v^{-1}(w^{-1}(R))\in J. \end{aligned}$$

(c') For $R,S\in\Omega_{\mathbf{T}}$ we reason as follows:

$$\begin{aligned} w\in j_J(R\cap S) &\leftrightarrow w^{-1}(R\cap S)\in J \\ &\leftrightarrow w^{-1}(R)\cap w^{-1}(S)\in J \\ &\leftrightarrow w^{-1}(R)\in J\wedge w^{-1}(S)\in J \qquad (5.1.2.e) \\ &\leftrightarrow w\in j_J(R)\wedge w\in j_J(S) \\ &\leftrightarrow w\in j_J(R)\cap j_J(S). \end{aligned}$$

(d') Assume for $S_1 \ldots S_n\in\Omega_{\mathbf{T}}$ such that $j_J(S_i)=F\mathbb{1}$ $(1\leq i\leq n)$.
Then it holds that $*^{-1}(S_i)\in J$, or simply $S_i\in J$ for $(1\leq i\leq n)$. Now for $w\in Fn$ it follows that $w^{-1}(S_1\oplus\ldots\oplus S_n)\in J$.
Hence $j_J(w^{-1}(S_1\oplus\ldots\oplus S_n))=F\mathbb{1}$ by (5.2.1.b).

Ad (b).
For $j\in LT\text{-}TOP$ and $R\in\Omega_{\mathbf{T}}$ we have

$$\begin{aligned} j_{(J_j)}(R) &= \{w\in F\mathbb{1} \mid w^{-1}(R)\in J_j\} \\ &= \{w\in F\mathbb{1} \mid w^{-1}(R)\in\{R\in\Omega_{\mathbf{T}} \mid jR=F\mathbb{1}\}\} \\ &= \{w\in F\mathbb{1} \mid jw^{-1}(R)=F\mathbb{1}\} \\ &= \{w\in F\mathbb{1} \mid *\in jw^{-1}(R)\} \\ &= \{w\in F\mathbb{1} \mid *\in w^{-1}j(R)\} \\ &= \{w\in F\mathbb{1} \mid w\in jR\} \\ &= jR. \end{aligned}$$

And for $J\in GG\text{-}TOP$ we have

$$\begin{aligned} J_{(j_J)} &= \{R\in\Omega_{\mathbf{T}} \mid j_J(R)=F\mathbb{1}\} \\ &= \{R\in\Omega_{\mathbf{T}} \mid \{w\in F\mathbb{1} \mid w^{-1}(R)\in J\}=F\mathbb{1}\} \\ &= \{R\in\Omega_{\mathbf{T}} \mid *\in\{w\in F\mathbb{1} \mid w^{-1}(R)\in J\}\} \\ &= \{R\in\Omega_{\mathbf{T}} \mid *^{-1}(R)\in J\} \\ &= \{R\in\Omega_{\mathbf{T}} \mid R\in J\} \\ &= J. \end{aligned}$$

□

5.1.8 Corollary
There is a 1-1 correspondence between the global elements of $\{j:\Omega_{\mathbb{T}}\to\Omega_{\mathbb{T}} \mid j$ is Lawvere-Tierney topology$\}$, the global elements of $\{J\subseteq\Omega_{\mathbb{T}}\to\Omega_{\mathbb{T}} \mid j$ is a Gabriel-Grothendieck topology$\}$ in a topos $\mathbb{E}$ and the universal closure operations on subcategory of algebras in $\mathbb{E}$.

□

5.2 Commutative and Semi-commutative algebras

In theorem (4.2.5) we encountered algebras in which unary operations commute with all operators of arbitrary arity of the algebraic theory. [Linton] considered algebras in which all operations commute with each other.

5.2.1 Definition. Let $\mathbb{T}$ be an algebraic theory.
(i) $\mathbb{T}$ is called *semi-commutative* if unary algebraic operations commute with n-ary algebraic operators, that is, if equations of the following kind are implied by the equational axioms of the theory $\mathbb{T}$:

$$p(w(x_1),\ldots,w(x_n))=w(p(x_1,\ldots,x_n))$$

where p is an n-ary operator and w a unary operation in $F\mathbb{1}$.
(ii) $\mathbb{T}$ is called *commutative* if operations of arbitrary arity commute with each other, that is, if equations of the following kind are implied by the equational axioms of the theory $\mathbb{T}$:

$$v(w(x_{11},\ldots x_{1m}),\ldots,w(x_{n1},\ldots x_{nm}))=w(v(x_{11},\ldots,x_{n1}),\ldots,v(x_{1m},\ldots,x_{nm}))$$

where $v\in Fn$ is an n-ary operation and $w\in Fm$ is an m-ary operation in $F\mathbb{1}$.

It follows that if $\mathbb{T}$ is a semi-commutative algebraic theory, then unary algebraic operations commute with *operations* of arbitrary arity.

Under the stronger condition of semi-commutativity the definitions of Lawvere-Tierney topology and Gabriel-Grothendieck can be weakened. Note that the basic notion of universal closure can not be weakened in a similar way.

5.2.2 Theorem. Let $\mathbb{T}$ be a semi-commutative theory.
(i) $j:\Omega_{\mathbb{T}}\to\Omega_{\mathbb{T}}$ is a Lawvere-Tierney topology, if and only if j satisfies:
  (a) $jF\mathbb{1}=F\mathbb{1}$
  (b) $\forall R\in\Omega_{\mathbb{T}}\ jjR=jR$

(c) $\forall R,S\in\Omega_{\mathbf{T}}\ j(R\cap S)=jR\cap jS$
(d) $\forall R\in\Omega_{\mathbf{T}}\ \forall f:F\mathbb{1}\rightarrow F\mathbb{1}\ jf^{-1}(R)=f^{-1}(jR)$.

(ii) $J\subseteq\Omega_{\mathbf{T}}$ is a Gabriel-Grothendieck topology, if and only if J satisfies:
(a) $F\mathbb{1}\in J$
(b) $\forall R\in J\ \forall S\in\Omega_{\mathbf{T}}\ [(\forall w\in R\ w^{-1}(S)\in J)\rightarrow S\in J]$
(c) $\forall S\in J\ \forall w\in F\mathbb{1}\ w^{-1}(S)\in J$

The following conditions can be derived from (ii)(a,b,c):
(d) $\forall R\in J\ \forall R\hookrightarrow S\ S\in J$
(e) $\forall R,S\in\Omega_{\mathbf{T}}\ [(R\cap S)\in J\leftrightarrow(R\in J\wedge S\in J)]$
(f) $\forall R\in J\ \forall S\in\Omega_{\mathbf{T}}\ [(\forall f:\Omega_{\mathbf{T}}\rightarrow\Omega_{\mathbf{T}}\ f^{-1}(R\cap S)\in J)\rightarrow S\in J]$

Proof.
(i) (Only if) is trivial. For (if), we give a similar proof as in [Borceux and Veit 86]. It suffice to show that we can derive

$$\forall S_1\ldots S_n\in\Omega_{\mathbf{T}}\forall w\in Fn\ [(jS_1=F\mathbb{1}\wedge\ldots\wedge jS_n=F\mathbb{1})\rightarrow jw^{-1}(S_1\oplus\ldots\oplus S_n)=F\mathbb{1}]$$

from (i) (a,b,c,d) under the assumption of semi-commutativity.
Assume for $S_1\ldots S_n\in\Omega_{\mathbf{T}}$ and $w\in Fn$ that $jS_1=F\mathbb{1}\wedge\ldots\wedge jS_n=F\mathbb{1}$.
Consider $w^{-1}(S_1\oplus\ldots\oplus S_n)$.

$$\begin{aligned} w^{-1}(S_1\oplus\ldots\oplus S_n) &= \{v\in F\mathbb{1}\mid v(w)\in S_1\oplus\ldots\oplus S_n\}\\ &= \{v\in F\mathbb{1}\mid w(v(*_1),\ldots v(*_n))\in S_1\oplus\ldots\oplus S_n\}\\ &\supseteq \{v\in F\mathbb{1}\mid v\in S_1\wedge\ldots\wedge v\in S_n\}\\ &\supseteq \cap_{1\le i\le n}S_i \end{aligned}$$

Hence $F\mathbb{1}\subseteq\cap_{1\le i\le n}jS_i\subseteq j\cap_{1\le i\le n}S_i\subseteq jw^{-1}(S_1\oplus\ldots\oplus S_n)\subseteq F\mathbb{1}$.
Therefore $jw^{-1}(S_1\oplus\ldots\oplus S_n)=F\mathbb{1}$.
(ii) A similar arguments applies.

□

5.3 Sheaves for a Gabriel-Grothendieck topology

In this section and the following a generalization of the Grothendieck associated sheaf constructions will be presented for *commutative* $\mathbf{T}$-algebras with respect to a Gabriel-Grothendieck topology on the subobject classifier $\Omega_{\mathbf{T}}$.

Needed for this algebraic approach is a generalized formulation of the notions involved apt for this particular situation: singleton, separated, closed, dense, sheaf etc. The general definitions should collapses to the usual meaning in case of the *trivial* algebra without symbols and defining axioms. Note that in case of the trivial algebra $F\mathbb{1}$ is isomorphic to $\mathbb{1}$ and hence $\Omega_{\mathbf{T}}$ is isomorphic to Ω. A Gabriel-Grothendieck topology on $\Omega_{\mathbf{T}}$ trivializes to a Grothendieck topology on Ω.

The usual classical notion of singleton on A is a subset of A containing exactly one element. Equivalently, but from a different perspective it is a morphism $\mathbb{1}\to A$. In the intuitionistic setting of chapter 3 we have seen a large family of singleton notions. Let us focus on what is called a vii-singleton, that is a subset $S\subseteq A$ such that:

(i) $\forall x,y\in A\ (x\in S\wedge y\in S\to x=y)$

(ii) $j\exists x\in A\ x\in S$.

Observe that vii-singletons are in one-one correspondence to functions $\mathbb{1}\restriction\omega\to A$ for truth values $\omega\in J_j$. In the present set up the subalgebras of $F\mathbb{1}$ have taken over the role of truth values. An element $a\in A$ defines a morphism $F\mathbb{1}\to A{:}w\mapsto w(a)$. So the counterpart of a function $\mathbb{1}\restriction\omega\to A$ with $\omega\in J_j$ is now a morphism $R\to A$ with $R\in J$.

5.3.1 Definition. Let $J{:}\Omega_{\mathbb{T}}\to\Omega_{\mathbb{T}}$ be a Gabriel-Grothendieck topology.

(i) A *singleton* on A is just a morphism $S\to A$, where $S\in J$.

(ii) Let $f{:}R\to A$ and $g{:}S\to A$ be singletons on A. Then we define $f\approx g$ if $\forall w\in R\cap S\ f(w)=g(w)$.

From now on we suppose a Gabriel Grothendieck topology $J\subseteq\Omega_{\mathbf{T}}$ to be given. The usual definitions of dense morphism, separated object and sheaf can be generalized to the present algebraic context.

5.3.2 Definition.

A monomorphism $m{:}B\rightarrowtail A$ is called dense if $\forall f{:}F\mathbb{1}\to A\ f^{-1}(B)\in J$.

The following lemma contains some useful properties of dense morphisms.

5.3.3 Lemma.

(i) A subalgebra $S\rightarrowtail F\mathbb{1}$ is dense if and only if $S\in J$.

(ii) The pullback of a dense morphism is again dense.

(iii) The monomorphisms $g{:}C\rightarrowtail B$ and $h{:}B\rightarrowtail A$ are dense if and only if the composite $g\circ f$ is dense.

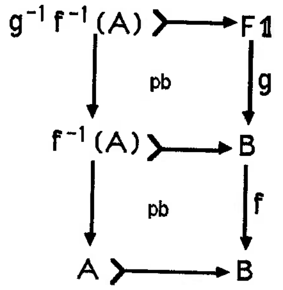
Proof.

(i) If $S \in J$, then $jS = F\mathbb{1}$. Consider $f: F\mathbb{1} \to F\mathbb{1}$.
Since

$$jf^{-1}(S) = f^{-1}(jS) = f^{-1}(F\mathbb{1}) = F\mathbb{1},$$

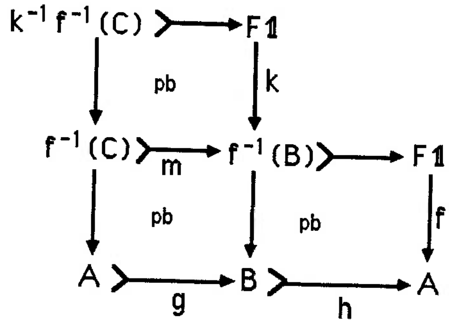
we get $f^{-1}(S) \in J$. Thus, $S \rightarrowtail F\mathbb{1}$ is a dense morphism. The reverse is trivial.

(ii) Let $A \rightarrowtail B$ be a dense morphism. Consider the pullback along $f: C \to B$. In order to prove that $f^{-1}(A) \rightarrowtail C$ is dense, we consider a morphism $g: F\mathbb{1} \to C$ and take another pullback:



As both squares are pullbacks, also the outer rectangle is a pullback.
Hence $g^{-1}f^{-1}(A) = (fg)^{-1}(A) \in J$. By (5.1.2.b) it follows that $f^{-1}(A) \to C$ is dense.

(iii) (only if) Let $g: C \to B$ and $h: B \to A$ be dense morphisms. We have to show that for each $f: F\mathbb{1} \to A$ the subalgebra $f^{-1}(C)$ is an element of J.
Let $k: F\mathbb{1} \to f^{-1}(B)$ be a morphism and consider the following diagram constructed by taking successive pullbacks as indicated in the squares of the diagram.



Note that by construction $m: f^{-1}(C) \to f^{-1}(B)$ is a genuine embedding.
Now $h: B \rightarrowtail A$ is dense implies $f^{-1}(B) \in J$. And $g: C \rightarrowtail B$ is dense implies with help of the second item of this lemma that $k^{-1}(f^{-1}(C)) \rightarrowtail F\mathbb{1}$ is also dense. But by the first item of this lemma this means that $k^{-1}(f^{-1}(C)) \in J$.
Hence $k^{-1}(f^{-1}(B) \cap f^{-1}(C)) = k^{-1}(f^{-1}(C)) \in J$.
It finally follows by (5.1.2.b) that $f^{-1}(C) \in J$.
(if) Suppose $g \circ h: C \rightarrowtail B \rightarrowtail A$ is a dense morphism. Then for all $f: F\mathbb{1} \to A$ we have $f^{-1}(C) \in J$. Since $f^{-1}(C) \subseteq f^{-1}(B)$ we get $f^{-1}(B) \in J$ by (5.1.2.c). Therefore $h: B \to A$ is dense.

Next, consider $k: F1 \to B$, and $hok: F1 \to A$. Then $k^{-1}(C)=(hk)^{-1}(C) \in J$, i.e. $g: B \to A$ is dense as well.

□

5.3.4 Definition.

(iii) An algebra A is J-separated if $\forall f,g: F1 \to A \ \forall R \in J \ [f{\restriction}R = g{\restriction}R \to f=g]$.

(iv) An algebra A is a J-sheaf if $\forall R \in J \ \forall f: R \to A \ \exists! g: F1 \to A \ g{\restriction}R = f$.

Observe that if the algebraic theory $\mathbb{T}$ is empty, i.e, when there are no functions and axioms, then an algebra is just a type like all others, and our new notions reduce to the old ones. The construction of the associated sheaf à la Johnstone can now proceed along similar lines as in chapter 3.

5.4 Associated sheaves à la Grothendieck.

We will now give the internal account of a Grothendieck construction for algebras along the lines set out in [Borceux and Van den Bossche 84]. Note, as Jaap Vermeulen pointed out to me, that only the very first step of the construction depends on the full commutativity of the algebraic theory. Elsewhere in the construction and proofs semi-commutativity suffices.

[Borceux and Van den Bossche 84] constructed associated sheaves with respect to a given Gabriel-Grothendieck topology in the special case of commutative algebras in toposes of sheaves $sh(\mathbb{H})$ on a complete Heyting algebra $\mathbb{H}$. Our internal description of the construction works for any elementary topos and circumvents the special properties of that particular topos which obscured their construction.

5.4.1 Definition.

(i) $\lambda A := \{f: S \to A \mid f \text{ is a singleton on } A\}/\approx$.

(ii) $\lambda: A \to \lambda A$ is the function that maps $a \in A$ to the equivalence class of singletons represented by $F1 \to A: w \mapsto w(a)$.

Note. For sheaves over a frame $\mathbb{H}$ this definition is exactly the internal content of the external notion of [Borceux and Van den Bossche 84] pointwise defined by:

$$\lambda A(u) := \varinjlim_{S \in J(u)} \hom(S, A), \quad \text{for } u \in \mathbb{H}$$

5.4.2 Lemma. Let A be an commutative algebra.
(i) λA is an algebra
(ii) $\lambda: A \to \lambda A$ is a morphism.

Proof. (i) Let $p: A^n \to A$ be an algebraic operator. We define $\lambda p: (\lambda A)^n \to \lambda A$ by defining its behavior on representatives: λp maps $(\bar{g}_1, \ldots, \bar{g}_n) \in (\lambda A)^n$ to the singleton represented by

$$\cap_{1 \le i \le n} \mathrm{dom}(g_i) \to A : w \mapsto p(g_1(w), \ldots, g_n(w)),$$

where $g_1, \ldots, g_n$ are representatives of respectively $\bar{g}_1, \ldots, \bar{g}_n$.
This definition is independent of representatives. $\lambda p: \cap_{1 \le i \le n} \mathrm{dom}(g_i) \to A$ should be an morphism as well.
Now for q an m-ary operator, and $v_1, \ldots, v_m \in \cap_{1 \le i \le n} \mathrm{dom}(g_i)$, we have

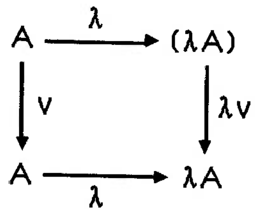
$$\begin{aligned} q(\lambda p(v_1), \ldots, \lambda pf(v_m)) &= q(p(g_1(v_1), \ldots, g_n(v_1)), \ldots, p(g_1(v_m), \ldots, g_n(v_m))) \\ &= p(q(g_1(v_1), \ldots, g_1(v_m)), \ldots, q(g_n(v_1), \ldots, g_n(v_m))) \\ &= p(g_1(q(v_1, \ldots v_m)), \ldots, g_n(q(v_1, \ldots v_m))) \\ &= \lambda p(q(v_1, \ldots v_m)) \end{aligned}$$

The second equation is of the form

$$q(p(v_{11}, \ldots, v_{1n}), \ldots, p(v_{m1}, \ldots, v_{mn})) = p(q(v_{11}, \ldots, v_{m1}), \ldots, q(v_{m1}, \ldots, v_{mn}))$$

and therefor depends on the commutativity of $\mathbb{T}$. Similarly the third equation is an instance of semi-commutativity.
Finally to check that that λA indeed is an algebra easy.

(ii) Let v be a n-ary algebraic operation. We have to show that the following diagram commutes:



Let $a_1, \ldots, a_n \in A$. Since $\lambda \circ v(a_1, \ldots a_n)$ is represented by $F\mathbb{1} \to A: w \mapsto w(v(a_1, \ldots, a_n))$ and $v(\lambda(a_1), \ldots, \lambda(a_n))$ is represented by $F\mathbb{1} \to A: w \mapsto v(w(a_1), \ldots w(a_n))$, the commuting of the diagram follows immediately from the assumption of the lemma.

□

The next lemma will be helpful in the proof that $\lambda\lambda A$ is a sheaf.

5.4.3 Lemma. Let A be an algebra such that unary operators on A commute with each other and let $f: F\mathbb{1} \to \lambda A$ be a morphism. Then $f_*: R \to A$ represents $f(*) \in \lambda A$ if and only if $\lambda \circ f_* = f \restriction R$.

Proof. (only if) Let $f_*: R \to A$ represent $f(*)$ and let $v \in R$. We will show $\lambda o f_*(v) = f(v)$. Observe that $\lambda o f_*(v)$ is represented by $F\mathbb{1} \to A: w \mapsto w(f_*(v))$. Hence $R \to A: w \mapsto w(f_*(v))$ is also a representative of $\lambda o f_*(v)$. However $f(v) = f(v(*)) = vf(*)$ and hence $f(v)$ is represented by $R \to A: w \mapsto v(f_*(w))$. By semi-commutativity we get $vf_*(w) = f_*(v(w)) = f_* w(v)) = wf_*(v)$. This implies that the two representatives coincide above $R \in J$. Hence $\lambda o f_*(v) = f(v)$.

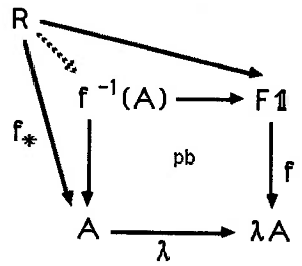
(if) $f(*)$ is represented by $R \to A: w \mapsto f_*(w)$. Also $\lambda o f_*(*)$ is represented by $F\mathbb{1} \to A$: $w \mapsto w(f_*(*))$. But $w(f_*(*)) = f_*(w(*)) = f_*(w)$, so $f(*)$ is represented by f_*.

□

5.4.4 Lemma. Let A be an algebra such that unary operators on A commute with each other. If A is J-separated, then $\lambda: A \to \lambda A$ is a dense morphism.

Proof. It follows from the separatedness of A that $\lambda: A \to \lambda A$ is a monomorphism.

Let $f: F\mathbb{1} \to \lambda A$ be some morphism and let $f_*: R \to A$ be a representative of $f(*)$ in λA. Set $f^{-1}(A) = \{w \in F\mathbb{1} \mid \exists a \in A\ \lambda(a) = f(w)\}$. Consider the following diagram:

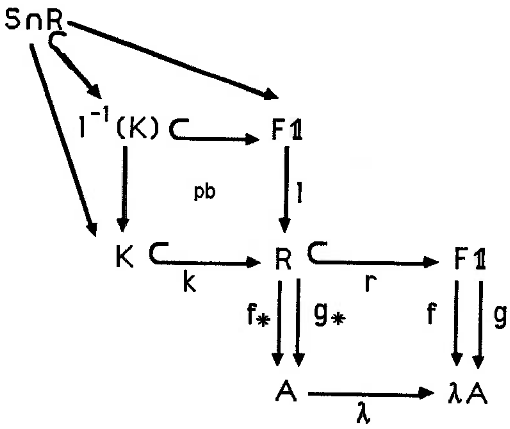


The innersquare is a pullback as $\lambda: A \to \lambda A$ is a mono. Because of lemma 5.4.3 the outersquare commutes, hence $R \rightarrowtail f^{-1}(A)$. Since $R \in J$ we get $f^{-1}(A) \in J$.

□

5.4.5 Theorem. Let A be an algebra such that unary operators on A commute with each other. Then λA is separated.

Proof. Let $f,g:F\mathbb{1}\to\lambda A$ be such that for some $R\in J$ we have $f{\restriction}R=g{\restriction}R$. If we are able to prove that $f(*)=g(*)$ then $f=g$ follows easily. So it suffices to prove that representatives of $f(*)$ and $g(*)$ are $\approx$-equivalent.
Let f_* and g_* represent respectively $f(*)$ and $g(*)$. For simplicity assume that the domains of f_* and g_* are equal to R. If not so, restrict the following argument to $R_*:=R\cap dom(f_*)\cap dom(g_*)$ which belongs to J.
Let $K:=\{w\in R\,|\,f_*(w)=g_*(w)\}$ be the equalizer of f_* and g_*. In order to show that $f_*\approx g_*$ it suffices to prove that $K\in J$, or by lemma (5.3.3) to prove that $K\hookrightarrow F\mathbb{1}$ is dense. That is, for all $l:F\mathbb{1}\to R$ we have to show that $l^{-1}(K)\in J$.
Consider the following diagram:

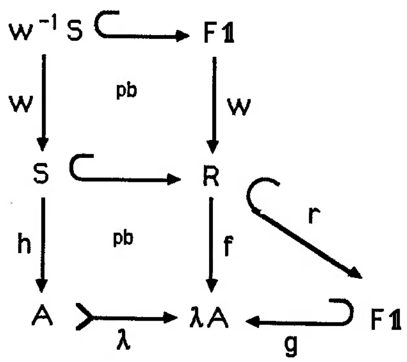


Because of lemma (5.4.3) we have $\lambda f_*l=frl=grl=\lambda g_*l$.
It follows from (5.4.3) that $lf_*\approx lg_*$.
Hence there is $S\in J$ such that $lf_*{\restriction}S=lg_*{\restriction}S$. Now $S\cap R\in J$.
By construction of the equalizer K we get the factorization of $S\cap R\hookrightarrow F\mathbb{1}\to R$ by $k:K\hookrightarrow R$.
Using the pullback property we find that $S\cap R\subseteq l^{-1}(K)$. Hence $l^{-1}(K)\in J$, and we are done.

□

5.4.6 Theorem. Let A be an algebra such that unary operators on A commute with each other. Then
(i) if A is a separated algebra, then λA is a sheaf,
(ii) $\lambda\lambda A$ is a sheaf.

Proof. (ii) is a trivial consequence of part (i) and theorem (5.4.5).
(i) Let $R\in J$ and let $f:R\to\lambda A$ be a morphism. If we are able to construct a morphism $g:F\mathbb{1}\to\lambda A$ such that $g{\restriction}R=f$, then it follows from the separatedness of λA that g is the only one such. The following diagram illustrates the construction:

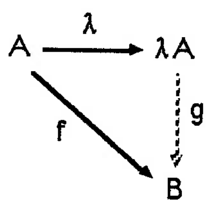


Let S be the pullback of f and λ. I.e., $S=\{w\in R \mid \exists a\in A\ \lambda(a)=f(w)\}$. Because λ is dense (lemma 5.4.4) and the preservation of density by pullbacks (lemma 5.3.3) we get $S\in J$.
To construct g it suffices to give the value of g on $*$, as $g(w) = gw(*) = wg(*)$ for $w\in R$. We define $g(*)$ to be the equivalence class represented by $f\restriction S$.
Recall that a word $w\in R$ defines a morphism $F\mathbb{1}\to R: v\mapsto w[*{=}v]$, that we also denote by w. We take the pullback $w^{-1}(S)=\{v\in F\mathbb{1} \mid w(v)\in S\}$. Also $w^{-1}(S)\in J$ as $w^{-1}(S)$ is the pullback of a dense morphism. By lemma (5.4.3) we see that $f(w)\in\lambda$ is represented by $w\circ h: w^{-1}(S)\to S\to A$. Similarly we get that $gr(w) = g(w) = gw(*) = wg(*)$ is represented by $w\circ h: S\to A\to A$, where again by abuse of notation $w: A\to A$ is the morphism $a\mapsto w[*{=}a]$. If we consider finally a word $v\in w^{-1}(S)$ we see that $hw(v)=wh(v)$. Hence $wh: S\to A$ and $hw: w^{-1}(S)\to A$ coincide on $w^{-1}(S)\in J$. Therefore $gr(w)=f(w)$, and so we get $gr=f$, as desired.

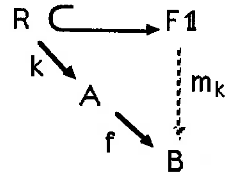
□

5.4.7 Theorem. Let A and B be *semi-commutative* algebras and B a sheaf. Let $f: A\to B$ be a morphism. Then f factors uniquely through $\lambda: A\to\lambda A$.

Proof. Let A, B and $f: A\to B$ be as assumed in the theorem. We have to construct $g:\lambda A\to B$ such that the following diagram commutes.



Let $k:R \hookrightarrow A$ be a representative for some arbitrary equivalence class $\overline{k} \in \lambda A$. Then $R \in J$. Because of the assumption that B a sheaf is, there is a unique $m_k: F\mathbb{1} \to B$ such that the following diagram commutes:



Now we define $g: \lambda A \to B: \overline{k} \mapsto m_k(*)$. This definition is independent of the choice of representative k. We have $g \circ \lambda(a) = g(\lambda(a)) = m_{F\mathbb{1} \to A: w \mapsto w(a)}(*) = f(a)$ for $a \in A$. Unicity of g follows from the unicity of m_k.

□

5.4.8 Theorem. If A be sheaf, then $\lambda: A \to \lambda A$ is an isomorphism.

Proof. Let A be a sheaf. Hence, it is separated and therefore $\lambda: A \to \lambda A$ is a monomorphism. Remains to prove that $\lambda: A \to \lambda A$ is an epimorphism. Let $\overline{k}$ be an equivalence class in λA with representative $k: R \to A$ for some $R \in J$. A is a sheaf, hence k extends uniquely to $l: F\mathbb{1} \to A$ by the previous lemma (5.4.7). But then $\lambda(l(*)) = \overline{k}$. And so we see that λ is an epimorphism.

□

5.4.9 Theorem. For a commutative algebraic theory, the above construction λ preserves products and equalizers of algebras.

Proof. Let A, B be algebras and let $A \times B$ be their (usual) product. We will show that the algebra $\lambda A \times \lambda B$ is isomorphic with the algebra $\lambda(A \times B)$.
Let us start with an element $<\overline{f}, \overline{g}> \in \lambda A \times \lambda B$. And let $f: R \to A$ and $g: S \to A$ be representatives of respectively $\overline{f} \in \lambda A$ and $\overline{g} \in \lambda B$.
Then the function $R \cap S \to A \times B: w \mapsto <f(w), g(w)>$ is representative of an element of $\lambda(A \times B)$.
The other way round, if we have an element $\overline{h} \in \lambda(A \times B)$ then we extract from a representative $h: R \to A \times B$ of $\overline{h}$ the obvious functions $f: R \to A: w \mapsto \pi_1(h(w))$ and $g: R \to A: w \mapsto \pi_2(h(w))$ with corresponding equivalence classes $\overline{f} \in \lambda A$ and $\overline{g} \in \lambda B$.
And so we get an element $<\overline{f}, \overline{g}> \in \lambda A \times \lambda B$.
It is easy to see that this describes two morphisms between $\lambda A \times \lambda B$ and $\lambda(A \times B)$ that are each others inverse.
For equalizers a similar argument works.

□

If we can summarize the above construction, for A,B be algebras and $f:A \to B$ a morphism we defined:

(i) $\lambda A = \{S \to A \mid S \text{ is a J-singleton of } A\}/\approx$
(ii) $\lambda f = \lambda A \to \lambda B: [g:S \to A] \mapsto [S \to A: w \mapsto f(g(w))]$
(iii) $L_J A = \lambda(\lambda A)$
(iv) $\gamma_A = A \to \lambda A: a \mapsto [F\mathbb{1} \to A: w \mapsto w(a)]$
(v) $\eta = \gamma_{\lambda A} \circ \gamma_A : A \to LA.$

And we have proved for it the following:

5.4.10 Corollary. If $\mathbb{T}$ is an commutative algebraic theory, then for an algebra A we have

(i) λA is a J-separated algebra,
(ii) if A is J-separated, then λA is a sheaf,
(iii) if A is a J-sheaf, then λA is isomorphic with A.

□

If we have an internal algebraic theory $\mathbb{T}$ in a topos $\mathbb{E}$, then we can look at the category $\mathbb{E}_{\mathbb{T}}$ of $\mathbb{T}$-algebras in $\mathbb{E}$. For a subobject $J \subseteq \Omega_{\mathbb{T}}$ that is a Gabriel-Grothendieck topology we can also consider the category $sh_J(\mathbb{E}_{\mathbb{T}})$ of J-sheaves in $\mathbb{E}_{\mathbb{T}}$.

5.4.11 Theorem. In the case of a commutative algebra $\mathbb{T}$ it holds that

(i) $L_J: \mathbb{E}_{\mathbb{T}} \to sh_J(\mathbb{E}_{\mathbb{T}})$ is left exact left adjoint of the inclusion $i: sh_J(\mathbb{E}_{\mathbb{T}}) \hookrightarrow \mathbb{E}_{\mathbb{T}}$ corresponding to the Gabriel-Grothendieck topology J.

□

Chapter 6

Generalized Gödel-translations

It is an old result of Gödel that classical first order arithmetic can be embedded in intuitionistic arithmetic via a so-called negative translation (cf. [Gödel] and the accompanying annotation by Troelstra in the same volume.) Similar translations have been made for other systems as well: second order Heyting arithmetic, type theory (not to be confused with the - let us say - topos type theory of this thesis) and set theory (cf. [Friedman], Troelstra's annotation in [Gödel], [Leivant] and [Troelstra and van Dalen]).

Two facts from topos theory hint at a generalization of Gödel's negative translation for type theories:

(i) toposes are in a natural sense nothing but type theories (cf. for instance [Lambek and Scott])
(ii) in each topos $\mathbb{E}$ a boolean topos $\mathcal{Sh}_{\neg\neg}\mathbb{E}$ is included via the geometric morphism $L_{\neg\neg}:\mathcal{Sh}_{\neg\neg}\mathbb{E}\rightarrow\mathbb{E}$ determined by the double negation topology (cf. for instance [Johnstone 77]).

So, we want a translation $(-)^G:L_H\rightarrow L_H$ for a type theory H with language L_H such that:

$$H+\text{Principle of Excluded Middle} \vdash \phi \;\Leftrightarrow\; H\vdash\phi^G.$$

together with a semantical proof in the following style:

"$\Rightarrow$" Assume $H \nvdash \phi^G$. Using completeness, there is a topos $\mathbb{E}$ modelling H and invalidating ϕ^G. Construct $\mathcal{Sh}_{\neg\neg}\mathbb{E}$. If $\mathcal{Sh}_{\neg\neg}\mathbb{E}$ models H and invalidates ϕ we can conclude $H+PEM \nvdash \phi$.
"$\Leftarrow$" In the presence of PEM the G-translation trivializes to the identity on L_H.

We have to take care of three steps in the proof of "$\Rightarrow$":

(i) Given an interpretation $\models$ of H in $\mathbb{E}$ we have to define an interpretation $\models_j$ of H in $\mathcal{Sh}_j\mathbb{E}$, that will satisfy:

(ii) $\mathbb{E}\not\models\phi^G \;\Rightarrow\; \mathcal{Sh}_{\neg\neg}\mathbb{E}\not\models_{\neg\neg}\phi$,

(iii) $\mathbb{E} \models H \Rightarrow \mathcal{Sh}_{\neg\neg}\mathbb{E} \models_{\neg\neg} H.$

To define an interpretation of a language in a topos, it suffices to indicate how the basic types and the functions have to be interpreted.
With respect to types it is easy to make the first step. A basic type A that in $\mathbb{E}$ is interpreted by $[\![A]\!]$ will be interpreted in $\mathcal{Sh}_j\mathbb{E}$ by its sheafification $[\![LA]\!]$. It follows that, for example, a type $A\times[B]$ with basic types A and B will then be interpreted in $\mathcal{Sh}_j\mathbb{E}$ by $[\![A]\!]\times\Omega_j^{[\![B]\!]}$. Care is needed for functions when domain or codomain are no basic types.

The observation that $\mathcal{Sh}_j\mathbb{E}$ is a subcategory of $\mathbb{E}$, can be made more precise by a translation $(-)^j$ of $L_{\mathcal{Sh}_j\mathbb{E}}$ in $L_{\mathbb{E}}$ with the property

$$\mathcal{Sh}_j\mathbb{E}\models\phi \Leftrightarrow \mathbb{E}\models\phi^j$$

Although for the topos theorist this translation may seem to go in the wrong direction, there are important instances where the topos of sheaves is well-known, in contrast to the base topos one started off from, see for example [Hyland 82]'s effective topos.

Now the second step can be taken. If we j-translate the just constructed interpretation of H in $\mathcal{Sh}_j\mathbb{E}$ we get the first extension of the Gödel-translation for an arbitrary Lawvere Tierney topology j. In case of the double negation topology this is a generalization of the original Gödel-translation to higher-order type theories. In case of open and closed topologies:

$$j_p:\Omega\rightarrow\Omega:\omega\mapsto p\rightarrow\omega$$
$$j^p:\Omega\rightarrow\Omega:\omega\mapsto p\vee\omega$$

we get generalizations of the Friedman-translations (cf. [Friedman]). Such an observation has been made for intuitionistic predicate logic by [de Jongh].

The last step can only be made for type theories H that satisfy $H \vdash H^G$, that is, type theories H that proves its own Gödel translation H^G. For example, in the case of higher-order Heyting arithmetic **HHA** $\vdash$ **HHA**^{G} will be a logical reformulation of the categorical preservation of a natural number object by a left exact functor like the associated sheaf functor (cf. [Freyd], [Johnstone 77]). Other examples we will discuss are geometric theories (Joyal and Reyes, cf. [Johnstone 77] and [Makkai and Reyes] and existential fixed point logic (cf. [Blass] and [Blass and Gurevich]).

Note that there is no need to work with axiom schemes because the intuitionistic higher-order logics we work with have an object of truth values over which we quantify. That means that we only need to investigate properties like $\phi \vdash \phi^G$ for formulas and not for schemes, as in [Friedman] and [Leivant].

The G-translation constructed by the foregoing recipe enjoys the pleasant property that it translates $\in$ and $=$ by $\in$ and $=$, and preserves the extensionality of $\in$, i.e., the relationship $\forall x,y \in [A](x=y \leftrightarrow \forall z \in A[z \in x \leftrightarrow z \in y])$ for any type A. The penalty is that the types are translated into subtypes of types of a higher complexity.

We will end this chapter with another Gödel-translation. This g-translation can be seen as the G-translation followed by a further translation. In contrast to the G-translation it has the good property that the types are preserved, and the bad property that the translation of the predicates $=$ and $\in$ becomes complicated.

By now we have promised to introduce three translations (respectively called j, G and g). We will begin this chapter with the j-translation from $\mathcal{Sh}_j\mathbb{E}$ into $\mathbb{E}$.

6.1 Translating from $\mathcal{Sh}_j\mathbb{E}$ into $\mathbb{E}$

For a topos $\mathbb{E}$ and topology $j: \Omega \rightarrow \Omega$ in $\mathbb{E}$ we will translate types and terms of the canonical language of $\mathcal{Sh}_j\mathbb{E}$ into types and terms of the canonical language $L_{\mathbb{E}}$ of $\mathbb{E}$. Let us denote by $[\![\,]\!]^j$ the canonical interpretation of $L_{\mathcal{Sh}_j\mathbb{E}}$ in $\mathcal{Sh}_j\mathbb{E}$ to be distinguished from the canonical interpretation $[\![\,]\!]$ of $L_{\mathbb{E}}$ in $\mathbb{E}$.

The basic types of $L_{\mathcal{Sh}_j\mathbb{E}}$ are also basic objects in $L_{\mathbb{E}}$. Power types are constructed differently. Inside $\mathcal{Sh}_j\mathbb{E}$ the power $P(A)$ of a basic type A is just the type of subtypes of A, but from the point of view of $\mathbb{E}$, it is $\Omega_j{}^A$, the type of j-closed subtypes (cf. (2.3.5-6)) of A. This can be generalized to arbitrary types:

6.1.1 Definition.
Define by induction on the type A of $L_{\mathbb{E}}$ a (sub)type A_j in L_E:

(i) $A_j{:=}A$, for basic types
(ii) $(\Omega)_j{=}(P\mathbb{1})_j{:=}\Omega_j{=}\{\omega \in \Omega \mid j\omega\}{=}\{\omega \in \Omega \mid \omega{=}j\omega\}$
(iii) $(PA)_j := \Omega_j{}^{A_j} {=} \{B \subseteq A_j \mid B \text{ is j-closed}\}$
(iv) $(A \times B)_j {:=} A_j \times B_j$
(v) $(B^A)_j := B_j{}^{A_j}$

For example, compare the types $P(A\times PB)$ and $(P(A\times PB))_j$. They are equal to respectively $\Omega^{A\times\Omega^B}$ and $\Omega_j^{A\times\Omega_j^B}$.

6.1.2 Lemma. For each type A of $L_{\mathcal{Sh}_j\mathbf{E}}$ we have $[\![A]\!]^j=[\![A_j]\!]$.

Proof. Trivial induction on the structure of the type.

□

6.1.3 Definition. For types A in $L_{\mathbf{E}}$ built up from basic types that are sheaves, we define $e_A:A_j\hookrightarrow A$ together with $d_A:A\rightarrow A_j$ by induction on A:

(i) $e_A:A_j\hookrightarrow A=id_A$ for basic sheaf types and $\mathbb{1}$
(ii) $e_\Omega:\Omega_j\hookrightarrow\Omega:\omega\mapsto\omega$
(iii) $e_{\Omega^A}:\Omega_j^{A_j}\hookrightarrow\Omega^A:B\mapsto B$
(iv) $e_{A\times B}:A_j\times B_j\hookrightarrow A\times B:\langle a,b\rangle\mapsto\langle e_A(a),e_B(b)\rangle$
(v) $e_{B^A}:B_j^{A_j}\hookrightarrow B^A:f\mapsto d_B\circ f\circ e_A$.

and

(vi) $d_A:A\rightarrow A_j=id_A$ for basic sheaf types and $\mathbb{1}$
(vii) $d_\Omega:\Omega\rightarrow\Omega_j:\omega\mapsto j\omega$
(viii) $d_{\Omega^A}:\Omega^A\rightarrow\Omega_j^{A_j}:B\subseteq A\mapsto\eta^{-1}L_j\{d_A(b)\in A_j\mid b\in B\}$
(ix) $d_{A\times B}:A\times B\rightarrow A_j\times B_j:\langle a,b\rangle\mapsto\langle d_A(a),d_B(b)\rangle$
(x) $d_{B^A}:B^A\hookrightarrow B_j^{A_j}:f\mapsto d_B\circ f\circ e_A$.

Note that e_A and d_A is well-defined. We check (iii) and (viii) for example.
Ad (iii). Observe that if $B\in\Omega_j^{A_j}$, that is $B\subseteq A_j$ But since $A_j\subseteq A$ we get $B\subseteq A$, i.e. $B\in\Omega^A$.
Ad (viii). Observe that A_j is a sheaf, whenever A is a type built up from basic types that are sheaves. Then $\{d_A(b)\in A_j\mid b\in B\}$ is a subtype of the sheaf A_j. For this A_j we have an isomorphism $\eta:A_j\rightarrow L_j(A_j)$ (cf. (2.4)). Hence

$$L_j\{d_A(b)\in A_j\mid b\in B\}\subseteq A_j$$

and so $\eta^{-1}L_j\{d_A(b)\in A_j\mid b\in B\}$ is a subsheaf of A_j, which implies (cf. (2.3.6)) that

$$\eta^{-1}L_j\{d_A(b)\in A_j\mid b\in B\}\in\Omega_j^{A_j}.$$

6.1.4 Lemma.; For types A in $L_{\mathbf{E}}$ built up from basic types that are sheaves we have that $d_A\circ e_A$ equal to id_A, i.e., e_A is a retract.

Proof. By induction, for example for $B\in\Omega_j^{A_j}$:

$$\begin{aligned} d_{\Omega A}\circ e_{\Omega A}(B) &= \eta^{-1}L_j\{d_A(b)\in A_j \mid e_A(b)\in B\} \\ &= \eta^{-1}L_j\{d_A(b)\in A_j \mid b\in B\} \\ &= \eta^{-1}L_jB \\ &= B. \end{aligned}$$

Or, for $f\in B_j{}^{A_j}$:

$$\begin{aligned} d_{B^A}\circ e_{B^A}(f) &= d_{B^A}(e_B\circ f\circ d_A) \\ &= d_B\circ e_B\circ f\circ d_A\circ e_A \\ &= id_B\circ f\circ id_A \\ &= f \end{aligned}$$

□

We have now built the machinery necessary to define a translation of terms of $L_{\mathcal{Sh}_j\mathbb{E}}$ in terms of $L_{\mathbb{E}}$. It suffices to define the translation on the kernel of $L_{\mathcal{Sh}_j\mathbb{E}}$. The translation is remarkable simple. For, all the basic elements of $L_{\mathcal{Sh}_j\mathbb{E}}$, sorts and function symbols, already occur as basic symbols in $L_{\mathbb{E}}$.

6.1.5 Definition. The j-translation $t^j\in L_{\mathbb{E}}$ of a term $t\in L_{\mathcal{Sh}_j\mathbb{E}}$ is obtained by simultaneously replacing every subterm in $t\in L_{\mathcal{Sh}_j\mathbb{E}}$ of the form $\{x{:}A \mid \phi\}$ by the term $\{x{:}A \mid x\in A_j\wedge\phi^j\}$ in $L_{\mathbb{E}}$.

We can now calculate the translation of terms of the *full* language $L_{\mathcal{Sh}_j\mathbb{E}}$ in $L_{\mathbb{E}}$.

6.1.6 Lemma.

Let ϕ,ψ be formulas in $L_{\mathcal{Sh}_j\mathbb{E}}$. Then in the canonical type theory $H_{\mathbb{E}}$ of the topos $\mathbb{E}$ the following holds:

(i) $\top^j\leftrightarrow\top$
(ii) $(\phi\wedge\psi)^j\leftrightarrow(\phi^j\wedge\psi^j)$
(iii) $(\phi\rightarrow\psi)^j\leftrightarrow(\phi^j\rightarrow\psi^j)$
(iv) $(\phi\leftrightarrow\psi)^j\leftrightarrow(\phi^j\leftrightarrow\psi^j)$
(v) $(\forall x{:}A\ \phi)^j\leftrightarrow\forall x{:}A\,(x\in A_j\rightarrow\phi^j)$
(vi) $\bot^j\leftrightarrow j\bot$ ($\leftrightarrow\bot$ in case of a dense topology)
(vii) $(\neg\phi)^j\leftrightarrow(\phi^j\rightarrow j\bot)$ ($\leftrightarrow\neg\phi^j$ in case of a dense topology)
(viii) $(\phi\vee\psi)^j\leftrightarrow j(\phi^j\vee\psi^j)$
(ix) $(\exists x{:}A\ \phi)^j\leftrightarrow j\exists x{:}A\,(x\in A_j\wedge\phi^j)$

Proof.

(i) $\top^j \leftrightarrow \langle *=*\rangle^j$
$\leftrightarrow \langle *=*\rangle$

$\leftrightarrow \top$

(ii) $(\phi\wedge\psi)^j \leftrightarrow (\langle\phi,\psi\rangle=\langle\top,\top\rangle)^j$
$\leftrightarrow \langle\phi^j,\psi^j\rangle=\langle\top,\top\rangle$
$\leftrightarrow (\phi^j\wedge\psi^j)$

(iii) $(\phi\to\psi)^j \leftrightarrow (\phi\wedge\psi=\phi)^j$
$\leftrightarrow \phi^j\wedge\psi^j=\phi^j$
$\leftrightarrow (\phi^j\to\psi^j)$

(iv) $(\phi\leftrightarrow\psi)^j \leftrightarrow (\phi\to\psi\wedge\psi\to\phi)^j$
$\leftrightarrow (\phi^j\to\psi^j\wedge\psi^j\to\phi^j)$
$\leftrightarrow (\phi^j\leftrightarrow\psi^j)$

(v) $(\forall x{:}A\ \phi)^j \leftrightarrow (\{x{:}A\,|\,\phi\}=\{x{:}A\,|\,\top\})^j$
$\leftrightarrow \{x{:}A\,|\,x\in A_j\to\phi^j\}=\{x{:}A\,|\,x\in A_j\to\top\}$
$\leftrightarrow \{x{:}A\,|\,x\in A_j\to\phi^j\}=\{x{:}A\,|\,\top\}$
$\leftrightarrow \forall x{:}A\,(x\in A_j\to\phi^j)$

(vi) $\bot^j \leftrightarrow (\forall\omega\in\Omega\ (\omega=\top))^j$
$\leftrightarrow \forall\omega\in\Omega\ (\omega\in\Omega_j\to\omega=\top)$
$\leftrightarrow \forall\omega\in\Omega\ j\omega$
$\leftrightarrow^* j\forall\omega\in\Omega\ \omega$
$\leftrightarrow j\bot$ ($\leftrightarrow\bot$ in case of a dense topology)
($\to$*) Substitute $\forall\omega\in\Omega\ \omega$.
($\leftarrow$*) Apply (2.2.3.vi).

(vii) $(\neg\phi)^j\leftrightarrow(\phi\to\bot)^j\leftrightarrow(\phi^j\to j\bot)$ ($\leftrightarrow\neg\phi^j$ in case of a dense topology)

(viii) $(\phi\vee\psi)^j \leftrightarrow (\forall\omega\in\Omega\ ((\phi\to\omega)\wedge(\psi\to\omega)\to\omega))^j$
$\leftrightarrow \forall\omega\in\Omega\ [\omega\in\Omega_j\to((\phi^j\to\omega)\wedge(\psi^j\to\omega)\to\omega))]$
$\leftrightarrow \forall\omega\in\Omega\ ((\phi^j\to j\omega)\wedge(\psi^j\to j\omega)\to j\omega))$
$\leftrightarrow^* j\forall\omega\in\Omega\ ((\phi^j\to\omega)\wedge(\psi^j\to\omega)\to\omega))$
$\leftrightarrow j(\phi^j\vee\psi^j)$

($\to$*) Substitute $\forall\omega\in\Omega\ ((\phi^j\to\omega)\wedge(\psi^j\to\omega)\to\omega)$), and apply the fact that $\phi^j\to\forall\omega\in\Omega\ ((\phi^j\to\omega)\wedge(\psi^j\to\omega)\to\omega))$ together with the corresponding fact for ψ^j.

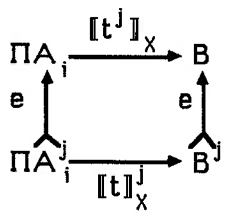
(ix) $(\exists x{:}A\ \phi)^j \leftrightarrow (\forall\omega\in\Omega\ [\forall x\in A\,(\phi\to\omega)\to\omega])^j$
$\leftrightarrow \forall\omega\in\Omega\ [\omega\in\Omega_j\to(\forall x\in A\,[x\in A_j\to(\phi^j\to\omega)])\to\omega]$
$\leftrightarrow \forall\omega\in\Omega\ [\forall x\in A\,(x\in A_j\to(\phi^j\to j\omega))\to j\omega]$

$$\leftrightarrow^* \; j\forall\omega\in\Omega\,[\forall x\in A\,(x\in A_j\rightarrow(\phi^j\rightarrow\omega))\rightarrow\omega]$$
$$\leftrightarrow \; j\forall\omega\in\Omega\,[\forall x\in A\,(x\in A_j\wedge\phi^j)\rightarrow\omega))\rightarrow\omega]$$
$$\leftrightarrow \; j\exists x{:}A\,(x\in A_j\wedge\phi^j)$$

($\rightarrow$*) Substitute $\forall\omega\in\Omega\,[\forall x\in A\,(x\in A_j\rightarrow(\phi^j\rightarrow\omega))\rightarrow\omega]$,
and note that $(x\in A_j\rightarrow(\phi^j\rightarrow \forall\omega\in\Omega\,[\forall x\in A\,(x\in A_j\rightarrow(\phi^j\rightarrow\omega))\rightarrow\omega])$.

□

6.1.7 Theorem. Let t be a term of $L_{\mathcal{Sh}_j\mathbb{E}}$, then we have the following commuting diagram in the topos $\mathbb{E}$:



where B is the type of the term t, and ΠA_i is the product of the types of the free variables in the term t.

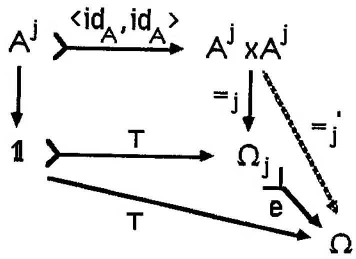
Proof. By induction on the complexity of t. It suffices to prove the theorem for t in the kernel of $L_{\mathrm{Sh}_j\mathbb{E}}$.

(i) $[\![(x_k)^j]\!]_X\circ e \;=\; \pi_k\circ e:\Pi(A_i)^j \rightarrowtail \Pi A_i \rightarrow A_k$
$\;=\; e\circ\pi_k:\Pi(A_i)^j \rightarrowtail (A_k)^j \rightarrow A_k$
$\;=\; e\circ[\![x_k]\!]_X^j$

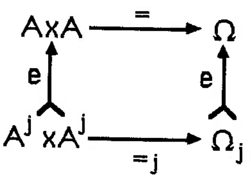
(ii) $[\![f(t)^j]\!]_X\circ e \;=\; [\![f(t^j)]\!]_X\circ e$
$\;=\; [\![f]\!]\circ[\![t^j]\!]_X\circ e$
$\;=\; [\![f]\!]\circ e\circ[\![t]\!]_X^j$ (by induction hypothesis)
$\;=\; e\circ[\![f]\!]^j\circ[\![t]\!]_X^j$ (f is a function symbol between basic types, i.e. sheaves)
$\;=\; e\circ[\![f(t)]\!]_X^j$

(iii) $[\![(t=s)^j]\!]_X\circ e \;=\; [\![t^j=s^j]\!]_X\circ e$
$\;=\; \langle[\![t^j]\!]_X,[\![s^j]\!]_X\rangle\circ[\![=]\!]\circ e$
$\;=^*\; \langle[\![t^j]\!]_X,[\![s^j]\!]_X\rangle\circ e\circ[\![=]\!]^j$
$\;=\; \langle[\![t^j]\!]_X\circ e,[\![s^j]\!]_X\circ e\rangle\circ[\![=]\!]^j$
$\;=\; \langle e\circ[\![t]\!]_X^j, e\circ[\![s]\!]_X^j\rangle\circ[\![=]\!]^j$ (induction hypothesis)
$\;=\; e\circ\langle[\![t]\!]_X^j,[\![s]\!]_X^j\rangle\circ[\![=]\!]^j$
$\;=\; e\circ[\![t=s]\!]_X^j$

(=*) Equality on an object A in a topos is defined as the map $=: A\times A \to \Omega$ that classifies the monomorphism $\langle id_A, id_A\rangle : A \rightarrowtail A\times A$. Hence in the topos $\mathbf{E}$ equality on A_j is the map $=_j: A_j\times A_j \to \Omega_j$ classified by $\langle id_{A_j}, id_{A_j}\rangle : A_j \rightarrowtail A_j\times A_j$. In the following diagram we see that the innersquare is a pullback, and hence also the outersquare. It follows that equality on A_j in $\mathbf{E}$ factors through the equality on A_j in $\mathcal{Sh}_j\mathbf{E}$ followed by $e: \Omega_j \to \Omega$.

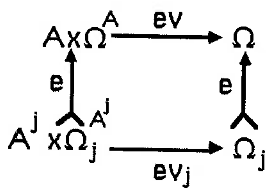


From the pullback property of the classifying diagram of $=: A\times A \to \Omega$ we get the desired:



(iv) $[\![(t\in s)^j]\!]_X \circ e$
$= [\![t^j \in s^j]\!]_X \circ e$
$= ev \circ \langle [\![t^j]\!]_X, [\![s^j]\!]_X\rangle \circ e$
$= ev \circ \langle [\![t^j]\!]_X \circ e, [\![s^j]\!]_X \circ e\rangle$
$= ev \circ \langle e \circ [\![t^j]\!]^j_X, e \circ [\![s^j]\!]^j_X\rangle$ (induction hypothesis)
$= ev_j \circ e \circ \langle [\![t]\!]^j_X, [\![s]\!]^j_X\rangle$
$=^* e \circ ev_j \circ \langle [\![t]\!]^j_X, [\![s]\!]^j_X\rangle$
$= e \circ [\![t\in s]\!]^j_X$

(=*) needs motivation: why does the following diagram commute in $\mathbf{E}$?



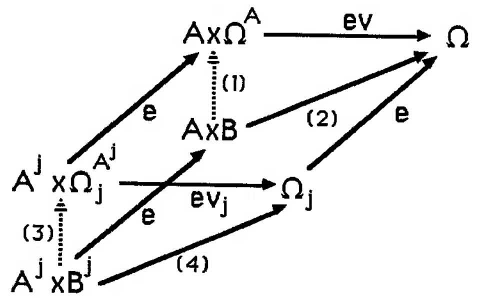
Recall that e is a genuine embedding. For an element (x,B) of $(A_j\times(A_j \to \Omega_j))$ we have $ev \circ e(x,B) = ev(x,B)$

$= x\in B$ (B is j-closed, hence $x\in B \leftrightarrow jx\in B$, i.e., $(x\in B)\in\Omega_j$)

$= e(x \in B)$
$= e \circ ev_j(x,B).$

(v) $[\![(z(x))^j]\!]_X \circ e = ev \circ \langle \pi_Z, \pi_X \rangle \circ e$
$= ev \circ \langle e \circ \pi_Z, e \circ \pi_X \rangle$
$= ev \circ e \circ \langle \pi_Z, \pi_X \rangle$
$= e \circ ev_j \circ \langle \pi_Z, \pi_X \rangle$
$= e \circ [\![z(x)]\!]^j_X$

(vi) The last case we have to prove is $[\![\{x:A \mid \phi\}^j]\!] \circ e = e \circ [\![\{x:A \mid \phi\}]\!]^j$. Assume for simplicity that ϕ contains only $x:A$ and $y:B$ as free variables and recall that the interpretation $[\![\{x:A \mid \phi\}]\!]_B : B \to \Omega^A$ of the type $\{x:A \mid \phi\}$ is defined as the transpose of $[\![\phi(x,y)]\!]_{A,B} : A \times B \to \Omega$.
Because exponentials are preserved by the associated sheaf functor, we get that from the point of view of $\mathbf{E}$ the morphism $[\![\{x:A \mid \phi\}]\!]^j_B : B^j \to (\Omega_j \to A_j)$, which is the interpretation of $\{x:A \mid \phi\}$ in $\mathcal{Sh}_j\mathbf{E}$, is constructed as the transpose of $[\![\phi(x,y)]\!]^j_{A,B} : A_j \times B_j \to \Omega_j$. Hence we can apply the induction hypothesis to the following diagram:



where

(1) = $id_A \times [\![\{x:A \mid x \in A_j \wedge \phi^j\}]\!]_B$
(2) = $[\![x \in A_j \wedge \phi^j(x,y)]\!]_{A \times B}$
(3) = $id_{A^j} \times [\![\{x:A \mid \phi\}]\!]^j_B$
(4) = $[\![\phi(x,y)]\!]^j_{A \times B}$

The induction hypothesis $e \circ [\![\phi(x,y)]\!]^j_{A \times B} = [\![\phi^j(x,y)]\!]_{A \times B} \circ e$ implies that the square with sides (e,2,e,4) commutes. Moreover we know that the square (e,ev,e,ev_j) commutes. Taking in consideration the definitions of (1) and (3) we get that the diagram (e,1,e,3) commutes. Hence we get $[\![\{x:A \mid \phi\}^j]\!] \circ e = e \circ [\![\{x:A \mid \phi\}]\!]^j$.

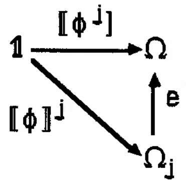
□

With help of this j-translation we can express in a topos $\mathbf{E}$ what it means that a formula ϕ is satisfied in the subcategory $\mathcal{Sh}_j\mathbf{E}$ of sheaves in $\mathbf{E}$:

6.1.8 Corollary. Let ϕ be a closed formula in $L_{\mathcal{Sh}_j\mathbf{E}}$ (i.e. there are no free variables in ϕ). Then $\mathbf{E} \models \phi^j$ if and only if $\mathcal{Sh}_j\mathbf{E} \models \phi$.

Proof. Observe the following easy facts:

(i) for a closed formula ϕ the diagram of theorem 6.1.7 reduces to



(ii) $\top = id_{1} \circ \top = e_{1} \circ \top = \top_j \circ e_\Omega$.

□

We apply (6.1.8) to get a simple proof for the following theorem of Lawvere and Tierney in topos theory (cf. (5.17) in [Johnstone 77]):

6.1.9 Corollary. For each topos $\mathbf{E}$ it holds that $\mathcal{Sh}_{\neg\neg}\mathbf{E}$ is a boolean topos (i.e., $\mathcal{Sh}_{\neg\neg}\mathbf{E} \models \forall \omega \in \Omega\ \omega \vee \neg\omega$, cf. [Johnstone 77])

Proof. Because of (6.1.8) we have that

$$\begin{aligned} \mathcal{Sh}_{\neg\neg}\mathbf{E} \models \forall \omega \in \Omega\ (\omega \vee \neg\omega) &\Leftrightarrow \mathbf{E} \models (\forall \omega \in \Omega\ (\omega \vee \neg\omega))^{\neg\neg} \\ &\Leftrightarrow \mathbf{E} \models \forall \omega \in \Omega_{\neg\neg}\ \neg\neg(\omega \vee \neg\omega) \\ &\Leftrightarrow \mathbf{E} \models \forall \omega \in \Omega\ \neg\neg(\neg\neg\omega \vee \neg\omega) \\ &\Leftrightarrow \mathbf{E} \models \forall \omega \in \Omega\ \neg(\neg\omega \wedge \neg\neg\omega) \\ &\Leftrightarrow \mathbf{E} \models \forall \omega \in \Omega\ \neg\bot \\ &\Leftrightarrow \mathbf{E} \models \top. \end{aligned}$$

The latter formula holds.

□

As another corollary of (6.1.8) we present a another, simple proof for the inverse of (6.1.9) that occurs as a lemma in [Blass and Scedrov]:

6.1.10 Lemma. Let j be a dense topology in a topos $\mathbf{E}$. If $\mathcal{Sh}_j\mathbf{E}$ is a boolean topos, then $j = \neg\neg$.

Proof. Assume for a topos $\mathbb{E}$ and a topology j in $\mathbb{E}$ that $\mathcal{Sh}_j\mathbb{E}$ is a boolean topos. Then by (6.1.8) we get

$$\mathbb{E}\models(\forall\omega\in\Omega\,(\omega\vee\neg\omega))^j,$$

i.e.,

$$\mathbb{E}\models\forall\omega\in\Omega_j\, j(\omega\vee(\omega\rightarrow j\bot)).$$

Hence

$$\mathbb{E}\models\forall\omega\in\Omega\, j(j\omega\vee(\omega\rightarrow j\bot)).$$

Now we argue internally in $\mathbb{E}$. The assumption $j\leq\neg\neg$ implies $j\bot\leftrightarrow\bot$. So, if we also suppose that $\neg\neg\omega$ holds, then

$$\begin{aligned} j(j\omega\vee(\omega\rightarrow j\bot)) &\rightarrow j(j\omega\vee\neg\omega) \\ &\rightarrow j(j\omega\vee(\neg\omega\wedge\neg\neg\omega)) \\ &\rightarrow j(j\omega\vee\bot) \\ &\rightarrow j(j\omega) \\ &\rightarrow j\omega \end{aligned}$$

Hence $\neg\neg\omega\rightarrow j\omega$, which implies $\neg\neg=j$ in $\mathbb{E}$.

□

6.2 The generalized Gödel-translation

The essence of the G-translation (cf. def. 6.2.7) is a translation of types. When the types are changed, it is natural that also the translation of functions and sets has to be considered. This is all that needs to be translated. The characteristic Gödel-translation of the logical connectives will follow naturally from the translation of the definitions of the connectives.

In proposition calculus we have an easy example of the way the G-translation operates in the case of the double negation topology.

Recall, that $\phi\vee\psi$ can be redefined in second order propositional calculus by $\forall\omega\in\Omega([\phi\rightarrow\omega\wedge\psi\rightarrow\omega]\rightarrow\omega)$. According to our recipe of the G-translation we will replace the type Ω by $\Omega_{\neg\neg}$, i.e., by $\{\omega\in\Omega\mid\omega=\neg\neg\omega\}$. The resulting formula is equivalent to $\neg\neg(\phi\vee\psi)$, which is the original Kolmogorow's translation of the disjunction (cf. [Troelstra and van Dalen 88]):

$$\begin{aligned} \forall\omega\in\Omega_{\neg\neg}([\phi\rightarrow\omega\wedge\psi\rightarrow\omega]\rightarrow\omega) &\leftrightarrow \forall\omega\in\Omega\,([\phi\rightarrow\neg\neg\omega\wedge\psi\rightarrow\neg\neg\omega]\rightarrow\neg\neg\omega) \\ &\leftrightarrow \forall\omega\in\Omega\,([\phi\rightarrow\omega\wedge\psi\rightarrow\omega]\rightarrow\neg\neg\omega) \\ &\leftrightarrow^{*} \neg\neg\forall\omega\in\Omega\,([\phi\rightarrow\omega\wedge\psi\rightarrow\omega]\rightarrow\omega) \\ &\leftrightarrow \neg\neg(\phi\vee\psi). \end{aligned}$$

$\leftrightarrow^*$ is one of the cases where double negation shift is acceptable from the intuitionistic point of view. The implication $\rightarrow^*$ follows by substitution of the truth value $\forall\omega\in\Omega\,([\phi\rightarrow\omega\wedge\psi\rightarrow\omega]\rightarrow\omega)$, i.e. $\phi\vee\psi$ for ω. We get

$$[(\phi\rightarrow(\phi\vee\psi))\wedge(\psi\rightarrow(\phi\vee\psi))]\rightarrow\neg\neg(\phi\vee\psi).$$

Hence $\neg\neg(\phi\vee\psi)$ follows by modus ponens as $[(\phi\rightarrow(\phi\vee\psi))\wedge(\psi\rightarrow(\phi\vee\psi)]$ is true.

6.2.1 Definition. For each type A of L_E we define a type A^G of L_E by induction on A:

(i) $A^G = LA$, for a basic type A,
(ii) $\Omega^G = \Omega_j$,
(iii) $(PA)^G = \Omega_j^{A^G} = \{B \subseteq A^G \mid B \text{ is j-closed}\}$,
(iv) $(A \times B)^G = A^G \times B^G$,
(v) $(B^A)^G = (B^G)^{A^G}$.

For each type A of L_E we define a type A^L of L_E by induction on A:

(vi) $A^L = LA$, for a basic type A,
(vii) $\Omega^L = \Omega$,
(viii) $(PA)^L = \Omega^{A^L}$,
(ix) $(A \times B)^L = A^L \times B^L$,
(x) $(B^A)^L = (B^L)^{A^L}$.

By induction on the structure of the type, one easily sees that

(i) A^G is a j-sheaf for each type A of L_E,
(ii) $e_{A^L} : A^G \hookrightarrow A^L$.
(iii) $A^G = (A^L)j$

After these preliminary steps we can define the G-translation.

6.2.2 Definition.
The G-translation $t^G \in L_H$ of a term t in the kernel of L_H is obtained inductively by replacing

(i) $x^G = x^L$ where x^L is a fresh variable of type A^L, where A is the type of x,
(ii) $f^G = Lf$, for function symbols $f: A \to B$,
(iii) $\{x:A \mid \phi\}^G = \{x:A^L \mid x \in A^G \wedge \phi^G\}$

We will now calculate the G-translation in L_H of terms of the full language L_H.

6.2.3 Corollary. Let ϕ, ψ be formulas in L_H.
Then the following holds in H:

(i) $T^G \leftrightarrow T$
(ii) $(\phi \wedge \psi)^G \leftrightarrow (\phi^G \wedge \psi^G)$
(iii) $(\phi \to \psi)^G \leftrightarrow (\phi^G \to \psi^G)$
(iv) $(\phi \leftrightarrow \psi)^G \leftrightarrow (\phi^G \leftrightarrow \psi^G)$

(v) $(\forall x{:}A\ \phi)^G \leftrightarrow \forall x{:}A^L\ (x \in A^G \rightarrow \phi^G)$
(vi) $\perp^G \leftrightarrow j\perp$ ($\leftrightarrow \perp$ in case of a dense topology)
(vii) $(\neg\phi)^G \leftrightarrow (\phi^G \rightarrow j\perp)$ ($\leftrightarrow \neg\phi^G$ in case of a dense topology)
(viii) $(\phi \vee \psi)^G \leftrightarrow j(\phi^G \vee \psi^G)$
(ix) $(\exists x{:}A\ \phi)^G \leftrightarrow j\exists x{:}A^L\ (x \in A^G \wedge \phi^G)$

Proof. As in (6.1.6).

□

6.2.4 Definition.
Let H be some type theory. If $\models$ is an interpretation of H in a topos $\mathbb{E}$, then an interpretation $\models_j$ of H in $\mathcal{S}h_j\mathbb{E}$ is defined by:
(i) $[\![A]\!]_j{:=}L[\![A]\!]$ for basic types A in H,
(ii) $[\![f]\!]_j{:}\,[\![A]\!]_j \rightarrow [\![B]\!]_j$ by $d_{B^L} \circ L[\![f]\!] \circ e_{A^L} : A^G \rightarrowtail LA \rightarrow LB \rightarrow B^G$.

6.2.5 Lemma. Let $\models$ be an interpretation of a type theory H in a topos $\mathbb{E}$. Then for all closed formulas ϕ (i.e. formulas with no free variables) of H we have:
$$\mathbb{E} \models \phi^G \Leftrightarrow \mathcal{S}h_j\mathbb{E} \models_j \phi.$$

Proof.
If we apply the j-translation to the interpretation $\models_j$ of ϕ in $\mathcal{S}h_j\mathbb{E}$ we get ϕ^G. We get the lemma as a corollary of (6.1.8).

□

6.2.6 Definition.
(i) A formula ϕ is *proper* if the types of its free variables are inhabited.
(ii) A type theory H is *proper* if all its defining axioms are proper.
(iii) A type theory H is G-*preservable* if $H \cup \{j\} \vdash \phi^G$ for all axioms ϕ of H, where $H \cup \{j\}$ is the type theory of H extended with one extra function symbol $j{:}\,\Omega \rightarrow \Omega$ satisfying the three axioms of a Lawvere-Tierney topology.

Proper formulas have the property that $\phi \leftrightarrow \forall x{:}A\ \phi$, if x is one of the free variables of ϕ. So, whenever we have a proper formula we may suppose that it is a closed formula. Likewise for a proper type theory we may assume that all its axioms are closed.
The notion G-preservable is the syntactic counterpart of the semantical notion being preserved by the associated sheaf functor. The preservability of Heyting arithmetic will be among the examples, that we will present in section (6.4).

6.2.7 Theorem.
A proper type theory H is G-preservable, if and only if for all toposes $\mathbb{E}$ it holds that $\mathbb{E} \models H$ implies $\mathcal{Sh}_j\mathbb{E} \models_j H$. The interpretation $\models_j$ of $H \cup \{j\}$ in $\mathcal{Sh}_j\mathbb{E}$ is obtained from the interpretation $\models$ of H in $\mathbb{E}$ by interpreting

(i) a basic type A by its sheafification $L[\![A]\!]$ in $\mathcal{Sh}_j\mathbb{E}$,
(ii) a function symbol $f{:}A \rightarrow B$ by its sheafification $L[\![f]\!]{:}LA \rightarrow LB$
(iii) interpreting j by the identity on Ω_j.

Proof. (cf. [de Vries 84])
(if) Assume for a proper type theory H holds that for all toposes $\mathbb{E}$ we have that $\mathbb{E} \models H\cup\{j\}$ implies $\mathcal{Sh}_j\mathbb{E} \models_j H$. Secondly, assume that H is not G-preservable. Then there is an axiom ϕ such that $H\cup\{j\} \not\vdash \phi^G$. By completeness there is a topos $\mathbb{E}$ with a topology $j{:}\Omega \rightarrow \Omega$ in $\mathbb{E}$, such that $\mathbb{E} \models H$ and $\mathbb{E} \not\models \phi^G$. The G-translation of ϕ corresponds by construction exactly with the j-translation of the interpretation $\models_j$ of L_H in $\mathcal{Sh}_j\mathbb{E}$. By (6.2.5) we have that $\mathcal{Sh}_j\mathbb{E} \not\models_j \phi$. It follows from the first assumption that $\mathcal{Sh}_j\mathbb{E} \models_j H$. But by soundness we have $\mathcal{Sh}_j\mathbb{E} \models_j \phi$. Contradiction.
Thus H is G-preservable.

(only if) Assume H is a proper, G-preservable type theory, and assume that for a topos $\mathbb{E}$ we have $\mathbb{E} \models H$. From both assumptions we get $\mathbb{E} \models H^G$. And so $\mathcal{Sh}_j\mathbb{E} \models_j H$.
□

Let PEM denote the *Principle of Excluded Middle* $\forall\omega\in\Omega\,(\omega\vee\neg\omega)$. Classically as well as intuitionistically it is equivalent to the principle $\forall\omega\in\Omega\,(\neg\neg\omega\rightarrow\omega)$. The latter may seem surprising if one recalls that in intuitionistic propositional calculus $A\vee\neg A$ is not implied by $\neg\neg A\rightarrow A$ (cf. for instance [Dummett]). Let PEM_j denote the principle $\forall\omega\in\Omega\,(j\omega\rightarrow\omega)$. It is easy to see that for open and closed topologies we have

$$PEM_{j_p} = \forall\omega\in\Omega\,[(p\vee\omega)\rightarrow\omega] = \neg p$$

and

$$PEM_{j^p} = \forall\omega\in\Omega\,[(p\rightarrow\omega)\rightarrow\omega] = p.$$

Observe that if we have an interpretation $\models$ of a typetheory $H\cup\{j\}$ in the topos $\mathbb{E}$, then $\mathcal{Sh}_j\mathbb{E} \models_j PEM_j$, for a very trivial reason: j is interpreted by the identity on Ω_j. Hence PEM_j trivializes to the tautology $\forall\omega\in\Omega(\omega\rightarrow\omega)$.

6.2.8 Notation.
In the case of the double negation topology we write $\phi^{\neg\neg}$ instead of ϕ^G.

Observe (cf. 6.2.3) that our translation $\phi^{\neg\neg}$ resembles the usual Gödel translations, but not quite! For example consider **HA**, Heyting's arithmetic. Our translation changes the type of the variables. For instance the translation will replace $\mathbb{N}$ by its sheafification $L_{\neg\neg}(\mathbb{N})$. The g-translation that we will define in section (6.5) will resemble the original Gödel-translation more closely in this aspect.

Finally, we can now give the semantical proof we aimed for in the introduction of this chapter. In its form it resembles a general theorem of [Friedman] for theories in many-sorted intuitionistic logic without identity.

6.2.9 Theorem. (A Gödel-Friedman Theorem for type theories)

(i) For any proper j-preservable type theory H we have

$$Hu\{j\}\vdash\phi^G \Leftrightarrow H+PEM_j\vdash\phi,$$

(ii) If $Hu\{j\}\vdash\phi^G \Leftrightarrow H+PEM_j\vdash\phi$ for a proper type theory H, then H is j-preservable.

Proof.

(i) Assume for a j-preservable type theory H that $Hu\{j\}\vdash\phi^G$. Then certainly $Hu\{j\}+PEM_j\vdash\phi^G$. Since $PEM_j\vdash\phi\leftrightarrow\phi^G$ we get $Hu\{j\}+PEM_j\vdash\phi$.
In the other direction assume that $Hu\{j\}+PEM_j\vdash\phi$. Because H is j-preservable we get $\mathcal{S}h_j\mathbb{E}\vDash H$. But we know already that $\mathcal{S}h_j\mathbb{E}\vDash_j PEM_j$. Hence by soundness $\mathcal{S}h_j\mathbb{E}\vDash\phi$. Applying (6.2.5) we get $\mathbb{E}\vDash\phi^G$. By an appeal to completeness we get $Hu\{j\}\vdash\phi^G$.

(ii) Assume H is a proper type theory such that $Hu\{j\}\vdash\phi^G \Leftrightarrow Hu\{j\}+PEM_j\vdash\phi$. And suppose that H is not j-preservable. Then for some axiom ϕ of H we have $Hu\{j\}\nvdash\phi^G$. By completeness there is a topos $\mathbb{E}$ such that $\mathbb{E}\vDash Hu\{j\}$ and $\mathbb{E}\nvDash\phi^j$. Then also $\mathbb{E}\vDash Hu\{j\}$. Thus $\mathcal{S}h_j\mathbb{E}\vDash_j H$ and $\mathcal{S}h_j\mathbb{E}\vDash_j\phi$. Because $Sh_j\mathbb{E}$ is a model for $Hu\{j\}+PEM_j$ we get $Hu\{j\}+PEM_j\nvdash\phi$. Contradiction with the initial assumption. Therefore $Hu\{j\}\vdash H^G$, i.e., H is j-preservable.

□

6.2.10 Corollary. (A Gödel-Friedman Theorem for type theories)

(i) For any proper $\neg\neg$-preservable type theory H we have $H\vdash\phi^{\neg\neg} \Leftrightarrow H+PEM\vdash\phi$,

(ii) If for a proper type theory H we have $(H\vdash\phi^{\neg\neg} \Leftrightarrow H+PEM\vdash\phi)$ then H is $\neg\neg$-preservable.

Proof. Note that for any type theory H we have $H\vdash Hu\{\neg\neg\}$ and apply (6.2.9)

□

Observe that in case of the open and closed topologies (2.2.2) for a truth value p, our G-translation results in higher-order extensions of well-known Friedman translations (cf. [Friedman] and [de Jongh]), but note again the difference in the treatment of types:

The open Friedman translation, (notation $(-)^p$) for a topology $j^p: \Omega \to \Omega: \omega \mapsto (p \to \omega)$ satisfying

(i) $\top^p \leftrightarrow \top$
(ii) $(\phi \wedge \psi)^p \leftrightarrow (\phi^p \wedge \psi^p)$
(iii) $(\phi \to \psi)^p \leftrightarrow (\phi^p \to \psi^p)$
(iv) $(\phi \leftrightarrow \psi)^p \leftrightarrow (\phi^p \leftrightarrow \psi^p)$
(v) $(\forall x{:}A\ \phi)^p \leftrightarrow \forall x{:}A^L\ (x \in A^p \to \phi^p)$
(vi) $\bot^p \leftrightarrow \neg p$
(vii) $(\neg\phi)^p \leftrightarrow (\phi^p \to (p \to \bot))$ ($\leftrightarrow \neg\phi^p$ in case of a dense topology)
(viii) $(\phi \vee \psi)^p \leftrightarrow [p \to (\phi^p \vee \psi^p)]$
(ix) $(\exists x{:}A\ \phi)^p \leftrightarrow [p \to \exists x{:}A^L\ (x \in A^p \wedge \phi^p)]$

The closed Friedman translation, (notation $(-)_p$), for a topology $j_p: \Omega \to \Omega: \omega \mapsto (p \vee \omega)$ satisfying

(i) $\top_p \leftrightarrow \top$
(ii) $(\phi \wedge \psi)_p \leftrightarrow (\phi_p \wedge \psi_p)$
(iii) $(\phi \to \psi)_p \leftrightarrow (\phi_p \to \psi_p)$
(iv) $(\phi \leftrightarrow \psi)_p \leftrightarrow (\phi_p \leftrightarrow \psi_p)$
(v) $(\forall x{:}A\ \phi)_p \leftrightarrow \forall x{:}A^L\ (x \in A_p \to \phi_p)$
(vi) $\bot_p \leftrightarrow p$
(vii) $(\neg\phi)_p \leftrightarrow (\phi_p \to p)$
(viii) $(\phi \vee \psi)_p \leftrightarrow [p \vee (\phi_p \vee \psi_p)]$
(ix) $(\exists x{:}A\ \phi)_p \leftrightarrow [p \vee \exists x{:}A^L\ (x \in A_p \wedge \phi_p)]$

The following corollary is related to results in [de Jongh] and [Visser 81 or 82, cf. (6.3.1)] for intuitionistic predicate calculus.

6.2.15 Corollary.

(i) For proper formulas ϕ and p we have $\vdash \phi^p \leftrightarrow (p \to \phi)$ and $\vdash \phi_p \leftrightarrow (\neg p \to \phi)$
(ii) For proper formulas ϕ and p we have $\vdash \phi^{\neg p} \leftrightarrow \phi_p$.

Proof. Application of (6.2.9) on an empty theory H plus the observation the for any type theory H it holds that $H \vdash H \cup \{j_p\} \cup \{j^p\}$. As follows:

(i) $\vdash \phi^p \Leftrightarrow \mathrm{PEM}_{j^p} \vdash \phi$

$\Leftrightarrow \; p \vdash \phi$

$\Leftrightarrow \; \vdash p \rightarrow \phi$, likewise for the closed topology j_p.

(ii) $\vdash \phi^{\neg p} \; \Leftrightarrow \; \neg p \vdash \phi$

$\Leftrightarrow \; \vdash \phi_p$, hence $\vdash \phi^{\neg p} \leftrightarrow \phi_p$.

□

6.3 G-Preservable formulas

In order to apply Gödel-Friedman theorem (6.2.9) to type theories it is useful to characterize syntactically formulas that are preserved by the G-translation. In this section we will consider a fixed type theory H *without function types.*
We start with some preliminary definitions and lemmas that will relate types and terms with the correspondent G-translation of these types and terms.

6.3.1 Definition. For each type A of L_E we define a function $\eta_A: A \to A^G$ of L_E by induction on A:

(i) $\eta_A(a)=\{a\}^j$, if A is a basic type,
(ii) $\eta_\Omega(\omega)=j\omega$, if $A=\Omega$,
(iii) $\eta_{[C]}(B)=\{\eta_C(b)\in C^G \mid b\in B\}^j$, if $A=[C]$,
(iv) $\eta_{B\times C}((b,c))=(\eta_B(b),\eta_C(c))$, if $A=B\times C$.

It is straightforward to verify that η_A is well-defined.

6.3.2 Lemma.
(i) For any function symbol $f:A\to B$ it holds that $\forall a\in A\ \eta_B(f(a))=f^G(\eta_A(a))$.
(ii) For any type A it holds that $\forall a\in A\ \forall X\in[A]\ (\eta_\Omega(a\in X)=(\eta_A(a)\in\eta_{[A]}(X)))$.

Proof. (i) For $f:A\to B$ and $a\in A$ we have:

$$\begin{aligned}\eta_B\circ f(a) &= \{f(a)\}^j\\ &= \{f(a) \mid \eta_A(a)=\eta_A(a)\}^j\\ &= f^G\circ\eta_A(a).\end{aligned}$$

(ii) For $a\in A$ and $X\in[A]$ we have that

$$\begin{aligned}\eta_A(a)\in\eta_{[A]}(X)) &= \eta_A(a)\in\{\eta_A(x)\in A^G \mid x\in X\}^j\\ &= ja\in X\\ &= \eta_\Omega(a\in X).\end{aligned}$$

□

6.3.3 Lemma. For all types A it holds that $\forall x\in A^G\ j\exists y\in A\ \eta(y)=x$.

Proof. By induction on A.

(i) If A is a basic type, then if $x\in A^G=LA$ then x is a Lawvere-singleton, and hence $j\exists y\in A(\eta(y)=\{y\}^j=x)$.
(ii) If A is Ω, then $\eta(x)=jx=x\in\Omega$, if $x\in\Omega^G=\Omega_j$.
(iii) Suppose A is of the form [B], and $X\in[B]^G$. Define $Y:=\{b\in B\mid\eta(b)\in X\}$. We will show $X=\eta(Y)$. Suppose $x\in X$. Then by induction hypothesis we have $j\exists y\in B\ \eta(y)=x$. Hence $x\in\{c\in B^G\mid j\exists b\in B[\eta(b)=c\wedge\eta(b)\in X]\}=\eta(Y)$. Hence $X\subseteq\eta(Y)$. On the other hand if $x\in\eta(Y)$ then $j\exists b\in B[\eta(b)=x\wedge\eta(b)\in X]$. Hence $jx\in X$. But $X\in[B]^G$ is j-stable, i.e., we get $x\in X$. Therefore $\eta(Y)=X$.
(iv) If $A=B\times C$, then for $x=(b,c)\in B^G\times C^G$ we apply the induction hypothesis to b and c.

□

After these preliminaries we can proceed.We will consider the following classes of formulae:

6.3.4 Definition.
(i) $Pres:=\{\phi\in L_H\mid H\cup\{j\}\vdash\forall x\in A[\phi(x)\rightarrow\phi^G[x^G:=\eta(x)]]\}$
(ii) $Crea:=\{\phi\in L_H\mid H\cup\{j\}\vdash\forall x\in A[\phi^G[x^G:=\eta(x)]\rightarrow\phi(x)]\}$ (G-creating formulas)
(iii) $Stab:=\{\phi\in L_H\mid H\cup\{j\}\vdash\forall x\in A^G[j\phi^G(x)\rightarrow\phi^G(x)]\}$
(iv) $Isol:=\{\phi\in L_H\mid H\cup\{j\}\vdash\forall x\in A[\phi^G[x^G:=\eta(x)]\rightarrow j\phi(x)]\}$ (G-isolating formulas)

In the context of the double negation topology [Leivant] and [Troelstra and van Dalen 88] use the terminology spreading and wiping for G-preservable, respectively G-creating.

We will just write $\phi^G(\eta(x))$ instead of $\phi^G[x^G:=\eta(x)]$ if no confusion can arise. Free variables will often be suppressed, i.e., we will write ϕ for $\phi(x)$.

For many intuitionistic logic ($1\leq k\leq\omega$) [Leivant] and [Troelstra and van Dalen 88] has systematically given a related classification of schema's instead of formulas with respect to the double negation translation. [Visser 81] has considered related classes of formulas for Friedman translations on **HA**, although he did not strive for a complete characterization.

We will show in the next section (6.4) that G-preservable extends the notion of geometric formula used in topos theory.

6.3.5 Stability Lemma. Any closed formula ϕ of L_H is G-stable.

Proof. The idea is to push j as deep as possible inside the formula ϕ^G, until it gets absorbed at the atomic level. It is left behind only in front of the disjunction and the existential quantifier.

□

For the benefit of the following lemmata, we introduce some terminology:
the *basic terms* of a type theory: a basic term is either a variable, or a function symbol followed by a basic term.

6.3.6 Creation Lemma.

(i) $\top \in \mathrm{Crea}$,
(ii) $\bot \in \mathrm{Crea}$, provided j is dense,
(iii) if $\phi,\psi \in \mathrm{Crea}$ then $(\phi \wedge \psi) \in \mathrm{Crea}$,
(iv) if $\phi,\psi \in \mathrm{Crea}$ then $(\phi \vee \psi) \in \mathrm{Crea}$,
(v) if $\phi \in \mathrm{Crea}$ then $(\forall x \in A\ \phi) \in \mathrm{Crea}$,
(vi) if $\phi \in \mathrm{Pres}$ and $\psi \in \mathrm{Crea}$ then $(\phi \rightarrow \psi) \in \mathrm{Crea}$,
(vii) if $\phi \in \mathrm{Pres}$ and j is dense then $(\neg\phi) \in \mathrm{Crea}$,
(viii) $(t=s) \in \mathrm{Crea}$, if t,s are basic terms of separated and basic type.

Proof.
(i), (ii), (iii) and (iv) follow trivially from (6.2.3).

(v) Suppose $\phi \in \mathrm{Crea}$ and $(\forall x \in A\ \phi)^G$. Then in particular $\phi^G(\eta(x))$, hence, by assumption $\phi(x)$, and so $\forall x \in A\ \phi$. Therefore $(\forall x \in A\ \phi) \in \mathrm{Crea}$.

(vi) Assume $\phi \in \mathrm{Pres}$ and $\psi \in \mathrm{Crea}$ and suppose $(\phi \rightarrow \psi)^G$ holds. Then $\phi^G \rightarrow \psi^G$. Hence if ϕ holds, we conclude ϕ^G, then ψ^G and finally ψ, using the assumptions. So we have shown $(\phi \rightarrow \psi)^G \rightarrow (\phi \rightarrow \psi)$. Therefore $(\phi \rightarrow \psi) \in \mathrm{Crea}$.

(vii) Trivial by (vi) and (ii).

(viii) Suppose $(t(x)=s(x))^G[x^G = \eta(x)]$. Then $t^G(\eta(x)) = s^G(\eta(x))$, and so by lemma (6.3.2) and the fact that s,t are basic terms we get $\eta(t(x)) = \eta(s(x))$. The type of s and t is basic and separated, hence $\{t(x)\}^j = \{s(x)\}^j$, and $jt(x)=s(x)$. Therefore $(t=s) \in \mathrm{Crea}$.

□

6.3.7 Isolation Lemma.

(i) $\top \in$ Isol,
(ii) $\bot \in$ Isol,
(iii) if $\phi, \psi \in$ Isol then $\phi \wedge \psi \in$ Isol,
(iv) if $\phi, \psi \in$ Isol then $\phi \vee \psi \in$ Isol,
(v) if $\phi \in$ Isol and $j\phi \rightarrow \phi$ then $(\forall x \in A\ \phi) \in$ Isol,
(vi) if $\phi \in$ Isol then $(\exists x \in A\ \phi) \in$ Isol,
(vii) if $\phi \in$ Pres and j is dense then $\neg\phi \in$ Isol,
(viii) $(t=s) \in$ Isol, if t,s are basic terms which type is basic,
(ix) $(t \in X) \in$ Isol, if t is a basic term and X a variable.
(x) Crea $\subseteq$ Isol.

Proof.

(i), (ii), (iii) follow trivially from (6.2.3).

(iv) Suppose $\phi, \psi \in$ Isol and $(\phi \vee \psi)^G$. Then $j(\phi^G \vee \psi^G)$. Hence by assumption we get $j(j\phi \vee j\psi)$, which implies $j(\phi \vee \psi)$.Therefore $(\phi \vee \psi) \in$ Isol.

(v) Suppose $\phi \in$ Isol and $(\forall x \in A\ \phi)^G$. Then in particular $\forall x \in A\ \phi^G(\eta(x))$, hence, by assumption $\forall x \in A\ j\phi(x)$, and by the second assumption $\forall x \in A\ \phi(x)$. Therefore $(\forall x \in A\ \phi) \in$ Isol.

(vi) Suppose $\phi \in$ Isol and $(\exists x \in A\ \phi)^G$. Then $j\exists x \in A^G\ \phi^G$. Hence by (6.3.3) we get $j\exists x \in A\ \phi^G(\eta(x))$. By assumption follows $j\exists x \in A\ j\phi$, and so $j\exists x \in A\ \phi$. Therefore $(\exists x \in A\ \phi) \in$ Isol.

(vii) Suppose $\phi \in$ Pres and assume $(\neg\phi)^G$, i.e., $\neg(\phi^G)$ under the assumption that j is dense. Then if ϕ holds, we get ϕ^G and so $\bot$. Hence $\neg\phi$. Therefore $(\neg\phi) \in$ Isol.

(viii) Suppose $(t(x)=s(x))^G[x^G = \eta(x)]$. Then $t^G(\eta(x)) = s^G(\eta(x))$, and so by lemma (6.3.2) and the fact that s,t are basic terms we get $\eta(t(x)) = \eta(s(x))$. The type of s and t is basic, $jt(x)=s(x)$. Therefore $(t=s) \in$ Isol.

(ix) Apply (6.3.2).

(x) Clearly if $\phi \in$ Crea, then ϕ^G implies ϕ, and hence also $\neg\phi$. Therefore $\phi \in$ Isol.

□

6.3.8 Preservation Lemma.

(i) $\top \in$ Pres,
(ii) $\bot \in$ Pres,

(iii) if $\phi,\psi \in \mathsf{Pres}$ then $\phi \wedge \psi \in \mathsf{Pres}$,
(iv) if $\phi,\psi \in \mathsf{Pres}$ then $\phi \vee \psi \in \mathsf{Pres}$,
(v) if $\phi \in \mathsf{Pres}$ then $(\forall x \in A\ \phi) \in \mathsf{Pres}$,
(vi) if $\phi \in \mathsf{Pres}$ then $(\exists x \in A\ \phi) \in \mathsf{Pres}$,
(vii) if $\phi \in \mathsf{Isol}$ and $\psi \in \mathsf{Pres}$ then $\phi \rightarrow \psi \in \mathsf{Pres}$,
(viii) if $\phi \in \mathsf{Isol}$ then $\neg\phi \in \mathsf{Pres}$,
(ix) if t,s are basic terms then $(t=s) \in \mathsf{Pres}$,
(x) if t is a basic term and X is a variable then $(t \in X) \in \mathsf{Pres}$.

Proof.
(i), (ii), (iii) and (iv) follow trivially from (6.2.3).

(v) Suppose $\phi \in \mathsf{Pres}$ and $\forall x \in A\ \phi$. Then $\phi(x)$. Hence $\phi^G(\eta(x))$ by assumption, and so $\forall x \in A\ \phi^G(\eta(x))$. However we must prove $\forall X \in A^G\ \phi^G(X)$. For $X \in A^G$ we have $j\exists x \in A\ \eta(x)=X$ by lemma (6.3.3). Hence $j\phi^G(X)$, and so by (6.3.5) we get $\phi^G(X)$ and $\forall X \in A^G\ \phi^G(X)$. Therefore $(\forall x \in A\ \phi) \in \mathsf{Pres}$.

(vi) Assume $\phi \in \mathsf{Pres}$ and $\exists x \in A\ \phi$. Then $j\phi(a)$ for some $a \in A$. Hence $j\phi^G(\eta(a))$, and $j\exists X \in A^G\ \phi^G(X)$, i.e., $(\exists x \in A\ \phi)^G$. Thus $(\exists x \in A\ \phi) \in \mathsf{Pres}$.

(vii) Assume $\phi \in \mathsf{Isol}$ and $\psi \in \mathsf{Pres}$. Suppose $\phi \rightarrow \psi$ and ϕ^G. Then we get $j\phi$, hence $j\psi$, and so $j\psi^G$, which equals ψ^G. Thus we see that $\phi^G \rightarrow \psi^G$, and therefore $(\phi \rightarrow \psi) \in \mathsf{Pres}$.

(viii) Apply (ii) and (vii).

(ix) Suppose $t(x)=s(x)$. Then $\eta(t(x))=\eta(s(x))$, and so because t,s are basic terms we get $t^G(\eta(x))=s^G(\eta(x))$ by lemma (6.3.2). So $(t(x)=s(x)) \in \mathsf{Pre}$.

(x) Apply (6.3.2).

□

6.3.9 Corollary. (extending [Visser 81]) Consider some type theory **H**.
(i) for $\phi \in \mathsf{Isol} \cap \mathsf{Pres}$ we have $\phi^G \leftrightarrow j\phi$,
(ii) if $\mathbf{H} \subseteq \mathsf{Isol}$ and $\phi \in \mathsf{Pres}$, then $\mathbf{H} \vdash \phi \Rightarrow \mathbf{H}^G \vdash \phi^G$,
(iii) if $\mathbf{H} \subseteq \mathsf{Isol} \cap \mathsf{Pres}$ and $\phi \in \mathsf{Isol} \cap \mathsf{Pres}$, then $\mathbf{H} \vdash \neg\neg\phi \Rightarrow \mathbf{H} \vdash \phi$,
(iv) if $\mathbf{H} \subseteq \mathsf{Isol} \cap \mathsf{Pres}$ and $\phi \in \mathsf{Isol} \cap \mathsf{Pres}$, then $\mathbf{H} \vdash \neg\neg\phi \rightarrow \phi \Rightarrow \mathbf{H} \vdash \phi \vee \neg\phi$.

Proof.

(i) Assume $\phi \in \mathsf{Isol} \cap \mathsf{Pres}$. If ϕ^G holds then we have $j\phi$ because $\phi \in \mathsf{Isol}$. And if $j\phi$ is the case, then we get $j\phi^G$ and ϕ^G, because $\phi \in \mathsf{Pres}$.

(ii) Assume $\mathbf{H} \subseteq \mathsf{Isol}$, $\phi \in \mathsf{Pres}$ and $\mathbf{H} \vdash \phi$. Now if we assume that $\mathbf{H}^G$ holds, we can conclude that we have $j\psi$ for any $\psi \in \mathbf{H}$. Hence $j\phi$ holds. And so $j\phi^G$ and ϕ^G follow. Therefore $\mathbf{H}^G \vdash \phi^G$.

(iii) Assume $\mathbf{H} \subseteq \mathsf{Isol} \cap \mathsf{Pres}$, $\phi \in \mathsf{Isol} \cap \mathsf{Pres}$ and $\mathbf{H} \vdash \neg\neg\phi$. Let F be the Friedman translation with respect to ϕ.

Then $\mathbf{H}$	$\vdash \mathbf{H}^F$	($\mathbf{H} \subseteq \mathsf{Pres}$)
	$\vdash (\neg\neg\phi)^F$	(by (iii))
	$\vdash (\phi^F \to \phi) \to \phi$	(by definition of F)
	$\vdash ((\phi \vee \phi) \to \phi) \to \phi$	(by (i))
	$\vdash \phi$	

(iv) Assume $\mathbf{H} \subseteq \mathsf{Isol} \cap \mathsf{Pres}$, $\phi \in \mathsf{Isol} \cap \mathsf{Pres}$ and $\mathbf{H} \vdash \neg\neg\phi \to \phi$. Let F be the Friedman translation with respect to $\neg\phi$.

Then $\mathbf{H}$	$\vdash \mathbf{H}^F$	($\mathbf{H} \subseteq \mathsf{Pres}$)
	$\vdash (\neg\neg\phi \to \phi)^F$	(by (iii))
	$\vdash [(\phi^F \to \neg\phi) \to \neg\phi] \to \phi^F$	(by definition of F)
	$\vdash [((\phi \vee \neg\phi) \to \neg\phi) \to \neg\phi] \to (\phi \vee \neg\phi)$	(by (i))
	$\vdash \phi \vee \neg\phi$.	

□

One should observe the verbatim similarity with the proofs by [Visser 81] for **HA**. A careful inspection of the definition of his classes A, Σ and Δ of formulas of **HA** reveals that they are well-chosen subsets of $\mathsf{Isol} \cap \mathsf{Pres}$, in order to satisfy the conditions of (6.3.9).

Another application of the preservation and creation lemmas is the following theorem:

6.3.10 Theorem. Let $\mathbf{H}$ be a $\neg\neg$-preservable type theory, and ϕ a $\neg\neg$-creating formula. Then $\mathbf{H} \vdash \phi \Leftrightarrow \mathbf{H} + \mathsf{PEM} \vdash \phi$.

Proof. Suppose $\mathbf{H} + \mathsf{PEM} \vdash \phi$ and assume for a topos $\mathbb{E}$ that $\mathbb{E} \models \mathsf{H}$. Then $\mathbb{E} \models \mathsf{H} \cup \{j\}$, and so $\mathcal{Sh}_{\neg\neg}\mathbb{E} \models_{\neg\neg} \mathsf{H}$. Thus by soundness $\mathcal{Sh}_{\neg\neg}\mathbb{E} \models_{\neg\neg} \phi$. Hence $\mathbb{E} \models \phi^G$ It follows by the creating property of ϕ that $\mathbb{E} \models \phi$. So by an appeal to completeness we get $\mathsf{H} \vdash \phi$.

□

6.4 G-Preservable theories.

The three lemmata of the last section concerning the preservation, creation and stability of formulas under the G-translation with respect to a certain geometric modality j are useful tools in showing that a particular type theory is G-preservable.

6.4.1 Theorem. Consider **HAH**, higher-order Heyting arithmetic.
(i) **HAH** is G-preservable,
(ii) for any formula ϕ of **HAH** we have $\mathbf{HAH} \vdash \phi^{\neg\neg} \Leftrightarrow \mathbf{HAH}+\mathsf{PEM} \vdash \phi$,
(iii) natural number objects in toposes are preserved by associated sheaf functors.

Proof.
(i) Checking that the following axioms are G-preservable is straightforward with the previous preservation and isolation lemma:

$$P_1 \quad \forall n,m{:}\mathbb{N}\ (s(n)=s(m) \to n=m)$$
$$P_2 \quad \forall n{:}\mathbb{N}\ (0=s(n) \to \bot)$$
$$P_2 \quad \forall X{:}[\mathbb{N}]\ [0\in X \wedge \forall n{:}\mathbb{N}(n\in X \to s(n)\in X) \to \forall x\in \mathbb{N}\ x\in X].$$

(ii) Apply (6.2.9).

(iii) Suppose $\langle \mathbb{N},0,s\rangle$ is a natural number object in the topos $\mathbf{E}$, and $j{:}\Omega \to \Omega$ is some Lawvere-Tierney topology in $\mathbf{E}$. If we apply the associated sheaf functor to $\langle \mathbb{N},0,s\rangle$ we get the triple $\langle L\mathbb{N},L0,Ls\rangle$ in $\mathcal{S}h_j\mathbf{E}$. Because $\langle \mathbb{N},0,s\rangle$ is a natural number object in the topos $\mathbf{E}$, we have an interpretation $\models$ of **HAH** in $\mathbf{E}$. Observe that the interpretation corresponding to $\langle L\mathbb{N},L0,Ls\rangle$ is just $\models_j$ as in the proof of (6.2.7). It follows from the G-preservability of **HAH** that $\mathbf{E} \models \mathbf{HAH}^{G}$. It follows from the previous remark on $\models_j$ and (6.1.8) that $\mathcal{S}h_j\mathbf{E} \models_j \mathbf{HAH}$. Therefore $\langle L\mathbb{N},L0,Ls\rangle$ is the natural number object in $\mathcal{S}h_j\mathbf{E}$.

□

Recall the following definition (cf. e.g. [Johnstone 77]).

6.4.2 Definition. Let **H** be some type theory
(i) The set $\mathbf{Pos_H}$ of *positive* formulas of **H** contains $\top,\bot$ and equality applied to basic terms which type is a basic type or a product of basic types, and is closed under $\wedge,\vee$ and $\exists$.

(ii) A *geometric* formula of **H** is a formula ϕ of the form

$\forall x_1{\in}A_1 \ldots \forall x_n{\in}A_n[\psi_1 \rightarrow \psi_2]$

containing no free variables, where $\psi_1, \psi_2 \in \mathbf{Pos_H}$.

6.4.3 Lemma.
Any geometric formula of a type theory **H** is G-preservable.

Proof. Observe that $\mathbf{Pos_H} \subseteq \mathsf{Iso} \cap \mathsf{Pres}$.

□

Note that our present set up of type theory does not contain relation symbols as primitives as in [Johnstone 77], we consider only function symbols as primitives. This omission is not serious. A relation symbol can be interpreted as a function symbol accompanied by an axiom expressing its monotonicity. Such axioms are G-preservable.

Lemma (6.4.3) implies that our notion of a G-preservable theory extends the well-known notion of a geometric theory. The extension is useful, for instance to derive the above result, which otherwise could not have been proved by this method, as Peano's axioms are not geometric. Note however that one important aspect of geometric formulas is not preserved by our G-preservable formulas, namely preservation by the inverse image of arbitrary geometric morphisms, not only the associated sheaf functors. It is an open question whether G-preservable formulas are preserved by left exact, inverse image functors.

Our final example of G-preservable theories will be the class of *existential fixed point logics* of Blass and Gurevich (cf. [Blass] and [Blass and Gurevich]).

6.4.4 Definition.

(i) The fragment $\square_H$ of a type theory H contains

(a) $\bot, \top$ and formulas of the form $t{=}s$, where t and s are basic terms

(b) formulas of the form $t{\in}P$, where t is a basic term which is basic or a product of basic types, and P a variable

and is closed under

(c) disjunction, conjunction and existential quantification over basic types or products of basic types,

(d) construction of second order formulas of the form
LET $P(x) \leftarrow \delta$ THEN ϕ defined as $\forall P{:}[A](\forall x{:}A(\delta \rightarrow x{\in}P) \rightarrow \phi)$, (we will say that P is *bounded* by the LET P(x)...THEN... construction)

(ii) The fragment EFP_H consists of all formulas $\forall x_1 \in A_1 \ldots \forall x_n \in A_n (\phi \rightarrow \psi)$, where ϕ and ψ are formulas of $\square_H$ in which no second order variable occurs that is not bounded by a LET constructor.

6.4.5 Lemma. Let H be a type theory. Then the formulas in EFP_H are G-preservable.

Proof. Easy proof by induction, using the lemmas in (6.3). E.g., in order to show that

$$\forall P{:}[A](\forall x{:}A(\delta \rightarrow x \in P) \rightarrow \phi)$$

is G-preservable, it suffices to show that δ and ϕ are G-preservable which is given by the induction hypothesis.

□

6.5 A second Gödel-translation preserving types

We will define a variant of the Gödel-translation, that leaves types in tact, but then distorts = and ∈. It is extensionality we have to worry about in this business... We will use this translation in the next chapter 7. In order to keep the inductive proofs simple, we will only consider type theory without explicit function types.

6.5.1 Definition.
The g-translation $t^g \in L_H$ of a term t in the kernel of L_H is obtained inductively by replacing

(i) $(t=s)^G := \eta(t^g) = \eta(s^g)$
(ii) $(t \in s)^G := \eta(t^g) \in \eta(s^g)$
(iii) $(\omega)^G := j\omega$, for $\omega{:}\Omega$
(iv) $(f(t))^G := f(t^g)$ for function symbols $f{:}A \rightarrow B$

6.5.2 Theorem.
(i) For all terms t in the kernel of L_H we have $\eta(t^g(x)) = t^G(\eta(x))$
(ii) For all formulas ϕ in the kernel of L_H we have $\phi^g(x) \leftrightarrow \phi^G(\eta(x))$

Proof.
(i) By induction on the term t in kernel type theory (cf. 1.1.2). There are only four cases of interest
(a) $\eta((s=t)^g(x)) = [\eta(s^g(x)) = \eta(t^g(x))]$

$= [s^G(\eta(x))=t^G(\eta(x))]$
$= (s=t)^G(\eta(x))$

(b) $\eta(\{x{:}A\,|\,\phi\}^g) = \eta(\{x{:}A\,|\,\phi^g\})$
$= \{\eta(x){:}A^L\,|\,\phi^g\}^j$
$= \{x{:}A^L\,|\,j\exists y\in A[\eta(y)=x\wedge\phi^g(y)]\}$
$= \{x{:}A^L\,|\,j\exists y\in A[\eta(y)=x\wedge j\phi^g(y)]\}$
$= \{x{:}A^L\,|\,j\exists y\in A[\eta(y)=x\wedge\phi^G(\eta(y))]\}$
$= \{x{:}A^L\,|\,j\exists y\in A[\eta(y)=x\wedge\phi^G(x)]\}$
$= \{x{:}A^L\,|\,x\in A^G\wedge\phi^G\}$ (6.3.3)
$= \{x{:}A\,|\,\phi\}^G$

(c) $\eta((s\in t)^g(x)) = [\eta(s^g(x))\in\eta(t^g(x))]$
$= [s^G(\eta(x))\in t^G(\eta(x))]$
$= (s\in t)^G(\eta(x))$

(d) $\eta((f(t(x)))^g) = \eta((f(t^g(x))))$
$= f^G(\eta(t^g(x)))$ (6.3.2)
$= f^G(t^G(\eta(x)))$
$= (f(t(y)))^G[y^G{=}\eta(x)]$

(ii) From (i) follows $j\phi^g(x)\leftrightarrow\phi^G(\eta(x))$. As for the G-translation we have $j\phi^g\leftrightarrow\phi^g$ (cf. 6.3.5).

□

6.5.3 Corollary.
(i) $\top^g\leftrightarrow\top$
(ii) $(\phi\wedge\psi)^g\leftrightarrow(\phi^g\wedge\psi^g)$
(iii) $(\phi\rightarrow\psi)^g\leftrightarrow(\phi^g\rightarrow\psi^g)$
(iv) $(\phi\leftrightarrow\psi)^g\leftrightarrow(\phi^g\leftrightarrow\psi^g)$
(v) $(\forall x{:}A\,\phi)^g\leftrightarrow\forall x{:}A\,\phi^g$
(vi) $\bot^g\leftrightarrow j\bot$ ($\leftrightarrow\bot$ in case of a dense topology)
(vii) $(\neg\phi)^g\leftrightarrow(\phi^g\rightarrow j\bot)$ ($\leftrightarrow\neg\phi^g$ in case of a dense topology)
(viii) $(\phi\vee\psi)^g\leftrightarrow j(\phi^g\vee\psi^g)$
(ix) $(\exists x{:}A\,\phi)^g\leftrightarrow j\exists x{:}A\,\phi^g$

Proof.
Almost similar as (6.1.6). For instance:
(i) $\top^g={<}*{=}*{>}^g=[\eta(*)=\eta(*)]=\top$
(ix) $(\exists x{:}A\,\phi)^g \leftrightarrow (\forall\omega\in\Omega\,[\forall x\in A\,(\phi\rightarrow\omega)\rightarrow\omega])^g$
$\leftrightarrow \forall\omega\in\Omega\,[\forall x\in A\,(\phi^g\rightarrow j\omega)\rightarrow j\omega]$

$\leftrightarrow^*$ $\forall\omega\in\Omega\ [\forall x\in A\ (\phi^g\to j\omega))\to j\omega]$
$\leftrightarrow$ $j\forall\omega\in\Omega\ [\forall x\in A\ (\phi^g\to j\omega)\to j\omega]$
$\leftrightarrow$ $j\forall\omega\in\Omega\ [\forall x\in A\ (\phi^g\to j\omega)\to j\omega]$
$\leftrightarrow$ $j\exists x{:}A\ \phi^g$

($\to$*) Substitute $\forall\omega\in\Omega\ [\forall x\in A\ (\phi^g\to j\omega))\to j\omega]$,
and realize that $\phi^g\to\forall\omega\in\Omega\ [\forall x\in A\ (\phi^g\to\omega))\to\omega]$.

□

6.6 On the relation between [A] and $[A]^G$

By induction on type A we define a predicate "X is j-stable" What is the relation between A, A^L and A^G? We know already that A^G is a subtype of A^L via the embedding $e_{AL}{:}A^G\to A^L$. We will construct a subtype of A isomorphic to A^G if A is a powertype. We will need a notion of stable set introduced by Myhill (cf. [Myhill]). The next lemma (6.6.2) will be needed in chapter 7. As in the previous section we will consider in this section type theories without explicit function types.

6.6.1 Definition. By induction on type A we define a predicate "X is j-stable" for $X\subseteq A$:

(i) A is a basic type:
X is j-stable:= T
(ii) A is Ω:
X is j-stable:= $\forall x\in\Omega\,[jx\in X\to x\in X]\wedge\forall x\in X\ \omega=j\omega$
(iii) A is [B]:
X is j-stable:= $\forall x\in[B][jx\in X\to x\in X]\wedge\forall x\in X\,(x$ is j-stable)
(iv) A is BxC:
X is j-stable:= $\forall x\in B\times C[jx\in X\to x\in X]\wedge\forall\langle x,y\rangle\in X\,[x$ is j-stable $\wedge$ y is j-stable].

6.6.2 Lemma. {X∈[A] | X is j-stable} is isomorphic to $[A]^G$ for any type A.

Proof. By induction on A:

(a) If A is a basic type then
{X∈[A] | X is j-stable} = {X∈[A] | X is j-closed}
≈ {Y⊆LA | Y is j-closed}
= $[A]^G$

The isomorphism ≈ needs some explanation, define

$p:\{Y\subseteq LA \mid Y \text{ is j-closed}\} \to \{X\in[A] \mid X \text{ is j-closed}\}: Y \mapsto \{x:A \mid j\exists S\in Y\ x\in S\}$,
and

$q:\{X\in[A] \mid X \text{ is j-closed}\} \to \{Y\subseteq LA \mid Y \text{ is j-closed}\}: X \mapsto \eta(X)$.

p and q are well-defined.
Now, if $X\subseteq A$ and X is j-closed, then $p(q(X))=\{x\in A \mid j\exists S\in\eta(X)\ x\in S\}=X$. And if $Y\subseteq LA$ and Y is j-closed, then

$$\begin{aligned} q(p(Y)) &= \eta(p(Y)) \\ &= \{T\subseteq A \mid T=T^j \wedge j\exists x\in p(Y)\ \{x\}^j=T\} \\ &= \{T\subseteq A \mid T=T^j \wedge j\exists x\in A[j\exists S\in Y\ x\in S \wedge \{x\}^j=T]\} \\ &= \{T\subseteq A \mid T=T^j \wedge j\exists x\in A[j\{x\}^j=T \wedge jT\in Y]\} \\ &= \{T\subseteq A \mid jT\in Y \wedge T=T^j \wedge j\exists x\in A\{x\}^j=T\} \\ &= \{T\in LA \mid jT\in Y\} \\ &= \{T\in LA \mid T\in Y\} \\ &= Y. \end{aligned}$$

Therefore p and q are inverse of each other.

(b) If A is Ω, then

$$\begin{aligned} \{X\in[\Omega] \mid X \text{ is j-stable}\} &= \{X\in[\Omega] \mid X \text{ is j-closed} \wedge \forall x\in X\ \omega=j\omega\} \\ &= \{X\in[\Omega_j] \mid X \text{ is j-closed}\} \\ &= [\Omega]^G \end{aligned}$$

(c) If A is [B], then

$$\begin{aligned} \{X\in[[B]] \mid X \text{ is j-stable}\} &= \\ &= \{X\in[[B]] \mid \forall x\in A[jx\in X\to x\in X] \wedge \forall x\in X\,(x \text{ is j-stable})\} \\ &\approx \{X\subseteq\{x\in[B] \mid x \text{ is j-stable}\} \mid X \text{ is j-closed}\} \\ &= [X\subseteq[B]^G \mid X \text{ is j-closed}\} \\ &= [[B]]^G \end{aligned}$$

(d) If A is BxC, then

$$\begin{aligned} \{X\in[BxC] \mid X \text{ is j-stable}\} &= \\ &= \{X\in[BxC] \mid X \text{ is j-closed} \wedge \forall\langle x,y\rangle\in X\,[x \text{ is j-stable} \wedge y \text{ is j-stable}]\} \\ &\approx \{X\subseteq(BxC)^G \mid X \text{ is j-closed}\} \\ &= [BxC]^G \end{aligned}$$

□

Chapter 7

Classical real numbers from an intuitionistic point of view

If we are working inside a type theory with natural numbers there are real numbers to construct. Let us focus on real numbers constructed by Dedekind cuts. In the constructive context there is a great variety of definitions: Dedekind reals, Dedekind-MacNeille reals (cf. [Johnstone 77], [Troelstra 80] and [Troelstra and van Dalen 88]), Troelstra's extended reals, Troelstra's classical reals (cf. [Troelstra 80] and [Troelstra and van Dalen 88]), Staples reals (cf. [Staples], [Troelstra 82] and [Troelstra and van Dalen 88]), van Dalen's singleton reals (cf. [van Dalen], [Troelstra 82] and [Troelstra and van Dalen 88]). All these different real number objects can be seen as linguistic variations of the standard definition (in the case of the singleton reals this true up to an isomorphism) obtained by scattering the double negation all over the standard definition.

We will give a systematic and general treatment of all these definitions using an arbitrary modal operator j instead of $\neg\neg$. We distinguish three different methods to construct Dedekind-real-like number objects:

(i) Take the usual definition, and vary it by putting a modal operator j on some places.
(ii) Take an Dedekind-real-like number object and apply the associated sheaf construction to it.
(iii) Take a linguistic reformulation, and apply the generalized Gödel translation to it.

In the context of toposes these methods of constructing Dedekind eals can be explained in categorical terms.
Let $\mathbb{E}$ be a topos with a natural number object.
(i) The interpretation of the linguistic reformulations gives us different real number objects in $\mathbb{E}$.
(ii) We can apply the associated sheaf construction to the objects of (i) to obtain a new set of objects that are sensible extensions of Dedekind real numbers.
(iii) If $\mathbb{E}$ has natural numbers, then the topos of j-sheaves has natural numbers as well. Hence in $\mathcal{Sh}_j\mathbb{E}$ too we can interpret the linguistic reformulations of natural numbers. Using the G-translation we can describe these objects of $\mathcal{Sh}_j\mathbb{E}$ in terms of

the natural numbers of the base topos $\mathbb{E}$. In this way we obtain a third class of real number objects in $\mathbb{E}$.

Dedekind reals can defined with help of left cuts of the rationals. Since have not yet constructed the rationals, we first will indicate how to construct the integers and rationals from the natural numbers. By usual induction methods we enrich the natural numbers with addition and multiplication. Then we define a predicate $\mathrm{int}: [\mathbb{N}\times\mathbb{N}] \longrightarrow \Omega$ by

$$z \mapsto \begin{cases} \exists\langle n,m\rangle \in \mathbb{N}\times\mathbb{N}\ \langle n,m\rangle \in z \\ \wedge \\ \forall\langle n_1,m_1\rangle,\langle n_2,m_2\rangle \in z\ n_1+n=m_1+m_2 \end{cases}$$

and a predicate $\mathrm{rat}: [\mathbb{N}\times\mathbb{N}\times\mathbb{N}] \longrightarrow \Omega$ by

$$r \mapsto \begin{cases} \exists\langle n,m,l\rangle \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} \langle n,m,l\rangle \in r \\ \wedge \\ \forall\langle n_1,m_1,l_1\rangle,\langle n_2,m_2,l_2\rangle \in r\ n_1(l_2+1)+n_2(l_1+1)=m_1(l_2+1)+m_2(l_1+1) \end{cases}$$

This gives us our familiar objects

$$\mathbb{Z} := \{z \in [\mathbb{N}\times\mathbb{N}] \mid \mathrm{int}(z)\}$$

and

$$\mathbb{Q} := \{r \in [\mathbb{N}\times\mathbb{N}\times\mathbb{N}] \mid \mathrm{rat}(r)\}.$$

7.1 Linguistic variations on Dedekind real numbers

We give a list of predicates for subsets $W \subseteq \mathbb{Q}$ that we will use in the definitions of left cut for the various real number objects.

7.1.1 Definition.

(i) $B_j := j\exists r \in \mathbb{Q}\ r \in W \wedge j\exists r' \in \mathbb{Q}\ \neg r' \in W$ (Boundedness)

(ii) $L_j := \forall r,r' \in \mathbb{Q}\ [r<r' \rightarrow j(r \in W \vee \neg r' \in W)]$ (Locatedness)

(iii) $M_j := \forall r,r' \in \mathbb{Q}\ [r<r' \rightarrow (jr' \in W \rightarrow r \in W)]$ (Monotonicity)

(iv) $O_j := \forall r \in W\ j\exists r' \in W\ r'>r$ (Openness)

(v) $S_j := \forall r \in \mathbb{Q}\ (jr \in W \rightarrow r \in W)$ (Stability)

Remark.

(i) $L \rightarrow M_j$, provided j is dense (cf. 7.1.3.i)

(ii) $B \rightarrow B_j$, $L \rightarrow L_j$ and $O \rightarrow O_j$

(iii) $M_j \to M$ and $S_j \to S$
(iv) In case of a dense topology ($j\omega \to \neg\neg\omega$ for all $\omega \in \Omega$, cf. (2.2.9)) $M_{\neg\neg} \to M_j$.

Notation. We will omit the subscript j if we want the use the predicates without a modal operator, and we replace j by $\neg\neg$ whenever we use the predicates with a double negation operator.

7.1.2 Definition.
Variations on L and M:

(i) $\mathbb{R} = \{W \subseteq \mathbb{Q} \mid B \wedge L \wedge O\}$ Dedekind reals
(ii) $\mathbb{R}^{LM} = \{W \subseteq \mathbb{Q} \mid B \wedge L_j \wedge M_j \wedge O\}$
(iii) $\mathbb{R}^{e} = \{W \subseteq \mathbb{Q} \mid B \wedge M_{\neg\neg} \wedge O\}$ MacNeille reals, extended reals
(iv) $\mathbb{R}^{M} = \{W \subseteq \mathbb{Q} \mid B \wedge M_j \wedge O\}$
(v) $\mathbb{R}^{*} = \{W \subseteq \mathbb{Q} \mid B \wedge M \wedge O\}$ Staples reals

Weakening of B:

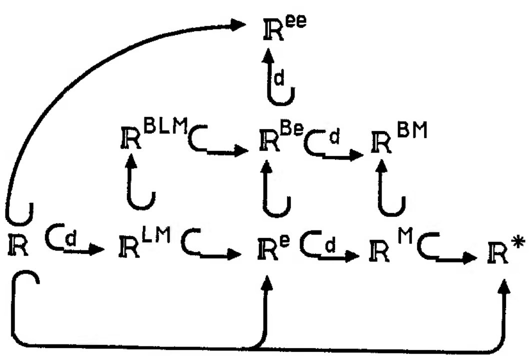
(vi) $\mathbb{R}^{BLM} = \{W \subseteq \mathbb{Q} \mid B_j \wedge L_j \wedge M_j \wedge O\}$
(vii) $\mathbb{R}^{Be} = \{W \subseteq \mathbb{Q} \mid B_j \wedge M_{\neg\neg} \wedge O\}$
(viii) $\mathbb{R}^{BM} = \{W \subseteq \mathbb{Q} \mid B_j \wedge M_j \wedge O\}$

Finally:

(ix) $\mathbb{R}^{ee} = \{W \subseteq \mathbb{Q} \mid B_{\neg\neg} \wedge M_{\neg\neg} \wedge O\}$ Troelstra's classical reals

Remark.
(i) $\mathbb{R} = \{W \subseteq \mathbb{Q} \mid B \wedge L \wedge O\}$ is identical with $\{W \subseteq \mathbb{Q} \mid B \wedge L \wedge M \wedge O\}$, as $L \to M$.
(ii) Because $M_{\neg\neg} \to L_{\neg\neg}$ we see that $\mathbb{R}^{LM}$ and $\mathbb{R}^{M}$ are identical with $\mathbb{R}^{e}$ in case of the double negation topology.

7.1.3 Lemma. The various objects of Dedekind real number objects are related according to the following picture (an embedding marked with a d means that the embedding exists only if the topology j happens to be dense):



Proof.
The embeddings $\mathbb{R}\subseteq\mathbb{R}^{e}\subseteq\mathbb{R}^{*}$ and $\mathbb{R}^{E}\subseteq\mathbb{R}^{ee}$ are easy to verify (cf. [Troelstra 80].)

(i) $\mathbb{R}\subseteq\mathbb{R}^{LM}$: it suffices to prove that $L\rightarrow M_j$ under the assumption that j is dense. For $W\in\mathbb{R}$ and $r<r'$ in $\mathbb{Q}$ assume $jr'\in W$. Locatedness of W tells us that $r\in W\vee\neg r'\in W$. But $\neg r'\in W$ and $jr'\in W$ imply that $j\bot$, and hence by density of j we get $\bot$. So we can conclude $r\in W$. And we have shown M_j

(ii) $\mathbb{R}^{LM}\subseteq\mathbb{R}^{e}$: it suffice to show that $L_j\wedge M_j\rightarrow M_{\neg\neg}$.
Hence for $W\in\mathbb{R}^{LM}$ and $r<r'$ in $\mathbb{Q}$ assume $\neg\neg r'\in W$. Thus $j(\neg\neg r'\in W)$. Define $r'':=(r+r')/2$. Then $r<r''<r'$, and so by L_j we get $j(r''\in W\vee\neg r'\in W)$. Hence $jr''\in W$. If we apply M_j we get $r\in W$. Thus we see that $M_{\neg\neg}$ holds for $W\in\mathbb{R}^{LM}$.

(iii) $\mathbb{R}^{e}\subseteq\mathbb{R}^{M}$: if j is dense then $M_{\neg\neg}\rightarrow M_j$, as we remarked above.

(iv) $\mathbb{R}^{M}\subseteq\mathbb{R}^{*}$, trivial as $M_j\rightarrow M$.

(v) $\mathbb{R}^{LM}\subseteq\mathbb{R}^{BLM}$, $\mathbb{R}^{e}\subseteq\mathbb{R}^{Be}$ and $\mathbb{R}^{M}\subseteq\mathbb{R}^{BM}$ are trivial, since $B_j\rightarrow B$.

(vi) $\mathbb{R}^{BLM}\subseteq\mathbb{R}^{Be}$: similar as (ii).

(vii) $\mathbb{R}^{Be}\subseteq\mathbb{R}^{BM}$: verbatim the same as (iii).

(viii) $\mathbb{R}^{Be}\subseteq\mathbb{R}^{ee}$: if j is dense then $B_j\rightarrow B_{\neg\neg}$.

□

7.1.4 Lemma.
(i) $\mathbb{R}^{BM}$ and all its subobjects have a j-stable equality.
(ii) $\mathbb{R}^{ee}$ and all its subobjects have a $\neg\neg$-stable equality.

Proof. The proof of the (i) is an easy generalization of [Troelstra 80]'s proof that $\mathbb{R}^{ee}$ has a $\neg\neg$-stable equality:

(i) Assume that $jV=W$ for $V,W\in\mathbb{R}^{BM}$. We will show $\forall r\in\mathbb{Q}\,(r\in V\leftrightarrow r\in W)$. Hence suppose $r\in V$ for some $r\in\mathbb{Q}$. Then by O there exists an $r'>r$ such that $r'\in V$. So from the assumption $jV=W$ we can conclude $jr'\in W$. Therefore by M_j we get $r\in W$. Thus we have shown $\forall r\in\mathbb{Q}\,(r\in V\rightarrow r\in W)$. The converse is similar. Hence $V=W$.

(ii) Substitute $\neg\neg$ for j.

□

7.1.5 Lemma. $\mathbb{R}\subseteq\mathbb{R}^{LM}$ if and only if j is dense.

Proof. "only if" cf. (7.1.3.i).
"if" Assume $\mathbb{R}\subseteq\mathbb{R}^{LM}$. We map $\mathbb{Q}$ into $\mathbb{R}$ via the assignment $[-]:r\mapsto\{r'\in\mathbb{Q}\mid r'<r\}$.
Now $[0]=[1]\ \rightarrow\ \forall r\in\mathbb{Q}\,(r<0\leftrightarrow r<1)$

$$\to (0<0 \leftrightarrow 0<1) \wedge (1<0 \leftrightarrow 1<1)$$
$$\to \neg 0<1 \wedge \neg 1<0$$
$$\to 0=1$$

by trichotomy of $\mathbb{Q}$.
Hence $0=1 \leftrightarrow [0]=[1]$. But $0=1 \leftrightarrow \bot$. So, if we have $j\bot$, then we get $j0=1$. Applying j-stability of equality of $\mathbb{R}^{LM}$ (7.1.4) we conclude $0=1$ and so we find $\bot$. Therefore j is dense.

□

7.1.6 Theorem. The object $\mathbb{R}^{ee}$ is a $\neg\neg$-sheaf.

Proof.
For $X \subseteq \mathbb{R}^{ee}$ assume

$$\neg\neg\exists! W \in \mathbb{R}^{ee}\, W \in X.$$

We define

$$V = \{r \in \mathbb{Q} \mid \exists r' > r\ \neg\neg\exists W \in X\, r' \in W\}.$$

In order to conclude that $\mathbb{R}^{ee}$ is a $\neg\neg$-sheaf we have to show that

(i) $V \in \mathbb{R}^{ee}$
(ii) $\neg\neg V \in X$
(iii) if $\neg\neg W \in X$ holds for $W \in \mathbb{R}^{ee}$, then $V = W$.

Proof of (i). We have to prove $B_{\neg\neg}$, $M_{\neg\neg}$ and O for V.
($B_{\neg\neg}$) Observe that

$$\exists! W \in \mathbb{R}^{ee}\, W \in X$$

implies

$$\neg\neg\exists r \in \mathbb{Q}\ r \in V,$$

or, equivalently,

$$\neg\neg\exists r \in \mathbb{Q}\ \exists r' > r\ \exists W \in X\, r' \in W.$$

by application of $B_{\neg\neg}$ to the unique $W \in \mathbb{R}^{ee}$ in X.
Hence we conclude that from the assumption

$$\neg\neg\exists! W \in \mathbb{R}^{ee}\, W \in X$$

follows

$\neg\neg\exists r \in \mathbb{Q}\ r \in V$, and similarly $\neg\neg\exists r \in \mathbb{Q}\ \neg r \in V$.

($M_{\neg\neg}$) For $r < r'$ in $\mathbb{Q}$ assume $\neg\neg r' \in V$. Put $r'' = (r'-r)/2$. Then

$$\neg\neg\exists r' > r''\ \neg\neg\exists W \in X\, r' \in W$$

and so we get

$$\neg\neg\exists r' > r''\ \exists W \in X\, r' \in W$$

and

$$\neg\neg\exists W \in X\, \exists r' > r''\ r' \in W.$$

By monotonicity it follows now that $\neg\neg\exists W\in X\, r''\in W$, i.e., $r\in V$. Therefore $M_{\neg\neg}$ holds for V.

(O) For $r\in V$ there is $r'>r$ such that $\neg\neg\exists W\in X\, r'\in W$. Put again $r''=(r'-r)/2$. Then by monotonicity of W we get $\neg\neg\exists W\in X\, r''\in W$, and we have $r''>r$ such that $r''\in V$. Therefore O holds for V.

Proof of (ii) Suppose $W\in\mathbb{R}^{ee}$ is the unique element in X. Then we claim that $W=V$. Hence $\neg\neg\exists! W\in\mathbb{R}^{ee}\, W\in X$ implies $\neg\neg\{V\}=X$

Proof of (iii). From (ii) follows: if for some $W\in\mathbb{R}^{ee}$ it holds that $\neg\neg W\in X$, then $\neg\neg V=W$. Hence since equality of $\mathbb{R}^{ee}$ is $\neg\neg$-stable (cf. 7.1.4) we get that $\neg\neg W\in X$ implies $V=W$ for $W\in\mathbb{R}^{ee}$.

□

7.1.7 Lemma. (cf. [Troelstra 80])
$\mathbb{R}^*$ is a quotient of $\mathbb{R}^M$. In particular $\mathbb{R}^*$ is a quotient of $\mathbb{R}^e$.

Proof.
For $W\subseteq\mathbb{Q}$ define
(i) $W^j:=\{r\in\mathbb{Q}\mid jr\in W\}$
(ii) $int(W):=\{r\in\mathbb{Q}\mid \exists r'\in\mathbb{Q}\,[r'>0\wedge\forall r''\in\mathbb{Q}\,(|r-r''|<r'\rightarrow r''\in W)]\}$.
Then $\approx$ defined by $V\approx W\leftrightarrow int(V^j)=int(W^j)$ is an equivalence relation on $\mathbb{R}^M$, and $\mathbb{R}^M/_\approx$ is isomorphic with $\mathbb{R}^*$. Hint: show that $W\approx\cup\{V\in\mathbb{R}^M\mid V\approx W\}$ and $\cup\{V\in\mathbb{R}^M\mid V\approx W\}\in\mathbb{R}^*$ for all $W\in\mathbb{R}^M$.

□

This spectrum of Dedekind reals appears to be inexhaustible: one can also weaken openness. The first isomorphism theorem will tell us that this does not result in new real number objects.

7.1.8 Definition.
(ix) $\mathbb{R}^{BLO}=\{W\subseteq\mathbb{Q}\mid B_j\wedge L_j\wedge O_j\wedge S_j\}$ $(=\{W\subseteq\mathbb{Q}\mid B_j\wedge L_j\wedge M_j\wedge O_j\wedge S_j\})$
(x) $\mathbb{R}^{BOe}=\{W\subseteq\mathbb{Q}\mid B_j\wedge M_{\neg\neg}\wedge O_j\wedge S_j\}$ $(=\{W\subseteq\mathbb{Q}\mid B_j\wedge M_{\neg\neg}\wedge M_j\wedge O_j\wedge S_j\})$
(xi) $\mathbb{R}^{BMO}=\{W\subseteq\mathbb{Q}\mid B_j\wedge M\wedge O_j\wedge S_j\}$ $(=\{W\subseteq\mathbb{Q}\mid B_j\wedge M_j\wedge O_j\wedge S_j\})$

This definition generalizes [Troelstra 80]'s definition of the real number object $\mathbb{R}_1^{ee}$. Troelstra proved that $\mathbb{R}_1^{ee}$ is bijective to $\mathbb{R}^{ee}$. For our generalizations we have similar bijections.

7.1.9 Definition.

(i) $\mathrm{int}: \mathbb{R}^{BMO} \to \mathbb{R}^{BM} := W \mapsto \{r \in \mathbb{Q} \mid \exists r' > r\ r' \in W\}$

(ii) $(-)^j: \mathbb{R}^{BM} \to \mathbb{R}^{BMO} := W \mapsto \{r \in \mathbb{Q} \mid jr \in W\}$

7.1.10 First isomorphism theorem. Let j be a dense topology.

(i) $\mathrm{int}: \mathbb{R}^{BMO} \to \mathbb{R}^{BM}$ and $(-)^j: \mathbb{R}^{BM} \to \mathbb{R}^{BMO}$ are each others inverse,

(ii) int and $(-)^j$ restrict to inverses on $\mathbb{R}^{BOe}$ and $\mathbb{R}^{Be}$,

(iii) int and $(-)^j$ restrict to inverses on $\mathbb{R}^{BLO}$ and $\mathbb{R}^{BLM}$.

Proof.

(i) The proof splits in three obvious parts. First we will show that int is well-defined, then we prove that $(-)^j$ is well-defined, and finally we establish that $(-)^j$ and int are are each others inverse.

First part. $\mathrm{int}: \mathbb{R}^{BMO} \to \mathbb{R}^{BM}$ is well-defined.

Let $W \in \mathbb{R}^{BMO}$. We have to show that $B_j \wedge M_j \wedge O$ holds for $\mathrm{int}(W)$, for then $\mathrm{int}(W) \in \mathbb{R}^{BM}$

(B_j) O^j implies $r \in W \to jr \in \mathrm{int}(W)$. Hence O_j also implies that $j\exists r \in \mathbb{Q}\ r \in \mathrm{int}(W)$ follows from $j\exists r \in \mathbb{Q}\ r \in W$. Since O_j and B_j hold for $W \in \mathbb{R}^{BMO}$ we conclude $j\exists r \in \mathbb{Q}\ r \in \mathrm{int}(W)$.

On the other hand note that $\neg r \in \mathrm{int}(W) \leftrightarrow \forall r' > r\ \neg r' \in W$. Hence $\neg r \in \mathrm{int}(W)$ follows from $M \wedge \neg r \in W$. A similar introduction of j enables us to conclude that $j\exists r \in \mathbb{Q}\ \neg r \in W$ implies $j\exists r \in \mathbb{Q}\ \neg r \in \mathrm{int}(W)$. Now, as M and B_j hold for $W \in \mathbb{R}^{BMO}$ we conclude $j\exists r \in \mathbb{Q}\ \neg r \in \mathrm{int}(W)$.

Thus B_j holds for $\mathrm{int}(W)$.

(M_j) Let $r < r'$. Clearly $r' \in \mathrm{int}(W) \to r' \in W \wedge r \in W$. So, if we assume $jr' \in \mathrm{int}(W)$, we obtain $jr' \in W$ and $jr \in W$. Then we have $r \in \mathrm{int}(W)$ by S_j.

Hence M_j holds for $\mathrm{int}(W)$.

(O) Let $r \in \mathrm{int}(W)$. I.e., we have an $r'' > r$ such that $r'' \in W$. Consider $r' := (r + r'')/2$. Then $r < r' < r''$, and so $r' \in W$ by M. We get that $r' \in \mathrm{int}(W)$. It follows that O holds for $\mathrm{int}(W)$.

Second part. $(-)^j: \mathbb{R}^{BM} \to \mathbb{R}^{BMO}$ is well-defined.

Let $W \in \mathbb{R}^{BM}$. We have to prove that $B_j \wedge M \wedge O_j \wedge S_j$ hold for W^j in order to conclude $W^j \in \mathbb{R}^{BMO}$.

(B_j) Clearly $j\exists r \in \mathbb{Q}\ r \in W \rightarrow j\exists r \in \mathbb{Q}\ jr \in W$. Similarly $j\exists r \in \mathbb{Q}\ \neg r \in W$ implies that $j\exists r \in \mathbb{Q}\ j(\neg r \in W)$. Using the density of j we get $j\exists r \in \mathbb{Q}\ \neg jr \in W$. Since B_j holds for $W \in \mathbb{R}^{BM}$ we see that B_j holds for W^j.

(M_j) Let $r \leq r$, and assume $r' \in W^j$, i.e., $jr' \in W$. Then $r \in W$ by M and so $r \in W^j$. Thus M holds for W^j.

(O_j) Let $r \in W^j$, i.e., $jr \in W$. We get $j\exists r'>r\ r' \in W$ by O, as O holds for W. Hence $j\exists r'>r\ jr' \in W$, and so $j\exists r'>r\ r' \in W$. Therefore O_j holds for W^j.

(S_j) The idempotency of j trivially implies that S_j holds for W^j.

Third part. int and $(-)^j$ are each others inverse.

For $W \in \mathbb{R}^{BMO}$ we have must show that $(int(W))^j = \{r \in \mathbb{Q} \mid j\exists r'>r\ r' \in W\} = W$.
It follows easily from O_j that $W \subseteq (int(W))^j$. Thus $(int(W))^j \subseteq W$ remains to be proved. M implies $\exists r'>r\ r' \in W \rightarrow r \in W$. Hence $j\exists r'>r\ r' \in W \rightarrow jr \in W$. Assuming that $j\exists r'>r\ r' \in W$ holds for $r \in \mathbb{Q}$ we see that S_j and M together imply $r \in W$. Whence $(int(W))^j \subseteq W$.

For $W \in \mathbb{R}^{BM}$ we have to prove that $int(W^j) = \{r \in \mathbb{Q} \mid \exists r'>r\ jr' \in W\} = W$. As above it follows easily from O that $W \subseteq int(W^j)$. And the reverse follows from M_j.

(ii) Let $W \in \mathbb{R}^{BOe}$. If we show that $M_{\neg\neg}$ holds for $int(W)$ then int restricts to a mapping from $\mathbb{R}^{BOe}$ to $\mathbb{R}^{Be}$.
So let $r \leq r'$, and assume $\neg\neg r' \in int(W)$, that is $\neg\neg\exists r''>r'\ r'' \in W$. Since $M_{\neg\neg}$ implies that M holds for W in $\mathbb{R}^{BOe}$ we get $\exists r'' \geq r'\ r'' \in W \rightarrow r' \in W$. Hence after suitable adding j and $\neg\neg$ we obtain $\neg\neg r' \in W$, using the assumption and S_j. Now by $M_{\neg\neg}$ we get $r \in W$ and $(r+r')/2 \in W$. Therefore $r \in int(W)$. I.e., $M_{\neg\neg}$ holds for $int(W)$.

Let $W \in \mathbb{R}^{Be}$. If we show that $M_{\neg\neg}$ holds for W^j then $(-)^j$ restricts to a mapping from $\mathbb{R}^{Be}$ to $\mathbb{R}^{BOe}$.
Let again $r<r$, and assume $\neg\neg r' \in W^j$, i.e., $\neg\neg jr' \in W$. Hence by density of j we get $\neg\neg r' \in W$, but then $r \in W$ by $M_{\neg\neg}$. And so $jr \in W$ and $r \in W^j$. That is $M_{\neg\neg}$ holds for W^j.

(iii) Let $W \in \mathbb{R}^{BLO}$. If we show that L_j holds for $int(W)$ then int restricts to a mapping from $\mathbb{R}^{BLO}$ to $\mathbb{R}^{BLM}$.
Let $r<r'$. Then $j(r \in W \vee \neg r' \in W)$. Now $r \in W$ implies $j\exists r''>r\ r'' \in W$ by O_j. And $\neg r' \in W$ implies $\forall r''>r\ \neg r'' \in W$ by L_j and the density of j, i.e., $\neg\exists r''>r\ r'' \in W$. Hence, putting things together we can conclude $j(jr \in int(W) \vee \neg r' \in int(W))$.

Therefore $j(r\in int(W)\vee\neg r'\in int(W))$. And so, L_j holds for $int(W)$.

Let $W\in\mathbb{R}^{BLM}$. If we show that L_j holds for W^j then $(-)^j$ restricts to a mapping from $\mathbb{R}^{BLM}$ to $\mathbb{R}^{BLO}$. Let $r<r'$, then as above we get $j(r\in W\vee\neg r'\in W)$. Hence $j(jr\in W\vee j\neg r'\in W)$. Which implies $j(r\in W^j\vee\neg r'\in W^j)$ using that j is dense. Therefore L_j holds for W^j.

□

7.2 Dedekind reals extended by singletons

Van Dalen used a $\neg\neg$-singleton construction to obtain an extension $\mathbb{R}^s$ of the Dedekind reals $\mathbb{R}$. $\mathbb{R}^s$ turned out to be isomorphic with a subobject of the classical reals $\mathbb{R}^{ee}$ of Troelstra. $\mathbb{R}^S$ is neither equal to $\mathbb{R}$ nor to $\mathbb{R}^{ee}$, but $\mathbb{R}^S=\{x\in\mathbb{R}^{ee}\mid\neg\neg x\in\mathbb{R}\}$ See [van Dalen 80] and [Troelstra 82] for a detailed treatment of $\mathbb{R}^s$.
Van Dalen's $\mathbb{R}^s$ consists of (vii)-singletons of $\mathbb{R}$ for the double negation topology subject to local equality (i.e., two singletons S,T are considered to be equal if $\neg\neg S=T$). This construction can be recognized as an internal version of a single step of the Johnstone-Grothendieck construction with (vii)-singletons. Since the type of Dedekind reals $\mathbb{R}$ is $\neg\neg$-separated, the execution on $\mathbb{R}$ of already a single step of the Johnstone-Grothendieck construction results in a $\neg\neg$-sheaf (cf. chapter 3).
In this section we will use the Lawvere construction L for a *dense* topology $j:\Omega\to\Omega$. It follows from the results in chapter 3 and (7.1.4) that $L\mathbb{R}$ is isomorphic to $\mathbb{R}^S$.

If we compare the definitions of the types of integers and rationals, respectively $\mathbb{Z}=\{z\in[\mathbb{N}x\mathbb{N}]\mid int(z)\}$ and $\mathbb{Q}=\{r\in[\mathbb{N}x\mathbb{N}x\mathbb{N}]\mid rat(r)\}$, with the definition of Dedekind reals, $\mathbb{R}=\{W\in[\mathbb{Q}]\mid W \text{ is a left cut}\}$, we observe that the three definitions are of the form $\{x\in[A]\mid\phi\}$.

7.2.1 Lemma. Let A be a type.
Then $L\{x\in[A]\mid\phi\}$ is isomorphic to $\{x\in[A]\mid x \text{ is j-closed}\wedge j\phi\}$.

Proof. The proof is essentially an application of (2.3.7), (3.2.2.iii) and the discussion in (3.4): for any A the type Ω_j^A is a sheaf, hence $L\Omega_j^A\approx\Omega_j^A$ via the following two inverses:

$L\Omega_j^A\to\Omega_j^A:S\mapsto\{x\in A\mid j\exists R\in[A]\, x\in R\in S\}$

and

$\Omega_j^A\to L\Omega_j^A:R\mapsto\{R\}^j$.

With help of these isomorphisms we calculate:

$$\begin{aligned} L\{x\in[A]\mid\phi\} &= \{S\subseteq\{x\in[A]\mid\phi\}\mid S \text{ is j-closed}\wedge j\exists x\in[A](\phi\wedge S=\{x\}^j)\} \\ &\approx \{y\in[A]\mid y \text{ is j-closed}\wedge j\exists x\in[A](\phi\wedge\{y\}^j=\{x\}^j)\} \\ &= \{y\in[A]\mid y \text{ is j-closed}\wedge j\exists x\in[A](\phi\wedge jx=y)\} \\ &= \{x\in[A]\mid x \text{ is j-closed}\wedge j\phi\} \end{aligned}$$

□

7.2.2 Corollary. For a dense topology j we have:

(i) $L(\mathbb{R})=L(\mathbb{R}^{LM})=\mathbb{R}^{BLO}\approx\mathbb{R}^{BLM}$

(ii) $L(\mathbb{R}^{e})=\mathbb{R}^{BOe}\approx\mathbb{R}^{Be}$

(iii) $L(\mathbb{R}^{M})=L(\mathbb{R}^{*})=\mathbb{R}^{BMO}\approx\mathbb{R}^{BM}$.

In particular for the double negation topology and van Dalen's singleton construction it holds that:

(iv) $\mathbb{R}^{s}\approx\{W\in\mathbb{R}^{ee}\mid\neg\neg W\in\mathbb{R}\}=\{W\in\mathbb{R}^{ee}\mid L_{\neg\neg}\}$

(v) $(\mathbb{R}^{e})^{s}\approx\mathbb{R}^{ee}$

(vi) $(\mathbb{R}^{*})^{ss}\approx\mathbb{R}^{ee}$.

Proof. Recall that W is j-closed is abbreviated by S_j. Observe that:

(i) $S_j\wedge j(B\wedge L\wedge O)=B_j\wedge L_j\wedge O_j\wedge S_j=j(B\wedge L_j\wedge M_j\wedge O)\wedge S_j$

(ii) $S_j\wedge j(B\wedge M_{\neg\neg}\wedge O)=B_j\wedge M_{\neg\neg}\wedge O_j\wedge S_j$

(iii) $S_j\wedge j(B\wedge M_j\wedge O)=B_j\wedge M\wedge O_j\wedge S_j=j(B\wedge M\wedge O)\wedge S_j$

Application of (7.1.10) gives the isomorphisms.

(iv)
$$\begin{aligned} \mathbb{R}^{s} &\approx \{W\in[\mathbb{Q}]\mid B_{\neg\neg}\wedge L_{\neg\neg}\wedge O_{\neg\neg}\wedge S_{\neg\neg}\} \\ &= \{W\in[\mathbb{Q}]\mid B_{\neg\neg}\wedge L_{\neg\neg}\wedge M_{\neg\neg}\wedge O_{\neg\neg}\wedge S_{\neg\neg}\} \\ &\approx \{W\in[\mathbb{Q}]\mid B_{\neg\neg}\wedge L_{\neg\neg}\wedge M_{\neg\neg}\wedge O\} && \text{(just as in 7.1.10 iii)} \\ &= \{W\in\mathbb{R}^{ee}\mid\neg\neg W\in\mathbb{R}\} \\ &= \{W\in\mathbb{R}^{ee}\mid L_{\neg\neg}\} \end{aligned}$$

(v)
$$\begin{aligned} (\mathbb{R}^{e})^{s} &\approx \{W\in[\mathbb{Q}]\mid B_{\neg\neg}\wedge M_{\neg\neg}\wedge O_{\neg\neg}\wedge S_{\neg\neg}\} \\ &= \mathbb{R}^{B\neg\neg e} \\ &\approx \mathbb{R}^{ee} && \text{(7.1.10)} \end{aligned}$$

(vi)
$$\begin{aligned} (\mathbb{R}^{*})^{ss} &\approx L(\mathbb{R}^{*}) && \text{(3.3)} \\ &\approx \{W\in[\mathbb{Q}]\mid B_{\neg\neg}\wedge M_{\neg\neg}\wedge O_{\neg\neg}\wedge S_{\neg\neg}\} && \text{(Troelstra's } \mathbb{R}_1^{ee}\text{)} \\ &\approx \{W\in[\mathbb{Q}]\mid B_{\neg\neg}\wedge M_{\neg\neg}\wedge O\} && \text{(7.1.10)} \\ &= \mathbb{R}^{ee} \end{aligned}$$

□

7.3 Dedekind reals obtained by Gödel translations

From the classical definition of a Dedekind cut Troelstra deduced the definition of the intuitionistic real number object $\mathbb{R}_1^{ee}$ by application of a double negation translation. Via a natural simplification Troelstra showed that $\mathbb{R}_1^{ee}$ is isomorphic to $\mathbb{R}^{ee}$.

We can generalize this approach using the G-translation for arbitrary geometric modality j.

Let $\mathbb{E}$ be some topos with a natural number object (for instance the free topos corresponding to the type theory we are working in). Associated sheaf functors preserve natural number objects, hence we have natural number objects in the sheaf toposes $\mathcal{Sh}_j\mathbb{E}$ and $\mathcal{Sh}_{\neg\neg}\mathbb{E}$. Thus we can construct real number objects in the sheaf toposes in exactly the same way as we construct them in $\mathbb{E}$. So in $\mathcal{Sh}_j\mathbb{E}$ we can construct $\mathbb{R}$, $\mathbb{R}^e$ and $\mathbb{R}^*$. Note that in case of the boolean topos $\mathcal{Sh}_{\neg\neg}\mathbb{E}$ all three constructions result in the same object of Dedekind reals $\mathbb{R}$.

In this section we will show by an application of the j-translation of $L_{\mathcal{Sh}_j\mathbb{E}}$ in $L_{\mathbb{E}}$ that the objects $\mathbb{R}$, $\mathbb{R}^e$ and $\mathbb{R}^*$ in $\mathcal{Sh}_j\mathbb{E}$ are isomorphic to the objects $\mathbb{R}^{BLM}$, $\mathbb{R}^{Be}$ and $\mathbb{R}^{BM}$ in $\mathbb{E}$. In particular the object of Dedekind real numbers in the topos $\mathcal{Sh}_{\neg\neg}\mathbb{E}$ is isomorphic to the object of classical reals $\mathbb{R}^{ee}$ in $\mathbb{E}$. This supports the slogan that the appropriate candidate for classical continuum from the intuitionistic point of view is the type of classical real numbers.

Hence from the point of view of $\mathcal{Sh}_j\mathbb{E}$ the objects $L\mathbb{N}$ and $\mathbb{R}^{BLM}$ are just $\mathbb{N}$ and $\mathbb{R}$. Although the object $L\mathbb{R}$ lives inside $\mathcal{Sh}_j\mathbb{E}$, it plays no natural rôle.

What happens if we G-translate the definition of $\mathbb{R}$? A first answer is provided by the following lemma.

7.3.1 Lemma.; $\mathbb{R}^G$ consists of subtypes $W \subseteq \mathbb{Q}^G$ satisfying

(i) W is j-closed,

(ii) $j\exists R\in\mathbb{Q}^G\, R\in W \wedge j\exists R'\in\mathbb{Q}^G\, \neg R'\in W$

(ii) $\forall R,R'\in\mathbb{Q}^G\, [R<R' \rightarrow j(R\in W \vee \neg R'\in W)]$

(iv) $\forall R\in W\, j\exists R'\in W\, R'>R$

□

What we really want, is a description of $\mathbb{R}^G$ in terms of $\mathbb{Q}$ instead of $\mathbb{Q}^G$. In category theory there seems to be a simple answer. Find a suitable construction of $\mathbb{Q}$ in terms of limits and colimits. Since limits and colimits are preserved by the associated shesaf functors it follows that $\mathbb{Q}^G$ is isomorphic to $L\mathbb{Q}$. It is instructive to give a syntactic proof of this phenomenon.

7.3.2 Lemma.; The constructions of $\mathbb{Z}$ and $\mathbb{Q}$ are preserved by associated sheaf functors based on a dense topology $j:\Omega\to\Omega$, in other words, $\mathbb{Z}^G$ and $\mathbb{Q}^G$ are isomorphic to $L\mathbb{Z}$, respectively $L\mathbb{Q}$.

Proof.

$$\begin{aligned}
\mathbb{Z}^G &= \{R\in[\mathbb{N}\times\mathbb{N}] \mid \mathrm{int}(R)\}^G \\
&= \{R\in[L\mathbb{N}\times L\mathbb{N}] \mid R \text{ is } j\text{-closed} \wedge \mathrm{int}^G(R)\} \\
&\approx \{R\in[\mathbb{N}\times\mathbb{N}] \mid R \text{ is } j\text{-closed} \wedge \mathrm{int}^G(\eta(R))\} && (6.5.2) \\
&= \{R\in[\mathbb{N}\times\mathbb{N}] \mid R \text{ is } j\text{-closed} \wedge \mathrm{int}^g(R)\} && (6.5.2) \\
&= \{R\in[\mathbb{N}\times\mathbb{N}] \mid R \text{ is } j\text{-closed} \wedge j(\mathrm{int}(R))\} \\
&\approx L\mathbb{Z} && (7.2.2)
\end{aligned}$$

Likewise, one can show that $\mathbb{Q}^G\approx L\mathbb{Q}$

□

Note, however, that in this proof $\{R\in[\mathbb{N}\times\mathbb{N}] \mid R \text{ is } j\text{-closed} \wedge j(\mathrm{int}(R))\}$ is the most illuminating description of $\mathbb{Z}^G$. Similarly $\{R\in[\mathbb{N}\times\mathbb{N}\times\mathbb{N}] \mid R \text{ is } j\text{-closed} \wedge j(\mathrm{rat}(R))\}$ is the clear description for $\mathbb{Q}^G$. The appeal to (6.5.2) is crucial to derive this result. If we apply the same trick to $\mathbf{R}^G$ we get the following theorem:

7.3.3 Theorem. If the Gödel-translation $(-)^G$ is based on a dense topology $j:\Omega\to\Omega$, then $\mathbb{R}^G\approx\mathbb{R}^{BLM}$, $(\mathbb{R}^e)^G\approx\mathbb{R}^{Be}$ and $(\mathbb{R}^*)^G\approx\mathbb{R}^{BM}$.

Proof.

$$\begin{aligned}
\mathbf{R}^G &= \{W\subseteq\mathbb{Q}^G \mid W \text{ is } j\text{-closed} \wedge (B\wedge L\wedge O)^G(W)\} \\
&\approx \{W\subseteq\mathbb{Q} \mid W \text{ is } j\text{-closed} \wedge (B\wedge L\wedge O)^G(q(W))\} && (6.5.2) \\
&= \{W\subseteq\mathbb{Q} \mid (B\wedge L\wedge O)^g(W)\wedge S^j(W)\} \\
&= \{W\subseteq\mathbb{Q} \mid (B_j\wedge L_j\wedge O_j)(W)\wedge S^j(W)\} \\
&= \mathbb{R}^{BLO} \\
&\approx \mathbb{R}^{BLM}. && (7.1.10)
\end{aligned}$$

$$\begin{aligned}
(\mathbb{R}^e)^G &= \{W\subseteq\mathbb{Q}^G \mid W \text{ is } j\text{-closed} \wedge (B\wedge M_{\neg\neg}\wedge O)^G(W)\} \\
&\approx \{W\subseteq\mathbb{Q} \mid W \text{ is } j\text{-closed} \wedge (B\wedge M_{\neg\neg}\wedge O)^G(q(W))\} && (6.5.2) \\
&= \{W\subseteq\mathbb{Q} \mid (B\wedge M_{\neg\neg}\wedge O)^g(W)\wedge S^j(W)\} \\
&= \{W\subseteq\mathbb{Q} \mid (B_j\wedge M_{\neg\neg}\wedge O_j)(W)\wedge S^j(W)\} \\
&= \mathbb{R}^{BOe} \\
&\approx \mathbb{R}^{Be}. && (7.1.10)
\end{aligned}$$

$$
\begin{aligned}
(\mathbb{R}^*)^G &= \{W \subseteq Q^G \mid W \text{ is j-closed} \wedge (B \wedge M \wedge O)^G(W)\} \\
&\approx \{W \subseteq Q \mid W \text{ is j-closed} \wedge (B \wedge M \wedge O)^G(q(W))\} && (6.5.2) \\
&= \{W \subseteq Q \mid (B \wedge M \wedge O)^g(W) \wedge S^j(W)\} \\
&= \{W \subseteq Q \mid (B_j \wedge M \wedge O_j)(W) \wedge S^j(W)\} \\
&= \mathbb{R}^{BMO} \\
&\approx \mathbb{R}^{BM}. && (7.1.10)
\end{aligned}
$$

□

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Samenvatting

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