Approximation Orders for Interpolation by Surface Splines to Rough Functions

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Abstract

In this paper we consider the approximation of functions by radial basic function interpolants. There is a plethora of results about the asymptotic behaviour of the error between appropriately smooth functions and their interpolants, as the interpolation points fill out a bounded domain in $\mathbb{R}^d$. In all of these cases, the analysis takes place in a natural function space dictated by the choice of radial basic function – the native space. In many cases, the native space contains functions possessing a certain amount of smoothness. We address the question of what can be said about these error estimates when the function being interpolated fails to have the required smoothness. These are the rough functions of the title. We limit our discussion to surface splines, as an exemplar of a wider class of radial basic functions, because we feel our techniques are most easily seen and understood in this setting.

[1]This author was supported by a studentship from the Engineering and Physical Sciences Research Council.

[2]The first author would like to dedicate this paper to the memory of Will Light.

1 INTRODUCTION

The process of interpolation by translates of a basic function is a popular tool for the reconstruction of a multivariate function from a scattered data set. The setup of the problem is as follows. We are supplied with a finite set of interpolation points $\mathcal{A} \subset \mathbb{R}^d$ and a function $f : \mathcal{A} \to \mathbb{R}$. We wish to construct an interpolant to f of the form

$$(Sf)(x) = \sum_{a \in \mathcal{A}} \mu_a \psi(x - a) + p(x), \qquad \text{for } x \in \mathbb{R}^d. \tag{1.1}$$

Here, ψ is a real-valued function defined on $\mathbb{R}^d$, and the principle ingredient of our interpolant is the use of the translates of ψ by the points in $\mathcal{A}$. The function ψ is referred to as the *basic function*. The function p in Equation (1.1) is a polynomial on $\mathbb{R}^d$ of total degree at most $k-1$. The linear space of all such polynomials will be denoted by Π_{k-1}. Of course, for Sf to interpolate f the real numbers μ_a and the polynomial p must be chosen to satisfy the system

$$(Sf)(a) = f(a), \qquad \text{for } a \in \mathcal{A}.$$

It is natural to desire a unique solution to the above system. However, with the present setup, there are less conditions available to determine Sf than there are free parameters in Sf. There is a standard way of determining the remaining conditions, which are often called the *natural boundary conditions*:

$$\sum_{a \in \mathcal{A}} \mu_a q(a) = 0, \qquad \text{for all } q \in \Pi_{k-1}.$$

It is now essential that $\mathcal{A}$ is Π_{k-1}–unisolvent. This means that if $q \in \Pi_{k-1}$ vanishes on $\mathcal{A}$ then q must be zero. Otherwise the polynomial term can be adjusted by any polynomial which is zero on $\mathcal{A}$. However, more conditions are needed to ensure uniqueness of the interpolant. The requirement that ψ should be strictly conditionally positive definite of order k is one possible assumption. To see explanations of why these conditions arise, the reader is directed to Cheney & Light (2000). In most of the common applications the function ψ is a radial function. That is, there is a function $\phi : \mathbb{R}_+ \to \mathbb{R}$ such that $\psi = \phi \circ |\cdot|$, where $|\cdot|$ is the Euclidean norm. In these cases we refer to ψ as a *radial basic function*.

Duchon (1976, 1978) was amongst the first to study interpolation problems of this flavour. His approach was to formulate the interpolation problem as a variational one. To do this we assume we have a space of continuous functions X which carries a seminorm $|\cdot|$. The so-called minimal norm interpolant to $f \in X$ on $\mathcal{A}$ from X is the function $Sf \in X$ satisfying

1. $(Sf)(a) = f(a)$, for all $a \in \mathcal{A}$;
2. $|Sf| \leq |g|$, for all $g \in X$ such that $g(a) = f(a)$ for all $a \in \mathcal{A}$.

The spaces that Duchon considers are in fact spaces of tempered distributions which he is able to embed in $C(\mathbb{R}^d)$. Let $\mathcal{S}'$ be the space of all tempered distributions on $\mathbb{R}^d$. The particular spaces of distributions that we will be concerned with are called Beppo-Levi spaces. The k^{th} order Beppo-Levi space is denoted by $BL^k(\Omega)$ and defined as

$$BL^k(\Omega) = \left\{ f \in \mathcal{S}' : D^\alpha f \in L_2(\Omega),\, \alpha \in \mathbb{Z}_+^d,\, |\alpha| = k \right\},$$

with seminorm

$$|f|_{k,\Omega} = \left(\sum_{|\alpha|=k} c_\alpha \int_\Omega |(D^\alpha f)(x)|^2 \, dx \right)^{1/2}, \qquad f \in BL^k(\Omega).$$

The constants c_α are chosen so that the seminorm is rotationally invariant:

$$\sum_{|\alpha|=k} c_\alpha x^{2\alpha} = |x|^{2k}, \qquad \text{for all } x \in \mathbb{R}^d.$$

We assume throughout the paper that $2k > d$, because this has the affect that $BL^k(\Omega)$ is embedded in the continuous functions (Duchon 1976). The spaces $BL^k(\mathbb{R}^d)$ give rise to minimal norm interpolants which are exactly of the form given in Equation (1.1), where the radial basic function is $x \mapsto |x|^{2k-d}$ or $x \mapsto |x|^{2k-d} \log |x|$, depending on the parity of d.

It is perhaps no surprise to learn that the related functions ψ are strictly conditionally positive definite of some appropriate order. The name given to interpolants employing these basic functions is surface splines. This is because they are a genuine multivariate analogue of the well-loved natural splines in one dimension.

It is of central importance to understand the behaviour of the error between a function $f : \Omega \to \mathbb{R}$ and its interpolant as the set $\mathcal{A} \subset \Omega$ becomes "dense" in Ω. The measure of

density we employ is the *fill-distance* $h = \sup_{x\in\Omega} \min_{a\in\mathcal{A}} |x-a|$. One might hope that for some suitable norm $\|\cdot\|$ there is a constant γ, independent of f and h, such that

$$\|f - Sf\| = \mathcal{O}(h^\gamma), \qquad \text{as } h \to 0.$$

In the case of the Beppo-Levi spaces, there is a considerable freedom of choice for the norm in which the error between f and Sf is measured. The most widely quoted result concerns the norm $\|\cdot\|_{L_\infty(\Omega)}$, but for variety we prefer to deal with the L_p-norm. To do this it is helpful to assume Ω is a bounded domain, whose boundary is sufficiently smooth. In this case there is a constant $C > 0$, independent of f and h, such that for all $f \in BL^k(\Omega)$,

$$\|f - Sf\|_{L_p(\Omega)} \leq \begin{cases} Ch^{k-\frac{d}{2}+\frac{d}{p}}|f|_{k,\Omega}, & 2 \leq p \leq \infty \\ Ch^k |f|_{k,\Omega}, & 1 \leq p < 2 \end{cases}, \qquad \text{as } h \to 0. \tag{1.2}$$

There has been considerable interest recently in the following very natural question. What happens if the function f does not possess sufficient smoothness to lie in $BL^k(\Omega)$? It may well be that f lies in $BL^m(\Omega)$, where $2k > 2m > d$. The condition $2m > d$ ensures that $f(a)$ exists for each $a \in \mathcal{A}$, and so Sf certainly exists. However, $|f|_{k,\Omega}$ is not defined. It is simple to conjecture that the new error estimate should be

$$\|f - Sf\|_{L_p(\Omega)} \leq \begin{cases} Ch^{m-\frac{d}{2}+\frac{d}{p}}|f|_{m,\Omega}, & 2 \leq p \leq \infty \\ Ch^m |f|_{m,\Omega}, & 1 \leq p < 2 \end{cases}, \qquad \text{as } h \to 0. \tag{1.3}$$

It is perhaps surprising to the uninitiated reader that this estimate is not true even with the reasonable restrictions we have placed on k and m. We are going to describe a recent result from Johnson (2002). To do that, we recall the familiar definition of a Sobolev space. Let $W_2^k(\Omega)$ denote the k^{th} order Sobolev space, which consists of functions all of whose derivatives up to and including order k are in $L_2(\Omega)$. It is a Banach space under the norm

$$\|f\|_{k,\Omega} = \left(\sum_{i=0}^{k} |f|_{i,\Omega}^2 \right)^{1/2}, \qquad \text{where } f \in W_2^k(\Omega).$$

We have already tacitly alluded to the Sobolev embedding theorem which states that when Ω is reasonably regular (for example, when Ω possesses a Lipschitz continuous boundary) and $k > d/2$, then the space $W_2^k(\Omega)$ can be embedded in $C(\Omega)$ (see Adams 1978, Theorem 5.4, p. 97). Now Johnson's result is as follows.

Theorem 1.1 (Johnson). *Let Ω be the unit ball in $\mathbb{R}^d$ and assume $d/2 < m < k$. For every $h_0 > 0$, there exists an $f \in W_2^m(\mathbb{R}^d)$ and a sequence of sets $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$ with the following properties:*

(i) each set $\mathcal{A}_n$ consists of finitely many points contained in Ω;

(ii) the fill-distance of each set $\mathcal{A}_n$ is at most h_0;

(iii) if $S_k^n f$ is the surface spline interpolant to f from $BL^k(\mathbb{R}^d)$ associated with $\mathcal{A}_n$, for each $n \in \mathbb{N}$, then $||S_k^n f||_{L_1(\Omega)} \to \infty$ as $n \to \infty$.

If the surface spline interpolation operator is unbounded, there is of course no possibility of getting an error estimate of the kind we conjectured. Johnson's proof uses point sets which have a special feature. We define the separation distance of $\mathcal{A}_n$ as $q_n = \min\{|a-b|/2 : a, b \in \mathcal{A}_n, a \neq b\}$. Let the fill-distance of each $\mathcal{A}_n$ be h_n. In Johnson's proof, the construction of $\mathcal{A}_n$ is such that $q_n/h_n \to 0$. We make this remark, because Johnson's result in one dimension refers to interpolation by natural splines, and in this setting the connection between the separation distance and the unboundedness of S_k^n has been known for some time. What is also known in the one-dimensional case is that if the separation distance is tied to the fill-distance, then a result of the type we are seeking is true. Theorem 3.5 is the definitive result we obtain, and is the formalisation of the conjectured bounds in Equation (1.3).

Subsequent to carrying out this work, we became aware of independent work by Yoon (2002). In that paper, error bounds for the case we consider here are also offered. Because of Yoon's technique of proof, which is considerably different to our own, he obtains error bounds for functions f with the additional restriction that f lies in $W_\infty^k(\Omega)$, so the results here have wider applicability. However, Yoon does consider the shifted surface splines, whilst in this paper we have chosen to consider only surface splines as an exemplar of what can be achieved. At the end of Section 3 we offer some comments on the difference between our approach and that of Yoon.

To close this section we introduce some notation that will be employed throughout the paper. The support of a function $\phi : \mathbb{R}^d \to \mathbb{R}$ is defined to be the closure of the set

$\{x \in \mathbb{R}^d : \ \phi(x) \neq 0\}$, and is denoted by $\mathrm{supp}\,(\phi)$. The volume of a bounded set Ω is the quantity $\int_\Omega dx$ and will be denoted $\mathrm{vol}(\Omega)$. We make much use of the space Π_{m-1}, so for brevity we fix ℓ as the dimension of this space. Finally, when we write $\widehat{f}$ we mean the Fourier transform of f. The context will clarify whether the Fourier transform is the natural one on $L_1(\mathbb{R}^d)$:

$$\widehat{f}(x) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} f(t) e^{-ixt}\, dt,$$

or one of its several extensions to $L_2(\mathbb{R}^d)$ or $\mathcal{S}'$.

2 SOBOLEV EXTENSION THEORY

In this section we intend to collect together a number of useful results, chiefly about the sorts of extensions which can be carried out on Sobolev spaces. We begin with the well-known result which can be found in many of the standard texts. Of course, the precise nature of the set Ω in the following theorem varies from book to book, and we have not striven here for the utmost generality, because that is not really a part of our agenda in this paper.

Theorem 2.1 (Adams 1978, Theorem 4.32, p. 91). *Let Ω be an open, bounded subset of $\mathbb{R}^d$ satisfying the uniform cone condition. For every $f \in W_2^m(\Omega)$ there is an $f^\Omega \in W_2^m(\mathbb{R}^d)$ satisfying $f^\Omega|_\Omega = f$. Moreover, there is a positive constant $K = K(\Omega)$ such that for all $f \in W_2^m(\Omega)$,*

$$\|f^\Omega\|_{m,\mathbb{R}^d} \leq K \|f\|_{m,\Omega}.$$

We remark that the extension f^Ω can be chosen to be supported on any compact subset of $\mathbb{R}^d$ containing Ω. To see this, we construct f^Ω in accordance with Theorem 2.1, then select $\eta \in C_0^m(\mathbb{R}^d)$ such that $\eta(x) = 1$ for $x \in \Omega$. Now, if we consider the compactly supported function $f_0^\Omega = \eta f^\Omega \in W_2^m(\mathbb{R}^d)$, we have $f_0^\Omega|_\Omega = f$. An elementary application of the Leibniz formula gives

$$\|f_0^\Omega\|_{m,\mathbb{R}^d} \leq C \|f\|_{m,\Omega}, \qquad \text{where } C = C(\Omega, \eta).$$

One of the nice features of the above extension is that the behaviour of the constant $K(\Omega)$ can be understood for simple choices of Ω. The reason for this is of course the choice of Ω and the way the seminorms defining the Sobolev norms behave under dilations and translations of Ω.

Lemma 2.2. *Let Ω be a measurable subset of $\mathbb{R}^d$. Define the mapping $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ by $\sigma(x) = a + h(x - t)$, where $h > 0$, and a, t, $x \in \mathbb{R}^d$. Then for all $f \in W_2^m(\sigma(\Omega))$,*

$$|f \circ \sigma|_{m,\Omega} = h^{m-d/2} |f|_{m,\sigma(\Omega)}.$$

Proof. We have, for $|\alpha| = m$,

$$(D^\alpha (f \circ \sigma))(x) = h^m (D^\alpha f)(\sigma(x)).$$

Thus,

$$\begin{aligned} |f \circ \sigma|_{m,\Omega}^2 &= \sum_{|\alpha|=m} c_\alpha \int_\Omega |(D^\alpha (f \circ \sigma))(x)|^2 \, dx \\ &= h^{2m} \sum_{|\alpha|=m} c_\alpha \int_\Omega |(D^\alpha f)(\sigma(x))|^2 \, dx. \end{aligned}$$

Now, using the change of variables $y = \sigma(x)$,

$$|f \circ \sigma|_{m,\Omega}^2 = h^{2m-d} \sum_{|\alpha|=m} c_\alpha \int_{\sigma(\Omega)} |(D^\alpha f)(y)|^2 \, dy = h^{2m-d} |f|_{m,\sigma(\Omega)}^2. \qquad \square$$

Unfortunately, the Sobolev extension refers to the Sobolev norm. We want to work with a norm which is more convenient for our purposes. This norm is in fact equivalent to the Sobolev norm, as we shall now see.

Lemma 2.3. *Let Ω be an open subset of $\mathbb{R}^d$ having the cone property and a Lipschitz-continuous boundary. Let $b_1, \ldots, b_\ell \in \Omega$ be unisolvent with respect to Π_{m-1}. Define a norm on $W_2^m(\Omega)$ via*

$$\|f\|_\Omega = \left(|f|_{m,\Omega}^2 + \sum_{i=1}^{\ell} |f(b_i)|^2 \right)^{1/2}, \qquad f \in W_2^m(\Omega).$$

There are positive constants K_1 and K_2 such that for all $f \in W_2^m(\Omega)$,

$$K_1 \|f\|_{m,\Omega} \le \|f\|_\Omega \le K_2 \|f\|_{m,\Omega}.$$

Proof. The conditions imposed on m and Ω ensure that $W_2^m(\Omega)$ is continuously embedded in $C(\Omega)$ (Adams 1978, Theorem 5.4, p. 97). So, given $x \in \Omega$, there is a constant C such that $|f(x)| \le C \|f\|_{m,\Omega}$ for all $f \in W_2^m(\Omega)$. Thus, there are constants $C_1, \ldots, C_\ell$ such that

$$\|f\|_\Omega^2 \le |f|_{m,\Omega}^2 + \sum_{i=1}^{\ell} C_i \|f\|_{m,\Omega}^2 \le \Big(1 + \sum_{i=1}^{\ell} C_i\Big) \|f\|_{m,\Omega}^2. \tag{2.1}$$

On the other hand, suppose there is no positive number K with $\|f\|_{m,\Omega} \leq K\|f\|_\Omega$ for all $f \in W_2^m(\Omega)$. Then there is a sequence $\{f_j\}$ in $W_2^m(\Omega)$ with

$$\|f_j\|_{m,\Omega} = 1 \qquad \text{and} \qquad \|f_j\|_\Omega \leq \frac{1}{j}, \qquad \text{for } j = 1, 2, \ldots.$$

The Rellich selection theorem (Braess 1997, Theorem 1.9, p. 32) states that $W_2^m(\Omega)$ is compactly embedded in $W_2^{m-1}(\Omega)$. Therefore, as $\{f_j\}$ is bounded in $W_2^m(\Omega)$, this sequence must contain a convergent subsequence in $W_2^{m-1}(\Omega)$. With no loss of generality we shall assume $\{f_j\}$ itself converges in $W_2^{m-1}(\Omega)$. Thus $\{f_j\}$ is a Cauchy sequence in $W_2^{m-1}(\Omega)$. Next, as $\|f_j\|_\Omega \to 0$ it follows that $|f_j|_{m,\Omega} \to 0$. Moreover,

$$\begin{aligned} \|f_j - f_k\|_{m,\Omega}^2 &= \|f_j - f_k\|_{m-1,\Omega}^2 + |f_j - f_k|_{m,\Omega}^2 \\ &\leq \|f_j - f_k\|_{m-1,\Omega}^2 + 2|f_j|_{m,\Omega}^2 + 2|f_k|_{m,\Omega}^2. \end{aligned}$$

Since $\{f_j\}$ is a Cauchy sequence in $W_2^{m-1}(\Omega)$, and $|f_j|_{m,\Omega} \to 0$, it follows that $\{f_j\}$ is a Cauchy sequence in $W_2^m(\Omega)$. Since $W_2^m(\Omega)$ is complete with respect to $\|\cdot\|_{m,\Omega}$, this sequence converges to a limit $f \in W_2^m(\Omega)$. By Equation (2.1),

$$\|f - f_j\|_\Omega^2 \leq \Big(1 + \sum_{i=1}^{\ell} C_i\Big)\|f - f_j\|_{m,\Omega}^2,$$

and hence $\|f - f_j\|_\Omega \to 0$ as $j \to \infty$. Since $\|f_j\|_\Omega \to 0$, it follows that $f = 0$. Because $\|f_j\|_{m,\Omega} = 1$, $j = 1, 2, \ldots$, it follows that $\|f\|_{m,\Omega} = 1$. This contradiction establishes the result. □

We are almost ready to state the key result which we will employ in our later proofs about error estimates. Before we do this, let us make a simple observation. Look at the unisolvent points $b_1, \ldots, b_\ell$ in the statement of the previous Lemma. Since $W_2^m(\Omega)$ can be embedded in $C(\Omega)$, it makes sense to talk about the interpolation projection $P : W_2^m(\Omega) \to \Pi_{m-1}$ based on these points. Furthermore, under certain nice conditions (for example Ω being a bounded domain), P is the orthogonal projection of $W_2^m(\Omega)$ onto Π_{m-1}.

Lemma 2.4. *Let B be any ball of radius h and center $a \in \mathbb{R}^d$, and let $f \in W_2^m(B)$. Whenever $b_1, \ldots, b_\ell \in \mathbb{R}^d$ are unisolvent with respect to Π_{m-1} let $P_b : C(\mathbb{R}^d) \to \Pi_{m-1}$ be the Lagrange interpolation operator on $b_1, \ldots, b_\ell$. Then there exists $c = (c_1, \ldots, c_\ell) \in B^\ell$ and $g \in W_2^m(\mathbb{R}^d)$ such that*

1. $g(x) = (f - P_c f)(x)$ *for all* $x \in B$;

2. $g(x) = 0$ *for all* $|x - a| > 2h$;

3. *there exists a* $C > 0$, *independent of* f *and* B, *such that* $|g|_{m,\mathbb{R}^d} \leq C|f|_{m,B}$.

Furthermore, $c_1, \ldots, c_\ell$ *can be arranged so that* $c_1 = a$.

Proof. Let B_1 be the unit ball in $\mathbb{R}^d$ and let $B_2 = 2B_1$. Let $b_1, \ldots, b_\ell \in B_1$ be unisolvent with respect to Π_{m-1}. Define $\sigma(x) = h^{-1}(x - a)$ for all $x \in \mathbb{R}^d$. Set $c_i = \sigma^{-1}(b_i)$ for $i = 1, \ldots, \ell$ so that $c_1, \ldots, c_\ell \in B$ are unisolvent with respect to Π_{m-1}. Take $f \in W_2^m(B)$. Then $(f - P_c f) \circ \sigma^{-1} \in W_2^m(B_1)$. Set $F = (f - P_c f) \circ \sigma^{-1}$. Let F^{B_1} be constructed as an extension to F on B_1. By Theorem 2.1 and the remarks following it, we can assume F^{B_1} is supported on B_2. Define $g = F^{B_1} \circ \sigma \in W_2^m(\mathbb{R}^d)$. Let $x \in B$. Since $\sigma(B) = B_1$ there is a $y \in B_1$ such that $x = \sigma^{-1}(y)$. Then,

$$g(x) = (F^{B_1} \circ \sigma)(x) = F^{B_1}(y) = ((f - P_c f) \circ \sigma^{-1})(y) = (f - P_c f)(x).$$

Also, for $x \in \mathbb{R}^d$ with $|x - a| > 2h$, we have $|\sigma(x)| > 2$. Since F^{B_1} is supported on B_2, $g(x) = 0$ for $|x - a| > 2h$. Hence, g satisfies properties *1* and *2*. By Theorem 2.1 there is a K_1, independent of f and B, such that

$$\|F^{B_1}\|_{m,B_2} = \|F^{B_1}\|_{m,\mathbb{R}^d} \leq K_1 \|F\|_{m,B_1}.$$

We have seen in Lemma 2.3 that if we endow $W_2^m(B_1)$ and $W_2^m(B_2)$ with the norms

$$\|v\|_{B_i} = \left(|v|_{m,B_i}^2 + \sum_{i=1}^{\ell} |v(b_i)|^2 \right)^{1/2}, \qquad i = 1, 2,$$

then $\|\cdot\|_{B_i}$ and $\|\cdot\|_{m,B_i}$ are equivalent for $i = 1, 2$. Thus, there are constants K_2 and K_3, independent of f and B, such that

$$\|F^{B_1}\|_{B_2} \leq K_2 \|F^{B_1}\|_{m,B_2} \leq K_1 K_2 \|F\|_{m,B_1} \leq K_1 K_2 K_3 \|F\|_{B_1}.$$

Set $C = K_1 K_2 K_3$. Since $F^{B_1}(b_i) = F(b_i) = (f - P_c f)(\sigma^{-1}(b_i)) = (f - P_c f)(c_i) = 0$ for $i = 1, \ldots, \ell$, it follows that $|F^{B_1}|_{m,B_2} \leq C|F|_{m,B_1}$. Thus, $|g \circ \sigma^{-1}|_{m,\mathbb{R}^d} \leq C|(f - P_c f) \circ \sigma^{-1}|_{m,B_1}$. Now, Lemma 2.2 can be employed twice to give

$$|g|_{m,\mathbb{R}^d} = h^{d/2-m} |g \circ \sigma^{-1}|_{m,\mathbb{R}^d} \leq C h^{d/2-m} |(f - P_c f) \circ \sigma^{-1}|_{m,B_1} = C|f - P_c f|_{m,B}.$$

Finally, we observe that $|f - P_c f|_{m,B} = |f|_{m,B}$ to complete the first part of the proof. The remaining part follows by selecting $b_1 = 0$ and choosing $b_2, \ldots, b_\ell$ accordingly in the above construction. □

Lemma 2.5 (Duchon 1978). *Let Ω be an open, bounded, connected subset of $\mathbb{R}^d$ having the cone property and a Lipschitz-continuous boundary. Let $f \in W_2^m(\Omega)$. Then there exists a unique element $f^\Omega \in BL^m(\mathbb{R}^d)$ such that $f^\Omega|_\Omega = f$, and amongst all elements of $BL^m(\mathbb{R}^d)$ satisfying this condition, $|f^\Omega|_{m,\mathbb{R}^d}$ is minimal. Furthermore, there exists a constant $K = K(\Omega)$ such that, for all $f \in W_2^m(\Omega)$,*

$$|f^\Omega|_{m,\mathbb{R}^d} \leq K|f|_{m,\Omega}.$$

3 ERROR ESTIMATES

We arrive now at our main section, in which we derive the required error estimates. Our strategy is simple. We begin with a function f in $BL^m(\mathbb{R}^d)$. We want to estimate $\|f - S_k f\|$ for some suitable norm $\|\cdot\|$, where S_k is the minimal norm interpolation operator from $BL^k(\mathbb{R}^d)$, and $k > m$. We suppose that we already have an error bound using the norm $\|\cdot\|$ for all functions $g \in BL^k(\mathbb{R}^d)$. Our proof now proceeds as follows. Firstly, we adjust f in a somewhat delicate manner, obtaining a function F, still in $BL^m(\mathbb{R}^d)$, and with seminorm in $BL^m(\mathbb{R}^d)$ not too far away from that of f. We then smooth F by convolving it with a function $\phi \in C_0^\infty(\mathbb{R}^d)$. The key feature of the adjustment of f to F is that $(\phi * F)(a) = f(a)$ for every point a in our set of interpolation points. It then follows that $F \in BL^k(\mathbb{R}^d)$. We then use the usual error estimate in $BL^k(\mathbb{R}^d)$. A standard procedure (Lemma 3.1) then takes us back to an error estimate in $BL^m(\mathbb{R}^d)$.

Lemma 3.1. *Let $m \leq k$ and let $\phi \in C_0^\infty(\mathbb{R}^d)$. For each $h > 0$, let $\phi_h(x) = h^{-d}\phi(x/h)$ for $x \in \mathbb{R}^d$. Then there exists a constant $C > 0$, independent of h, such that for all $f \in BL^m(\mathbb{R}^d)$,*

$$|\phi_h * f|_{k,\mathbb{R}^d} \leq Ch^{m-k}|f|_{m,\mathbb{R}^d}.$$

Furthermore, we have $|\phi_h * f|_{k,\mathbb{R}^d} = o(h^{m-k})$ as $h \to 0$.

Proof. The chain rule for differentiation gives $(D^\gamma \phi_h)(x) = h^{-(d+|\gamma|)}(D^\gamma \phi)(x/h)$ for all $x \in \mathbb{R}^d$, and $\gamma \in \mathbb{Z}_+^d$. Thus, for $\beta \in \mathbb{Z}_+^d$ with $|\beta| = m$ we have

$$\begin{aligned}
\int_{\mathbb{R}^d} |(D^\gamma \phi_h * D^\beta f)(x)|^2 \, dx &= \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} (D^\gamma \phi_h)(x-y)(D^\beta f)(y) \, dy \right|^2 dx \\
&= h^{-2(d+|\gamma|)} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} (D^\gamma \phi)\Big(\frac{x-y}{h}\Big)(D^\beta f)(y) \, dy \right|^2 dx \\
&= h^{-2|\gamma|} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} (D^\gamma \phi)(t)(D^\beta f)(x-ht) \, dt \right|^2 dx \\
&= h^{-2|\gamma|} \int_{\mathbb{R}^d} \left| \int_{K} (D^\gamma \phi)(t)(D^\beta f)(x-ht) \, dt \right|^2 dx, \qquad (3.1)
\end{aligned}$$

where $K = \operatorname{supp}(\phi)$. An application of the Cauchy-Schwartz inequality gives

$$\int_{\mathbb{R}^d} |(D^\gamma \phi_h * D^\beta f)(x)|^2 \, dx \le h^{-2|\gamma|} \int_{\mathbb{R}^d} \left(\int_K |(D^\gamma \phi)(t)|^2 \, dt \right) \left(\int_K |(D^\beta f)(x-ht)|^2 \, dt \right) dx,$$

and so,

$$\int_{\mathbb{R}^d} |(D^\gamma \phi_h * D^\beta f)(x)|^2 \, dx \le h^{-2|\gamma|} \int_{\mathbb{R}^d} |(D^\gamma \phi)(t)|^2 \, dt \int_{\mathbb{R}^d} \int_K |(D^\beta f)(x-ht)|^2 \, dt dx. \quad (3.2)$$

The Parseval formula together with the relation $(D^\alpha(\phi_h * f))\hat{} = (i\,\cdot)^\alpha (\phi_h * f)\hat{}$ provide us with the equality

$$\begin{aligned}
\sum_{|\alpha|=k} c_\alpha \int_{\mathbb{R}^d} |(D^\alpha(\phi_h * f))(x)|^2 \, dx &= \sum_{|\alpha|=k} c_\alpha \int_{\mathbb{R}^d} |(ix)^\alpha (\phi_h * f)\hat{}(x)|^2 \, dx \\
&= \int_{\mathbb{R}^d} \sum_{|\alpha|=k} c_\alpha x^{2\alpha} |(\phi_h * f)\hat{}(x)|^2 \, dx. \qquad (3.3)
\end{aligned}$$

Now, when Equation (3.3) is used in conjunction with the relation

$$\sum_{|\alpha|=k} c_\alpha x^{2\alpha} = |x|^{2k} = |x|^{2(m+k-m)} = \sum_{|\beta|=m} c_\beta x^{2\beta} \sum_{|\gamma|=k-m} c_\gamma x^{2\gamma},$$

we obtain

$$\begin{aligned}
\sum_{|\alpha|=k} c_\alpha \int_{\mathbb{R}^d} |(D^\alpha(\phi_h * f))(x)|^2 \, dx &= \int_{\mathbb{R}^d} \sum_{|\beta|=m} c_\beta x^{2\beta} \sum_{|\gamma|=k-m} c_\gamma x^{2\gamma} |(\phi_h * f)\widehat{\ }(x)|^2 \, dx \\
&= \sum_{|\beta|=m} c_\beta \int_{\mathbb{R}^d} \sum_{|\gamma|=k-m} c_\gamma x^{2\gamma} |(ix)^\beta (\phi_h * f)\widehat{\ }(x)|^2 \, dx \\
&= \sum_{|\beta|=m} c_\beta \int_{\mathbb{R}^d} \sum_{|\gamma|=k-m} c_\gamma x^{2\gamma} |(D^\beta(\phi_h * f))\widehat{\ }(x)|^2 \, dx \\
&= \sum_{|\beta|=m} c_\beta \sum_{|\gamma|=k-m} c_\gamma \int_{\mathbb{R}^d} |(ix)^\gamma (D^\beta(\phi_h * f))\widehat{\ }(x)|^2 \, dx \\
&= \sum_{|\beta|=m} c_\beta \sum_{|\gamma|=k-m} c_\gamma \int_{\mathbb{R}^d} |(D^\gamma(D^\beta(\phi_h * f)))\widehat{\ }(x)|^2 \, dx \\
&= \sum_{|\beta|=m} c_\beta \sum_{|\gamma|=k-m} c_\gamma \int_{\mathbb{R}^d} |(D^\gamma(D^\beta(\phi_h * f)))(x)|^2 \, dx.
\end{aligned}$$

Since the operation of differentiation commutes with convolution, we have that

$$\sum_{|\alpha|=k} c_\alpha \int_{\mathbb{R}^d} |(D^\alpha(\phi_h * f))(x)|^2 \, dx = \sum_{|\beta|=m} c_\beta \sum_{|\gamma|=k-m} c_\gamma \int_{\mathbb{R}^d} |(D^\gamma \phi_h * D^\beta f)(x)|^2 \, dx. \quad (3.4)$$

Combining Equation (3.2) with Equation (3.4) we deduce that

$$\begin{aligned}
&\sum_{|\alpha|=k} c_\alpha \int_{\mathbb{R}^d} |(D^\alpha(\phi_h * f))(x)|^2 \, dx \\
&\qquad \le \sum_{|\beta|=m} c_\beta \sum_{|\gamma|=k-m} c_\gamma h^{-2|\gamma|} \int_{\mathbb{R}^d} |(D^\gamma \phi)(t)|^2 \, dt \int_{\mathbb{R}^d} \int_K |(D^\beta f)(x-ht)|^2 \, dt dx \\
&\qquad = h^{2(m-k)} |\phi|^2_{k-m,\mathbb{R}^d} \sum_{|\beta|=m} c_\beta \int_{\mathbb{R}^d} \int_K |(D^\beta f)(x-ht)|^2 \, dt dx.
\end{aligned}$$

Fubini's theorem permits us to change the order of integration in the previous inequality. Thus,

$$\sum_{|\alpha|=k} c_\alpha \int_{\mathbb{R}^d} |(D^\alpha(\phi_h * f))(x)|^2 \, dx \le h^{2(m-k)} |\phi|^2_{k-m,\mathbb{R}^d} \sum_{|\beta|=m} c_\beta \int_K \int_{\mathbb{R}^d} |(D^\beta f)(x-ht)|^2 \, dx dt.$$

Finally, a change of variables in the inner integral above yields

$$\sum_{|\alpha|=k} c_\alpha \int_{\mathbb{R}^d} |(D^\alpha(\phi_h * f))(x)|^2 \, dx \le h^{2(m-k)} |\phi|^2_{k-m,\mathbb{R}^d} \sum_{|\beta|=m} c_\beta \int_K \int_{\mathbb{R}^d} |(D^\beta f)(z)|^2 \, dz dt.$$

Setting $C = |\phi|_{k-m,\mathbb{R}^d} \sqrt{\mathrm{vol}(K)}$ we conclude that $|\phi_h * f|_{k,\mathbb{R}^d} \le C h^{m-k} |f|_{m,\mathbb{R}^d}$ as required.

To deal with the remaining statement of the lemma, we observe that for $\gamma \neq 0$ we have

$$\int_K (D^\gamma \phi)(t) \, dt = \int_{\mathbb{R}^d} (D^\gamma \phi)(t) \, dt = (\widehat{D^\gamma \phi})(0) = ((i \cdot)^\gamma \widehat{\phi})(0) = 0.$$

Then it follows from Equation (3.1) that for $|\beta| = m$,

$$\int_{\mathbb{R}^d} |(D^\gamma \phi_h * D^\beta f)(x)|^2 \, dx = h^{-2|\gamma|} \int_{\mathbb{R}^d} \left| \int_K (D^\gamma \phi)(t)((D^\beta f)(x - ht) - (D^\beta f)(x)) \, dt \right|^2 dx.$$

Now, if we continue in precisely the same manner as before, we obtain

$$\sum_{|\alpha|=k} c_\alpha \int_{\mathbb{R}^d} |(D^\alpha(\phi_h * f))(x)|^2 \, dx$$
$$\leq h^{2(m-k)} |\phi|^2_{k-m,\mathbb{R}^d} \sum_{|\beta|=m} c_\beta \int_K \int_{\mathbb{R}^d} |(D^\beta f)(x - ht) - (D^\beta f)(x)|^2 \, dxdt.$$

Since $D^\beta f \in L^2(\mathbb{R}^d)$ for each $\beta \in \mathbb{Z}^d_+$ with $|\beta| = m$, it follows that for almost all $t, x \in \mathbb{R}^d$,

$$|(D^\beta f)(x - ht) - (D^\beta f)(x)| \to 0, \qquad \text{as } h \to 0.$$

Furthermore, setting

$$g(x,t) = 2|(D^\beta f)(x - ht)|^2 + 2|(D^\beta f)(x)|^2, \qquad \text{for almost all } x, t \in \mathbb{R}^d,$$

we see that

$$|(D^\beta f)(x - ht) - (D^\beta f)(x)|^2 \leq g(x,t),$$

for almost all $x, t \in \mathbb{R}^d$ and each $h > 0$. It follows by calculations similar to those used above that

$$\int_K \int_{\mathbb{R}^d} g(x,t) \, dxdt = 4\text{vol}(K) \int_{\mathbb{R}^d} |(D^\beta f)(x)|^2 \, dx < \infty.$$

Applying Lebesgue's dominated convergence theorem, we obtain

$$\int_K \int_{\mathbb{R}^d} |(D^\beta f)(x - ht) - (D^\beta f)(x)|^2 \, dxdt \to 0, \qquad \text{as } h \to 0.$$

Hence, for $m \leq k$, $|\phi_h * f|_{k,\mathbb{R}^d} = o(h^{m-k})$ as $h \to 0$. □

Lemma 3.2. *Suppose $\phi \in C_0^\infty(\mathbb{R}^d)$ is supported on the unit ball and satisfies*

$$\int_{\mathbb{R}^d} \phi(x) \, dx = 1 \qquad \textit{and} \qquad \int_{\mathbb{R}^d} \phi(x) x^\alpha \, dx = 0, \qquad \textit{for all } 0 < |\alpha| \leq m - 1.$$

For each $\varepsilon > 0$ and $x \in \mathbb{R}^d$, let $\phi_\varepsilon(x) = \varepsilon^{-d}\phi(x/\varepsilon)$. Let B be any ball of radius h and center $a \in \mathbb{R}^d$. For a fixed $p \in \Pi_{m-1}$ let f be a mapping from $\mathbb{R}^d$ to $\mathbb{R}$ such that $f(x) = p(x)$ for all $x \in B$. Then $(\phi_\varepsilon * f)(a) = p(a)$ for all $\varepsilon \leq h$.

Proof. Let B_1 denote the unit ball in $\mathbb{R}^d$. We begin by employing a change of variables to deduce

$$\begin{aligned}(\phi_\varepsilon * f)(a) &= \int_{\mathbb{R}^d} \phi_\varepsilon(a-y)f(y)\ dy \\ &= \varepsilon^{-d}\int_{\mathbb{R}^d} \phi\Big(\frac{a-y}{\varepsilon}\Big)f(y)\ dy \\ &= \int_{\mathbb{R}^d} \phi(x)f(a-x\varepsilon)\ dx \\ &= \int_{B_1} \phi(x)f(a-x\varepsilon)\ dx.\end{aligned}$$

Then, for $x \in B_1$, $|(a-x\varepsilon)-a| \leq \varepsilon \leq h$. Thus, $f(a-x\varepsilon) = p(a-x\varepsilon)$ for all $x \in B_1$. Moreover, there are numbers b_α such that $p(a-x\varepsilon) = p(a) + \sum_{0<|\alpha|\leq m-1} b_\alpha x^\alpha$. Hence,

$$\begin{aligned}(\phi_\varepsilon * f)(a) &= \int_{B_1} \phi(x)p(a-x\varepsilon)\ dx \\ &= \int_{\mathbb{R}^d} \phi(x)\bigg(p(a) + \sum_{0<\alpha\leq m-1} b_\alpha x^\alpha\bigg)\ dx \\ &= p(a).\end{aligned}$$

□

Definition 3.3. *Let Ω be an open, bounded subset of $\mathbb{R}^d$. Let $\mathcal{A}$ be a set of points in Ω. The quantity $\sup_{x\in\Omega}\inf_{a\in\mathcal{A}}|x-a| = h$ is called the fill-distance of $\mathcal{A}$ in Ω. The separation of $\mathcal{A}$ is given by the quantity*

$$q = \min_{\substack{a,b\in\mathcal{A}\\ a\neq b}} \frac{|a-b|}{2}.$$

The quantity h/q will be called the mesh-ratio of $\mathcal{A}$.

Theorem 3.4. *Let $\mathcal{A}$ be a finite subset of $\mathbb{R}^d$ of separation $q > 0$ and let $d < 2m \leq 2k$. Then for all $f \in BL^m(\mathbb{R}^d)$ there exists an $F \in BL^k(\mathbb{R}^d)$ such that*

1. *$F(a) = f(a)$ for all $a \in \mathcal{A}$;*
2. *there exists a $C > 0$, independent of f and q, such that $|F|_{m,\mathbb{R}^d} \leq C|f|_{m,\mathbb{R}^d}$ and $|F|_{k,\mathbb{R}^d} \leq Cq^{m-k}|f|_{m,\mathbb{R}^d}$.*

Proof. Take $f \in BL^m(\mathbb{R}^d)$. For each $a \in \mathcal{A}$ let $B_a \subset \mathbb{R}^d$ denote the ball of radius $\delta = q/4$ centered at a. For each B_a let g_a be constructed in accordance with Lemma 2.4. That is, for each $a \in \mathcal{A}$ take $c' = (c_2, \ldots, c_\ell) \in B_a^{\ell-1}$ and $g_a \in W_2^m(\mathbb{R}^d)$ such that

1. $a, c_2, \ldots, c_\ell$ are unisolvent with respect to Π_{m-1};

2. $g_a(x) = (f - P_{(a,c')}f)(x)$ for all $x \in B_a$;

3. $P_{(a,c')}f \in \Pi_{m-1}$ and $(P_{(a,c')}f)(a) = f(a)$;

4. $g_a(x) = 0$ for all $|x - a| > 2\delta$;

5. there exists a $C_1 > 0$, independent of f and B_a, such that $|g_a|_{m,\mathbb{R}^d} \le C_1 |f|_{m,B_a}$.

Note that if $a \neq b$, then $\operatorname{supp}(g_a)$ does not intersect $\operatorname{supp}(g_b)$, because if $x \in \operatorname{supp}(g_a)$ then

$$|x - b| > |b - a| - |x - a| \ge 2q - 2\delta = 6\delta.$$

Using the observation above regarding the supports of the g_a's it follows that

$$\begin{aligned}
\left|\sum_{a\in\mathcal{A}} g_a\right|^2_{m,\mathbb{R}^d} &= \sum_{|\alpha|=m} c_\alpha \int_{\mathbb{R}^d} \left|\sum_{a\in\mathcal{A}} (D^\alpha g_a)(x)\right|^2 dx \\
&= \sum_{|\alpha|=m} c_\alpha \sum_{b\in\mathcal{A}} \int_{\operatorname{supp}(g_b)} \left|\sum_{a\in\mathcal{A}} (D^\alpha g_a)(x)\right|^2 dx \\
&= \sum_{|\alpha|=m} c_\alpha \sum_{b\in\mathcal{A}} \int_{\operatorname{supp}(g_b)} |(D^\alpha g_b)(x)|^2 \, dx \\
&= \sum_{a\in\mathcal{A}} |g_a|^2_{m,\mathbb{R}^d}.
\end{aligned}$$

Applying Condition 5 to the above equality we have

$$\left|\sum_{a\in\mathcal{A}} g_a\right|^2_{m,\mathbb{R}^d} \le C_1^2 \sum_{a\in\mathcal{A}} |f|^2_{m,B_a} \le C_1^2 |f|^2_{m,\mathbb{R}^d}.$$

Now set $H = f - \sum_{a\in\mathcal{A}} g_a$. It then follows from Condition 2 above that $H(x) = (P_{(a,c')}f)(x)$ for all $x \in B_a$, and from Condition 3 that $H(a) = f(a)$ for all $a \in \mathcal{A}$. Let $\phi \in C_0^\infty(\mathbb{R}^d)$ be supported on the unit ball and enjoy the properties

$$\int_{\mathbb{R}^d} \phi(x)\, dx = 1 \qquad \text{and} \qquad \int_{\mathbb{R}^d} \phi(x) x^\alpha \, dx = 0, \qquad \text{for all } 0 < |\alpha| \le m - 1.$$

Now set $F = \phi_\delta * H$. Using Lemma 3.1, there is a constant $C_2 > 0$, independent of q and f, such that

$$\begin{aligned}
|F|^2_{k,\mathbb{R}^d} &\le C_2 \delta^{2(m-k)} \left| f - \sum_{a\in\mathcal{A}} g_a \right|^2_{m,\mathbb{R}^d} \\
&\le 2C_2 \delta^{2(m-k)} \left(|f|^2_{m,\mathbb{R}^d} + \left|\sum_{a\in\mathcal{A}} g_a\right|^2_{m,\mathbb{R}^d} \right) \\
&\le 2C_2(1 + C_1^2) \delta^{2(m-k)} |f|^2_{m,\mathbb{R}^d}.
\end{aligned}$$

Similarly, there is a constant $C_3 > 0$, independent of q and f, such that

$$\begin{aligned} |F|^2_{m,\mathbb{R}^d} &\leq C_3 \left| f - \sum_{a\in\mathcal{A}} g_a \right|^2_{m,\mathbb{R}^d} \\ &\leq 2C_3(1+C_1^2)|f|^2_{m,\mathbb{R}^d}. \end{aligned}$$

Thus $|F|_{k,\mathbb{R}^d} \leq Cq^{m-k}|f|_{m,\mathbb{R}^d}$ and $|F|_{m,\mathbb{R}^d} \leq C|f|_{m,\mathbb{R}^d}$ for some appropriate constant $C > 0$. Since $F = \phi_\delta * H$ and $H|_{B_a} \in \Pi_{m-1}$ for each $a \in \mathcal{A}$, it follows from Lemma 3.2 that $F(a) = H(a) = f(a)$ for all $a \in \mathcal{A}$. □

Theorem 3.5. *Let Ω be an open, bounded, connected subset of $\mathbb{R}^d$ satisfying the cone property and having a Lipschitz-continuous boundary. Suppose also $d < 2m \leq 2k$. For each $h > 0$, let $\mathcal{A}_h$ be a finite, Π_{k-1}–unisolvent subset of Ω with fill-distance h. Assume also that there is a quantity $\rho > 0$ such that the mesh-ratio of each $\mathcal{A}_h$ is bounded by ρ for all $h > 0$. For each mapping $f : \mathcal{A}_h \to \mathbb{R}$, let $S^h_k f$ be the minimal norm interpolant to f on $\mathcal{A}_h$ from $BL^k(\mathbb{R}^d)$. Then there exists a constant $C > 0$, independent of h, such that for all $f \in BL^m(\Omega)$,*

$$\|f - S^h_k f\|_{L_p(\Omega)} \leq \begin{cases} Ch^{m-\frac{d}{2}+\frac{d}{p}}|f|_{m,\Omega}, & 2 \leq p \leq \infty \\ Ch^m|f|_{m,\Omega}, & 1 \leq p < 2 \end{cases}, \qquad \text{as } h \to 0.$$

Proof. Take $f \in BL^m(\Omega)$. By Duchon (1976), $f \in W^m_2(\Omega)$. We define f^Ω in accordance with Lemma 2.5. For most of this proof we wish to work with f^Ω and not f, so for convenience we shall write f instead of f^Ω. Construct F in accordance with Theorem 3.4 and set $G = f - F$. Then $F(a) = f(a)$ and $G(a) = 0$ for all $a \in \mathcal{A}_h$. Furthermore, there is a constant $C_1 > 0$, independent of f and h, such that

$$|F|_{k,\mathbb{R}^d} \leq C_1 \left(\frac{h}{\rho}\right)^{m-k} |f|_{m,\mathbb{R}^d}, \tag{3.5}$$

$$|G|_{m,\mathbb{R}^d} \leq |f|_{m,\mathbb{R}^d} + |F|_{m,\mathbb{R}^d} \leq (1+C_1)|f|_{m,\mathbb{R}^d}. \tag{3.6}$$

Thus $S^h_k f = S^h_k F$ and $S^h_m G = 0$, where we have adopted the obvious notation for S^h_m. Hence,

$$\|f - S^h_k f\|_{L_p(\Omega)} = \|f - S^h_k F\|_{L_p(\Omega)} = \|F + G - S^h_k F\|_{L_p(\Omega)} \leq \|F - S^h_k F\|_{L_p(\Omega)} + \|G - S^h_m G\|_{L_p(\Omega)}.$$

Now, employing Duchon's (1978) error estimates for surface splines (1.2), there are positive constants $C_2 > 0$ and $C_3 > 0$, independent of h and f, such that

$$||f - S_k^h f||_{L_p(\Omega)} \leq C_2 h^{\beta(k)} |F|_{k,\Omega} + C_3 h^{\beta(m)} |G|_{m,\Omega}, \qquad \text{as } h \to 0,$$

where we have defined

$$\beta(j) = \begin{cases} j - \frac{d}{2} + \frac{d}{p}, & 2 \leq p \leq \infty \\ j, & 1 \leq p < 2 \end{cases}.$$

Finally, using the bounds in Equations (3.5) and (3.6) we have

$$||f - S_k^h f||_{L_p(\Omega)} \leq C_4 h^{\beta(m)} |f|_{m,\mathbb{R}^d}, \qquad \text{as } h \to 0,$$

for some appropriate $C_4 > 0$. To complete the proof we remind ourselves that we have substituted f^Ω with f, and so an application of Lemma 2.5 shows that we can find $C_5 > 0$ such that

$$||f - S_k^h f||_{L_p(\Omega)} \leq C_4 h^{\beta(m)} |f^\Omega|_{m,\mathbb{R}^d} \leq C_4 C_5 h^{\beta(m)} |f|_{m,\Omega}, \qquad \text{as } h \to 0. \qquad \square$$

We conclude this section with a brief commentary on the approach of Yoon (2002). It is hardly surprising that Yoon's technique also utilises a smoothing via convolution with a smooth kernel function corresponding closely to our function ϕ used in the proof of Theorem 3.5. However, Yoon's approach is simply to smooth at this stage, obtaining the equivalent of our function F in the proof of Theorem 3.5. Because there is no preprocessing of f to H, Yoon's function F does not enjoy the nice property $F(a) = f(a)$ for all $a \in \mathcal{A}$. It is this property which makes the following step, where we treat $G = f - F$, a fairly simple process. Correspondingly, Yoon has considerably more difficulty treating his function G. Our method also yields the same bound as that in Yoon, but for a wider class of functions. Indeed we would suggest that $BL^m(\Omega)$ is the natural class of functions for which one would wish an error estimate of the type given in Theorem 3.5.

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