On the Design of *hp*–Adaptive Finite Element Methods for Elliptic Partial Differential Equations

Paul Houston [1]

Department of Mathematics, University of Leicester, Leicester LE1 7RH, UK (email: `Paul.Houston@mcs.le.ac.uk`*)*

Christoph Schwab

Seminar for Applied Mathematics, ETH Zürich, CH-8092 Zürich, Switzerland (email: `schwab@math.ethz.ch`*)*

Endre Süli

University of Oxford, Computing Laboratory, Wolfson Building, Parks Road, Oxford OX1 3QD, UK (email: `Endre.Suli@comlab.ox.ac.uk`*)*

Abstract

We introduce an *hp*–adaptive finite element algorithm based on a combination of reliable and efficient residual error indicators and a new *hp*–extension control technique which assesses the local regularity of the underlying analytical solution on the basis of its local Legendre series expansion. Numerical experiments confirm the robustness and reliability of the proposed algorithm.

Key words:
hp–finite element methods, *a posteriori* error analysis, adaptivity

1 Introduction

Over the last few decades, considerable progress has been made on both the *a posteriori* error analysis of finite element methods for a wide range of partial differential equations of practical interest, and the theoretical and

[1] Funded by the EPSRC (Grant GR/R76615).

computational assessment of local refinement indicators; see, for example, [2,6,13,27,30,32] and the references cited therein. We note that once a computable error bound has been established for a particular discretisation of a partial differential equation, the refinement indicator is generally constructed by simply 'localising' the *a posteriori* bound over an element, or patch of elements. On the other hand, the state of development of 'optimal' mesh modification strategies which are capable of delivering the greatest reduction in the error for the least amount of computational cost, is far less advanced. Indeed, the majority of adaptive finite element software will simply subdivide elements where the local refinement indicator is large, while keeping the polynomial degree fixed at some low value. Clearly, if the analytical solution to the underlying partial differential equation is smooth, or at least locally smooth, then this may not provide the most efficient adaptive strategy. In this case, an enrichment of the polynomial degree (p–refinement) may be much more effective in reducing the error per unit cost.

The aim of this paper is to focus on the design of an automatic mesh modification strategy that is capable of exploiting both local h– and p–refinement. Such general hp–adaptive finite element methods offer greater flexibility and improved efficiency than mesh refinement methods which only incorporate either local mesh subdivision of the computational domain $\Omega \subset \mathbb{R}^d$ with the degree of the approximating polynomial fixed, or global polynomial degree variation on a fixed coarse mesh. Indeed, in recent years there has been tremendous interest in the development of automatic hp–mesh refinement algorithms, see [1,3,8,9,11,14,16–23,26–29,31], for example. The key step in the design of such an adaptive algorithm is the local decision taken on each element κ in the computational mesh as to which refinement strategy (i.e., h-refinement *via* local mesh subdivision or p-refinement by increasing the degree of the local polynomial approximation) should be employed on κ in order to obtain the greatest reduction in the error per unit cost.

On the basis of results from approximation theory, cf. [5,24], for example, hp–adaptive finite element strategies aim to exploit local p-refinement on elements κ where the analytical solution u to the partial differential equation under consideration is *smooth*, and local mesh subdivision on those elements κ where u is non-smooth. Thereby, regions in the computational domain where u is locally non-smooth are isolated from regions of smoothness, thus reducing the influence of singularities/discontinuities as well as making p–refinement more effective. Of course, since u is in general unknown analytically, the local smoothness of the solution cannot be determined. Motivated by the lack of precise information about the local regularity of the analytical solution, various algorithms have been developed in the literature with the aim to identify those parts of the computational domain where u may be perceived as being 'smooth' and regions where u is 'non-smooth'. Below we provide a brief review of existing methods and present an outline of the rest of the paper.

- *Use of a priori information.* For a linear elliptic boundary–value problem with piecewise analytic coefficients, forcing functions and boundary data, on a computational domain Ω with a piecewise analytic boundary surface $\partial\Omega$, the solution will be an analytic function everywhere, except in the neighbourhood of singularities in the data. Thereby, h–refinement may be employed in those elements in the computational domain whose closures contain such singularities, with p–refinement performed elsewhere. This approach has been employed by Owens and co-workers, for example; cf. [31,8].
- *Type–parameter.* In this strategy it is assumed that on each element κ in the computational mesh on Ω, one has a local refinement indicator $\eta_\kappa(u_{h,p}, h_\kappa, p_\kappa)$, which depends on the numerical approximation $u_{h,p}$, the local mesh-size h_κ and the local polynomial degree p_κ. Then, assuming that $\eta_\kappa(u_{h,p-1}, h_\kappa, p_\kappa - 1) \neq 0$, the perceived smoothness of the solution may be estimated using the ratio $\zeta_\kappa = \eta_\kappa(u_{h,p}, h_\kappa, p_\kappa)/\eta_\kappa(u_{h,p-1}, h_\kappa, p_\kappa - 1)$, cf. Adjerid *et al.* [1] and Gui & Babuška [14], for example. If $\zeta_\kappa \leq \gamma$, $0 < \gamma < 1$, the error is decreasing as the polynomial degree is increased, indicating that p–enrichment should be performed. On the other hand, if $\zeta_\kappa > \gamma$ then the element κ is subdivided. Here, γ is referred to as a *type–parameter* [14].
- *Predicted error reduction.* A very closely related technique to the type–parameter strategy is based on refining each element κ in the computational mesh according to the refinement history of κ; cf. [20]. To this end, a predicted (local) error indicator $\eta_\kappa^{\mathrm{pred}}$ is computed on the basis of the elemental error indicator η_κ calculated on the previous mesh, together with *a priori* estimates of the expected decay of η_κ after the refinement step has been performed, assuming that the underlying analytical solution is locally smooth. If the error indicator computed on the new mesh is larger than $\eta_\kappa^{\mathrm{pred}}$, then κ is subdivided; otherwise p–enrichment is performed, cf., also, [16].
- *'Texas 3–step'.* This strategy was first introduced by J.T. Oden and co-workers [22]; here, the smoothness of the solution to the underlying partial differential equation is not directly taken into account. Step 1 involves initialising various parameters, as well as setting intermediate and final error tolerances $\mathtt{TOL}_I$ and $\mathtt{TOL}_F$, respectively. Then, keeping the polynomial degree fixed, in Step 2 the mesh is adaptively h–refined in order to ensure that the error (measured in some appropriate norm) is less than $\mathtt{TOL}_I$. In the final third step, the mesh is kept fixed, while the local polynomial degrees are increased to achieve the final error tolerance $\mathtt{TOL}_F$. For related work, we refer to the articles [9,21], and the references cited therein.
- *Mesh optimisation strategy.* In this strategy an optimal refinement is determined for each element in the mesh by directly employing results from approximation theory. More precisely, a *reference solution* $\hat{u}$ is computed on a refined finite element space, where all the elements have been uniformly refined and the polynomial degree p has been globally incremented by one. Then, on each element κ in the original finite element mesh, elemental norms of the projection error between $\hat{u}$ and some suitable finite element projection $\Pi(\hat{u})$ may be computed; here, the error is computed by

projecting $\hat{u}$ onto a finite element space employing the original mesh, but with a local polynomial degree $p+1$, as well as on a sequence of finite element spaces corresponding to a local h–refinement of κ that results in the same increase in the number of degrees of freedom as the p–enrichment. The optimal refinement of κ is then chosen to be the one which leads to the smallest projection error; elements in the mesh are then refined based on those that will lead to the greatest decrease in the projection error per degree of freedom. This strategy was first introduced by Rachowicz *et al.* [23]; see also [11,26] for more recent work.

- *Decay rate of Legendre expansion coefficients.* Mavriplis [19] proposed determining whether the solution is locally smooth or non-smooth by calculating the decay rate of the Legendre expansion coefficients of the solution. More precisely, writing a_i, $i = 0, 1, \ldots$, to denote the ith Legendre coefficient in a one–dimensional expansion of the solution, it is assumed that $a_i \sim C\mathrm{e}^{-\sigma i}$, where C and σ are constants determined by a least–squares best fit. In [19], p–refinement was employed when $\sigma > 1$; otherwise h–refinement was used.
- *Local regularity estimation.* Here, the idea is to directly approximate the local Sobolev regularity index k_κ of the (unknown) analytical solution on each element κ in the computational mesh; then p–refinement is performed on elements where $k_\kappa > p_\kappa + 1$, otherwise h–refinement is employed. This strategy was first proposed by Ainsworth & Senior [3] in the context of norm control for second–order elliptic problems. In [3], the local Sobolev regularity index k_κ was estimated by employing a local error indicator η_κ which was computed by solving a series of local problems with different polynomial degrees; k_κ could then be extracted by employing local *a priori* error bounds for η_κ. Extensions of this method to linear and nonlinear hyperbolic problems were considered in the series of papers [18,28,29].

Stimulated by the last two strategies, in this paper we introduce two techniques for assessing local smoothness. By monitoring the decay rate of the sequence of coefficients in the Legendre series expansion of a square–integrable function u, we first develop a strategy for estimating the size of the Bernstein ellipse of u on a given interval in one–dimension, thereby determining whether u is analytic. In the case when u is not analytic, the second strategy, based on the work developed in [17], seeks to directly compute the local Sobolev index of u; here, we only develop this algorithm based on applying the Root Test to an infinite series involving the Legendre coefficients a_i; for the application of the Ratio Test, we refer to [17]. The proposed approach has a number of attractions from the computational point of view: firstly, it is entirely general in the sense that it does not depend on the choice of the particular local refinement indicator, and can be applied to any square–integrable function; secondly, in higher dimensional space, it is capable of estimating the smoothness of the function u in each coordinate direction separately which is an important feature in the design of anisotropic hp–adaptive finite element algorithms; finally, the technique is computationally inexpensive as it only requires information about the elemental Legendre coefficients (or approximations to these) of u, which are readily computed by solving a local projection problem.

The main focus of this paper will be to provide a complete account of the design of a fully automatic hp–adaptive algorithm for the finite element approximation to the one–dimensional Poisson equation. In this context the first $p+1$ elemental Legendre coefficients of the of the unknown analytical solution may be exactly determined from the computed numerical approximation. However, we stress that the hp–adaptive finite element algorithm developed in this article is more generally applicable, to both other model problems in higher space dimensions and other numerical methods. In these cases only approximations to the elemental Legendre coefficients of the underlying analytical solution are known.

2 *hp*–Adaptivity

In this section we develop a general hp–adaptive finite element algorithm based on a combination of reliable and efficient residual error indicators and a new hp–extension control technique which assesses the local regularity of the underlying analytical solution. To this end, we employ the following notation: for an interval $I=(a,b)\subset\mathbb{R}$ and an integer $p\geq 0$, we denote by $\mathcal{P}^p(I)$ the set of polynomials of degree p or less on I. Further, by $\|u\|_{k,I}$ and $|u|_{k,I}$ we denote the usual Sobolev norm and seminorm, respectively, of index $k\geq 0$ on I. For $k=0$, we simply write $\|u\|_I$ in lieu of $\|u\|_{0,I}$; moreover, the $L_2(I)$ inner product will be denoted by $(\cdot,\cdot)$.

2.1 Model Problem

Given the interval $I=(0,1)$, we consider the following boundary value problem (in weak form): find $u\in H_0^1(I)$ such that

$$(u',v')=(f,v)\quad\forall v\in H_0^1(I)\,. \tag{1}$$

In order to define the hp–finite element discretisation of (1), we first introduce the necessary notation. To this end, let $0=x_0<x_1<x_2<\ldots<x_M=1$ be a sequence of $M+1$ points in $\bar{I}$ to which we may associate the subdivision $\mathcal{T}=\{I_j : j=1,\ldots,M\}$ of I into M elements $I_j=(x_{j-1},x_j)$. The local mesh function h_j, defined on element I_j, $j=1,\ldots,M$, respectively, is defined by $2h_j=|I_j|=x_j-x_{j-1}$; additionally, the midpoint m_j of I_j is given by $m_j=(x_{j-1}+x_j)/2$, $j=1,\ldots,M$. We remark that h_j is defined to be half of the length of I_j, $j=1,\ldots,M$, since h_j will be required in the definition

of the elemental Bernstein ellipse, cf. Section 2.4 below. On $\mathcal{T}$, we define the distribution $\mathbf{p} = (p_1, \ldots, p_M)^{\mathrm{T}}$ of polynomial degrees $p_j \geq 1$; in the following, we shall refer to the mesh-degree combination $\mathcal{M} = (\mathcal{T}, \mathbf{p})$ as the *hp–mesh*. With this notation, the *hp*–finite element space $V(\mathcal{M})$ is defined by

$$V(\mathcal{M}) = \left\{ u \in H_0^1(I) : u|_{I_j} \in \mathcal{P}^{p_j}(I_j),\ j = 1, \ldots, M \right\} .$$

When the choice of $\mathcal{T}$ and $\mathbf{p}$ is clear from the context, we write simply V in lieu of $V(\mathcal{M})$.

Given an *hp*–mesh $\mathcal{M}$, the finite element approximation $u_{h,p}$ in V of (1) is defined in the usual way: find $u_{h,p} \in V$ such that

$$(u'_{h,p}, v') = (f, v) \quad \forall v \in V . \tag{2}$$

By a simple application of the Lax-Milgram lemma, we deduce that the finite element solution $u_{h,p}$ exists and is unique. Given that the error $e = u - u_{h,p}$ satisfies the Galerkin orthogonality property

$$(e', v') = 0 \quad \forall v \in V , \tag{3}$$

it can be shown that the finite element solution $u_{h,p}$ is the optimal approximation to u from V in the sense that $|u - u_{h,p}|_{1,I} \leq |u - v|_{1,I}$ for all $v \in V$. Additionally, $u_{h,p}$ is nodally exact; that is,

$$u_{h,p}(x_j) = u(x_j), \quad j = 0, \ldots, M. \tag{4}$$

This allows the following characterisation of the finite element solution.

Lemma 2.1 *Given an element $I_j \in \mathcal{T}$, $1 \leq j \leq M$, the finite element solution $u_{h,p}$ admits the following representation:*

$$u_{h,p}(x)|_{I_j} = u(x_{j-1}) + \int_{x_{j-1}}^{x} \Pi_j^{p_j - 1} [u'] (\xi) \, \mathrm{d}\xi ,$$

where $\Pi_j^{p_j-1}$ denotes the $L_2(I_j)$–orthogonal projector onto polynomials of degree $p_j - 1$.

Proof. By exploiting the nodal exactness of the numerical solution $u_{h,p}$, we shall first show that the Galerkin orthogonality property (3) holds locally on each element I_j, $j = 1, \ldots, M$, in the mesh $\mathcal{T}$. To this end, we first note that for any element I_j, $j = 1, \ldots, M$, a function $v \in V$ may be written in the following form

$$v|_{I_j} = \sum_{i=1}^{p_j+1} v_i^{[j]} \varphi_i^{[j]}(x) ,$$

where $\left\{ v_i^{[j]} \right\}_{i=1}^{p_j+1}$ denote the (modal) coefficients of v on I_j and $\left\{ \varphi_i^{[j]} \right\}_{i=1}^{p_j+1}$ is a given basis for $\mathcal{P}^{p_j}(I_j)$. In particular, we may select an hierarchical basis for $\mathcal{P}^{p_j}(I_j)$, such that $\varphi_1^{[j]}$ and $\varphi_2^{[j]}$ are the standard linear 'hat' functions and $\left\{\varphi_i^{[j]}\right\}_{i=3}^{p_j+1}$ are selected to be integrals of the Legendre basis functions of degree $i-1$, respectively, subject to some appropriate scaling; see, for example, Akin [4], pp. 79–85. We note that $\varphi_i^{[j]}(x_{j-1}) = \varphi_i^{[j]}(x_j) = 0$ for $i = 3, \ldots, p_j + 1$; thereby, writing $\hat{\varphi}_i^{[j]}$ to denote the extension of $\varphi_i^{[j]}$ to the entire interval I such that $\hat{\varphi}_i^{[j]}$ is zero on each connected component of $I \backslash I_j$, we deduce that $\hat{\varphi}_i^{[j]}$ belongs to the finite element space V for $i = 3, \ldots, p_j+1$ and $j = 1, \ldots, M$. Thus, from (3), we have that

$$\int_I e' \left(\hat{\varphi}_i^{[j]}\right)' \mathrm{d}\xi = \int_{I_j} e' \left(\varphi_i^{[j]}\right)' \mathrm{d}\xi = 0. \tag{5}$$

Furthermore, exploiting the nodal exactness of the finite element solution $u_{h,p}$, cf. (4) above, we note that

$$\int_{I_j} e' \left(\varphi_i^{[j]}\right)' \mathrm{d}\xi = \left(\varphi_i^{[j]}\right)' \int_{I_j} e' \,\mathrm{d}\xi = \left(\varphi_i^{[j]}\right)' (e(x_j) - e(x_{j-1})) = 0\,, \tag{6}$$

for $i = 1, 2$. Thus, we conclude from (5) and (6) that

$$\int_{I_j} (u - u_{h,p})' w \,\mathrm{d}\xi = 0 \qquad \forall w \in \mathcal{P}^{p_j-1}(I_j).$$

Hence, $\Pi_j^{p_j-1}[(u-u_{h,p})'] = \Pi_j^{p_j-1}[u'] - u'_{h,p} = 0$. On integrating this over the interval $[x_{j-1}, x]$ with $x_{j-1} < x \le x_j$, we get

$$\int_{x_{j-1}}^{x} \Pi_j^{p_j-1}[u'](\xi)\,\mathrm{d}\xi = u_{h,p}(x) - u_{h,p}(x_{j-1}), \qquad x \in I_j.$$

As (4) implies that $u_{h,p}(x_{j-1}) = u(x_{j-1})$, the result follows. $\square$

Remark 1 *Given the finite element solution $u_{h,p}$ corresponding to a particular hp–mesh $\mathcal{M}$, Lemma 2.1 implies that the coefficients $b_i^{[j]}$, $i = 0, 1, \ldots, p_j - 1$ of the Legendre series expansion for u' in element I_j, $j = 1, \ldots, M$, are also available. Moreover, employing the nodal exactness of the finite element approximation $u_{h,p}$ to u, this result also implies that the first p_j+1 coefficients $a_i^{[j]}$, $i = 0, 1, \ldots, p_j$, of the Legendre series expansion for u in I_j, $j = 1, \ldots, M$, may be exactly determined.*

2.2 hp–refinement strategy

The objective of an *hp*–adaptive finite element algorithm is to select a particular sequence $\mathcal{M}^{(l)}$, $l = 1, 2, \ldots$, of *hp*–meshes such that the corresponding sequence $u_{h,p}^{(l)} \in V(\mathcal{M}^{(l)})$ of finite element solutions converge at a high, preferably exponential, rate to the analytical solution u. In adaptive finite element methods, refinement of the subdivision $\mathcal{T}$ is usually steered on the basis of some appropriately chosen element error indicators, cf. Section 2.3 below, for example, where we derive residual–based element error indicators. The possibility of simultaneously changing the subdivision $\mathcal{T}$ as well as the degree distribution $\mathbf{p}$ of the hp–mesh requires the *a posteriori* extraction of additional regularity information from the computed finite element solution $u_{h,p}$. Therefore, in contrast with h–adaptive finite element methods, an hp–adaptive strategy must have two ingredients:

a) a (residual–based) error estimation procedure; and
b) an *hp–steering criterion* for deciding whether to subdivide element I_j, $j = 1, \ldots, M$, (h–refinement) or to increase the local polynomial order p_j (p–refinement).

The generic form of our hp–adaptive finite element method is given below.

Given a tolerance $\texttt{TOL} > 0$ and steering parameters $0 \leq \gamma \leq 1$ and $0 < \theta < 1$, an initial mesh $\mathcal{T}^{(0)}$, and the initial degree distribution $\mathbf{p}^{(0)} = (2, 2, \ldots, 2)$, set $l = 0$ and do

(1) Calculate the finite element solution $u_{h,p}^{(l)}$ satisfying (2) based on the hp–mesh $\mathcal{M}^{(l)} = (\mathcal{T}^{(l)}, \mathbf{p}^{(l)})$.
(2) Calculate the elemental error indicators η_j, $j = 1, \ldots, M$, and the global (*a posteriori*) error estimator

$$\mathcal{E}(u_{h,p}^{(l)}, \mathcal{M}^{(l)}) = \left(\sum_{j=1}^{M} \eta_j^2 \right)^{1/2}$$

which is hp–reliable (and, preferably, also hp–efficient).
(3) Test whether the stopping criterion has been satisfied, i.e. if $\mathcal{E}(u_{h,p}^{(l)}, \mathcal{M}^{(l)}) < \texttt{TOL}$ then **STOP**.
(4) If $\eta_j \geq \gamma \eta_{max}$, mark element I_j for refinement.
(5) Decide in marked elements I_j whether to perform an h– or p–refinement, based on the steering parameter θ.
(6) Construct the new hp–mesh $\mathcal{M}^{(l+1)}$, set $l = l + 1$ and **GOTO** 1.

Algorithm 1. General form of an hp–adaptive finite element method.

We note that $\gamma = 0$ is admissible; in this case, all elements are refined. We remark that in step 4, alternative refinement strategies may also be employed, see [27], for example; moreover, derefinement of the local mesh or polynomial degree may also be incorporated into steps 4 and 5 above. The basic algorithm is very similar to standard h–adaptive finite element feedback strategies, cf. e.g. [32], except for the hp–extension control in step 5 which decides, in marked elements, whether to perform h– or p–refinement. This issue will be addressed in Section 2.4 below; firstly, however, we derive a residual–based *a posteriori* error estimator for our model problem which is used as a stopping criterion in step 3 above.

2.3 Residual error estimator

The derivation of the residual–based *a posteriori* error estimator proceeds as usual (cf. e.g. [32]). Given the finite element solution $u_{h,p} \in V$, the error $e = u - u_{h,p}$ satisfies the problem: find $e \in H_0^1(I)$ such that

$$(e', v') = R(v) \quad \forall v \in H_0^1(I)\,,$$

where the weak residual is given by

$$R(v) = \sum_{j=1}^{M} \int_{I_j} (f + u''_{h,p}) v \,\mathrm{d}x + \sum_{j=1}^{M-1} [u'_{h,p}](x_j) v(x_j)\,.$$

Below we write $\omega_j(x)$, $j = 1, \ldots, M$, to denote the bubble function corresponding to element I_j given by

$$\omega_j(x) = (x_j - x)(x - x_{j-1})\,. \tag{7}$$

With this notation, we have the following result.

Theorem 2.1 *For any hp–mesh $\mathcal{M} = (\mathcal{T}, \mathbf{p})$, the error corresponding to the finite element solution $u_{h,p}$ satisfies*

$$|u - u_{h,p}|_{1,I} \le \mathcal{E}(u_{h,p}, \mathcal{M})\,, \tag{8}$$

where the computable a posteriori *error estimator $\mathcal{E}(u_{h,p}, \mathcal{M})$ is given by*

$$\mathcal{E}(u_{h,p}, \mathcal{M}) = \left(\sum_{j=1}^{M} \eta_j^2 \right)^{1/2}$$

and the elemental error indicators η_j are defined by

$$\eta_j^2 = \frac{1}{p_j(p_j+1)} \int_{I_j} (f + u''_{h,p})^2 \omega_j(x) \,\mathrm{d}x\,.$$

Moreover, $\mathcal{E}(u_{h,p}, \mathcal{M})$ is locally hp–efficient in the sense that there exists an efficiency constant $0 < c_{\mathrm{eff}} < 1$*, which is independent of $\mathcal{T}$ and $\mathbf{p}$, such that*

$$|u - u_{h,p}|_{1,I_j}^2 \ge c_{\mathrm{eff}}^2 \eta_j^2, \quad j = 1, \ldots, M\,. \tag{9}$$

We remark that the formulation of the elemental error indicators η_j assumes that we can evaluate integrals over I_j with the data f explicitly. If quadrature formulae are employed, then the statement of Theorem 2.1 can be modified to include such data oscillation errors, cf. [7,20,24]. Residual–based *a posteriori* error estimators of the type presented in Theorem 2.1 are well known by now; however, Theorem 2.1 is remarkable in the sense that the estimators are reliable and efficient *independent of the polynomial degrees* $\mathbf{p}$ – the efficiency constant c_{eff} in (9) is absolute. In order to achieve this, the inclusion of the bubble functions ω_j, $j = 1, \ldots, M$, in the definition of the elemental error indicators is essential.

Remark 2 *We note that once the stopping criterion in Algorithm 1 has been satisfied, i.e. when*

$$\mathcal{E}(u_{h,p}, \mathcal{M}) < \mathtt{TOL},$$

the efficiency bound (9) implies that the error $e = u - u_{h,p}$ *satisfies*

$$c_{\text{eff}}\, \mathcal{E}(u_{h,p}, \mathcal{M}) \leq |e|_{1,I} \leq \mathcal{E}(u_{h,p}, \mathcal{M}) < \mathtt{TOL},$$

i.e., the actual error $|e|_{1,I}$ *is smaller than the estimator* $\mathcal{E}(u_{h,p}, \mathcal{M})$*, but at most by a factor of* $1/c_{\text{eff}}$*.*

2.4 hp–extension control

To complete the definition of the *hp*–adaptive finite element method presented in Algorithm 1, we need a suitable (automatic) decision mechanism for determining when local *h*– or local *p*–refinement should be performed in step 5. Based on the arguments presented in Section 1, local mesh subdivision should be employed in elements where the solution is locally non-smooth, while local polynomial enrichment should be performed in elements where the underlying solution is locally smooth. When considering the (local) smoothness of a function u, we may wish to determine two things:

(1) Is u a real analytic function, and therefore C^∞?
(2) If u is not C^∞, then what is its (finite) Sobolev regularity index k?

Thereby, $u|_{I_j}$, for some $1 \leq j \leq M$, may be classified as being smooth on element I_j if either u is a real analytic function on I_j, or the local Sobolev regularity index k_j of u defined on I_j, i.e. $u|_{I_j} \in H^{k_j}(I_j)$, satisfies $k_j \geq p_j + 1$; otherwise, u is defined as being non-smooth on I_j. If the element I_j has been flagged for refinement, then in the former case *p*–refinement will be employed, while a local *h*–refinement of I_j will be performed otherwise.

As noted in Remark 1, cf. also Lemma 2.1, the coefficients $a_i^{[j]}$, $i = 0, 1, \ldots, p_j$ of the Legendre series expansion for u in element I_j, $j = 1, \ldots, M$, may be exactly determined once $u_{h,p}$ has been computed (assuming that round-off and quadrature errors are absent). In the following sections we introduce an

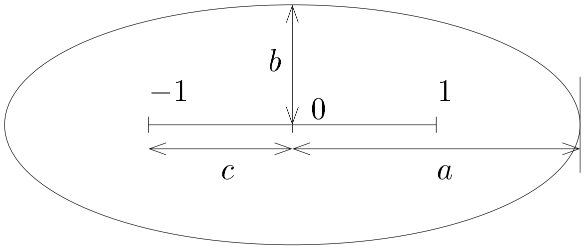


Fig. 1. Bernstein ellipse on the interval $[-1, 1]$.

hp–extension control algorithm based on the solution regularity inferred from this information. In Section 2.4.1 we consider question 1 above; the issue of estimating the finite regularity of a non-analytic function (or, more precisely, of a non-C^∞ function), cf. question 2, will be dealt with in Section 2.4.2.

2.4.1 Analyticity Estimation

In this section we are concerned with determining whether the analytical solution u is locally analytic on each element I_j in the mesh $\mathcal{T}$. To address this question, we observe the fact that Legendre coefficients of analytic functions decay to zero at an exponential rate. To describe this precisely, we associate to a function v, defined on the reference domain $\hat{I} = (-1, 1)$, its Bernstein ellipse $\hat{\mathcal{E}}_\rho$ with foci $x = \pm 1$ and radius $\rho = (a + b)/c \geq 1$, where a and b are the lengths of the semi–major and semi–minor axes, respectively, and c is equal to half the length of the interval $\hat{I}$, i.e. $c = 1$, cf. Fig. 1. We remark that $\rho = 1$ corresponds to the degenerate case of $a = 1$, $b = 0$ and $\hat{\mathcal{E}}_\rho = [-1, 1]$; thereby, v is singular in $\hat{I}$. With this notation, we have the following result.

Theorem 2.2 *Let $z \mapsto v(z)$ be analytic in the interior of $\hat{\mathcal{E}}_\rho$, $\rho > 1$, but not in the interior of any $\hat{\mathcal{E}}_{\rho'}$ with $\rho' > \rho$. Then the Legendre series*

$$v(z) = \sum_{i=0}^{\infty} b_i L_i(z) \, , \quad b_i = \frac{2i + 1}{2} \int_{-1}^{1} v(z) L_i(z) \, \mathrm{d}z \tag{10}$$

converges absolutely and uniformly on any closed set in the interior of $\hat{\mathcal{E}}_\rho$ and diverges in the exterior to $\hat{\mathcal{E}}_\rho$. Moreover,

$$\frac{1}{\rho} = \limsup_{i \to \infty} |b_i|^{1/i} \, . \tag{11}$$

Conversely, if $(b_i)_{i \geq 0}$ is a sequence satisfying (11) with some $\rho > 1$, then the Legendre series (10) converges absolutely and uniformly on any closed set inside of $\hat{\mathcal{E}}_\rho$ to an analytic function $z \mapsto v(z)$ satisfying (10)–(11). The series diverges in the exterior of $\hat{\mathcal{E}}_\rho$.

Proof. See Davis [10], Theorem 12.4.7, for details. □

This result can be localised to intervals $I_j \in \mathcal{T}$, $j = 1, \ldots, M$. To this end, we need the family $\left\{L_i^{[j]}(x)\right\}_{i=0}^{\infty}$ of $L_2(I_j)$–orthogonal polynomials, $j = 1, \ldots, M$. Using the orthogonality properties of the Legendre polynomials, we find that

$$L_i^{[j]}(x) = (1/h_j)^{1/2}\, L_i\left((x - m_j)/h_j\right) \, .$$

By the completeness of $\left\{L_i^{[j]}(x)\right\}_{i=0}^{\infty}$ in $L_2(I_j)$, we may write

$$u(x)|_{I_j} = \sum_{i=0}^{\infty} a_i^{[j]} L_i^{[j]}(x)\, , \quad \text{where} \quad a_i^{[j]} = \frac{2i+1}{2} \int_{I_j} u(x) L_i^{[j]}(x)\, \mathrm{d}x\, . \tag{12}$$

Moreover, by Lemma 2.1, cf. Remark 1, we also find that

$$u_{h,p}(x)|_{I_j} = \sum_{i=0}^{p_j} a_i^{[j]} L_i^{[j]}(x)\, .$$

With this notation, the analogue of Theorem 2.2 on a given element I_j in the mesh $\mathcal{T}$ holds verbatim; in this case the elemental Bernstein ellipse $\hat{\mathcal{E}}_{\rho_j}$ has foci at x_{j-1}, x_j and radius $\rho_j = (a_j + b_j)/h_j$, where $a_j \geq h_j$ and b_j are the lengths of the semi-major and semi-minor axes, respectively. Moreover, with the elemental Legendre coefficients of u being defined as in (12), if u is analytic in the interior of $\hat{\mathcal{E}}_{\rho_j}$, but not in the interior of any $\hat{\mathcal{E}}_{\rho_j'}$ with $\rho_j' > \rho_j$, the elemental Bernstein radius satisfies

$$\frac{1}{\rho_j} = \limsup_{i \to \infty} \left|a_i^{[j]}\right|^{1/i} \tag{13}$$

with some $\rho_j > 1$.

This result suggests that

$$\theta_j = \frac{1}{\rho_j}, \quad j = 1, \ldots, M, \tag{14}$$

is a measure of size of the domain of analyticity of u *relative to element* I_j. From the elemental analogue of Theorem 2.2, we deduce that $0 \leq \theta_j \leq 1$; $\theta_j = 0$ corresponds to an entire analytic function, whereas $\theta_j = 1$ corresponds to functions with singular support in $\overline{I}_j$. We also note that for a fixed, analytic function u, h–refinement of I_j asymptotically will always increase the relative size of the domain of analyticity of u, and hence decrease the value of θ_j.

The decision whether to h–refine or to increase p in step 5 of Algorithm 1 may therefore take the following form:

5(a). *In marked elements I_j, compute θ_j defined in (14). If $\theta_j \leq \theta$, increase p_j, otherwise bisect I_j and keep the polynomial degree equal to p_j on the resulting sub-elements.*

Strictly speaking, Algorithm 1 with step 5(a) is not well-defined, in the sense that the definition (14) of θ_j assumes knowledge of *all* the Legendre coefficients $a_i^{[j]}$ of u on each element I_j in the mesh $\mathcal{T}$. Thereby, to obtain a practical algorithm, we must compute an approximation of θ_j (or, equivalently, of ρ_j) based on the available local Legendre coefficients $a_i^{[j]}$, $i = 0, 1, \ldots, p_j$ of u in I_j. Indeed, motivated by (13), one possible approach would be to approximate θ_j by $\hat{\theta}_j = |a_{p_j}^{[j]}|^{1/(p_j)}$. In practice, this definition may not provide a suitably accurate approximation to θ_j, particularly for functions whose Legendre series expansion have repeating patterns of zero coefficients (occurring, for example, for functions which are locally symmetric or antisymmetric about the midpoint of I_j or for functions which have lacunary series expansions), since only the highest computed Legendre coefficient is included into the criterion.

Thereby, we propose an alternative approach which takes into account *all* of the computed Legendre coefficients on I_j. To this end, employing (13) we deduce that if u is analytic in $\overline{I}_j$, then $|a_i^{[j]}| \sim (1/\rho_j)^i$, as $i \to \infty$. This implies that $\log |a_i^{[j]}| \sim i \log(1/\rho_j)$, as $i \to \infty$. We compute an approximate value for θ_j by fitting the slope m_j in $|\log |a_i^{[j]}|| = i m_j + b_j$ by linear regression to the already computed $\log |a_i^{[j]}|$ for $i = 0, 1, \ldots, p_j$ (note that for $p_j \geq 1$ there are at least two Legendre coefficients of u per element available). Indeed, the slope m_j of the regression line of the data $\{i, y_i = |\log |a_i^{[j]}||\}_{i=0}^{p_j}$ is computed by

$$m_j = 6 \frac{2 \sum_{i=0}^{p_j} i y_i - p_j \sum_{i=0}^{p_j} y_i}{(p_j + 1)\left((p_j + 1)^2 - 1\right)};$$

thereby, the following approximation $\hat{\theta}_j$ to θ_j may be determined:

$$\hat{\theta}_j = e^{-m_j}. \tag{15}$$

With this approximation, step 5(a) above now fully completes the specification of the *hp*–adaptive algorithm outlined in Section 2.2. In the following section, we consider an alternative algorithm based on estimating the local Sobolev regularity index of the underlying analytical solution.

2.4.2 Regularity estimation

In this section we briefly outline how to estimate the (local) Sobolev regularity of $u : I_j \to \mathbb{R}$, on each element I_j in the mesh $\mathcal{T}$; for a complete account of this work, we refer to the article [17]. We first recall some basic properties of Legendre polynomials defined on I_j; for further details, we refer to [24], and the references cited therein. Suppose that $u \in L_2(I_j)$; then u can be expanded into the Legendre series

$$u(x)|_{I_j} = \sum_{i=0}^{\infty} a_i^{[j]} L_i^{[j]}(x) \qquad \text{with} \qquad a_i^{[j]} = \frac{2i+1}{2} \int_{I_j} u(x) L_i^{[j]}(x)\,\mathrm{d}x\,, \tag{16}$$

which converges in (the norm of) $L_2(I_j)$. Here $L_i^{[j]}(x)$ denotes the (scaled) Legendre polynomial in x of degree $i \geq 0$ defined in the previous section. Due to the orthogonality properties of Legendre polynomials in the inner product of $L_2(I_j)$, the following Parseval's identity holds (cf. Schwab [24], Lemma 3.10):

$$\int_{I_j} |u(x)|^2\,\mathrm{d}x = \sum_{i=0}^{\infty} |a_i^{[j]}|^2 \frac{2}{2i+1}\,. \tag{17}$$

By monitoring the decay of the sequence of coefficients $(|a_i^{[j]}|^2/(2i+1))_{i\geq 0}$ in Parseval's identity (17), the (weighted) Sobolev regularity index of the function $u|_{I_j} \in L_2(I_j)$ may be determined; here, we work in the weighted spaces $H^k_{\omega_j}(I_j)$, $k > 0$, where ω_j is defined in (7), cf. [17]. Consequently, the sequence $(c_i^{[j]})_{i\geq 1}$, where $c_i^{[j]}$ is defined by

$$c_i^{[j]} = \left(|a_i^{[j]}|^2 \frac{2}{2i+1}\right)^{1/i}, \qquad i \geq 1\,, \tag{18}$$

is bounded. Let us define

$$\Theta = \overline{\lim}_{i\to\infty} c_i^{[j]} = \overline{\lim}_{i\to\infty} \left(|a_i^{[j]}|^2 \frac{2}{2i+1}\right)^{1/i}. \tag{19}$$

According to the Root Test (cf. [12], for example) $\Theta \in (0,1]$, for otherwise (17) would not be finite, thereby contradicting the basic hypothesis that $u \in L_2(I_j)$. With this notation, we have the following result.

Proposition 2.1 *With the above notation, the following two cases hold:*

Case 1. *If $\Theta \in (0,1)$, then $u|_{I_j} \in \bigcap_{k_j \geq 0} H^{k_j}_{\omega_j}(I_j)$. Hence, $u|_{I_j} \in C^\infty(I_j)$ and is therefore regular in I_j.*

Case 2. *Given $\Theta = 1$, we suppose that there exists a positive real number ℓ_j such that*

$$\ell_j = \lim_{i\to\infty} \frac{\log\left(\frac{2i+1}{2|a_i^{[j]}|^2}\right)}{2\log i}\,. \tag{20}$$

Then, $u|_{I_j} \in H^{\ell_j - 1/2 - \varepsilon}_{\omega_j}(I_j)$, for $0 < \varepsilon \leq \ell_j - 1/2$; thereby, $u|_{I_j} \in H^{\ell_j - 1/2 - \varepsilon}_{\mathrm{loc}}(I_j)$, for $0 < \varepsilon \leq \ell_j - 1/2$.

Proof. See [17] for details. □

Remark 3 *We note that the local Sobolev regularity index of u on I_j may also be estimated in a similar manner by application of the ratio test, cf. [17]; however, numerical experiments performed in [17] indicated that the latter approach was computationally inferior to the root test.*

From this discussion, we see that the local regularity index of u on element I_j may be estimated by employing Proposition 2.1; thereby, step 5 in Algorithm 1 may take the following alternative form:

5(b). In marked elements I_j, compute ℓ_j defined in (20), and set $k_j = \ell_j - 1/2$. If $k_j \geq p_j + 1$, increase p_j, otherwise bisect I_j and keep the polynomial degree equal to p_j on the resulting sub-elements.

In order to ensure that Algorithm 1 with step 5(b) is well defined, a practical algorithm was outlined in [17] for the computation of an approximation to ℓ_j, and hence also k_j, based on the available local Legendre coefficients of the analytical solution u. Finally, we remark that an adaptive algorithm based on a combination of 5(a) and 5(b) may be employed. In this case, the analyticity estimation employed in 5(a) would first be used to determine whether the solution is locally analytic. If this is the case, then clearly p–refinement would be employed; otherwise the local regularity of the underlying function would be estimated, and p–refinement employed if $k_j \geq p_j + 1$. Otherwise, if neither test is passed, an h–refinement of the element is performed.

3 Numerical experiments

In this section we present a series of numerical experiments to highlight the practical performance of the proposed hp–adaptive finite element algorithm. In particular, we illustrate the flexibility of this strategy by considering its application to both the standard (conforming) finite element approximation of the model problem outlined in Section 2.1 using the energy norm *a posteriori* error estimator (8), as well as for the discontinuous Galerkin finite element approximation of a mixed elliptic–hyperbolic problem using a goal–oriented *a posteriori* error bound.

3.1 Poisson's equation in 1D

In this example, we consider Poisson's equation, where the forcing function f and non-homogeneous Dirichlet boundary conditions are chosen so that the analytical solution is given by $u(x) = 0$ for $-1 \leq x < \beta$ and $u(x) = (x - \beta)^\alpha$ for $\beta \leq x \leq 1$, in the computational domain $I = (-1, 1)$, where $\alpha \geq 0$ and

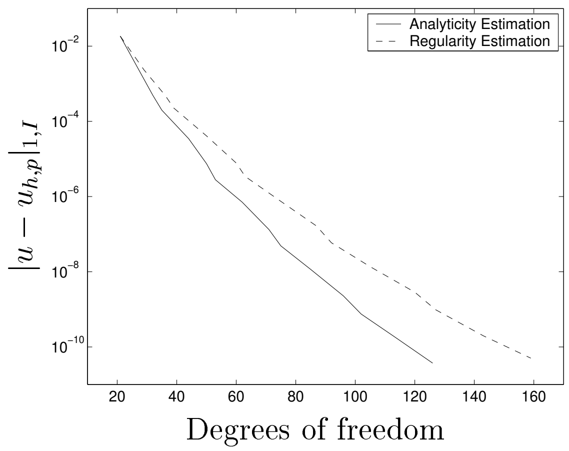


Fig. 2. *Example 1.* Comparison between employing analyticity estimation and Sobolev regularity index estimation.

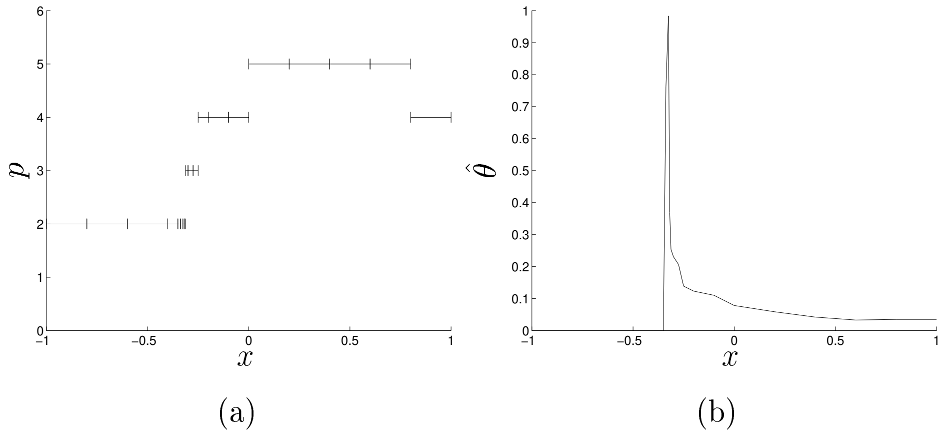
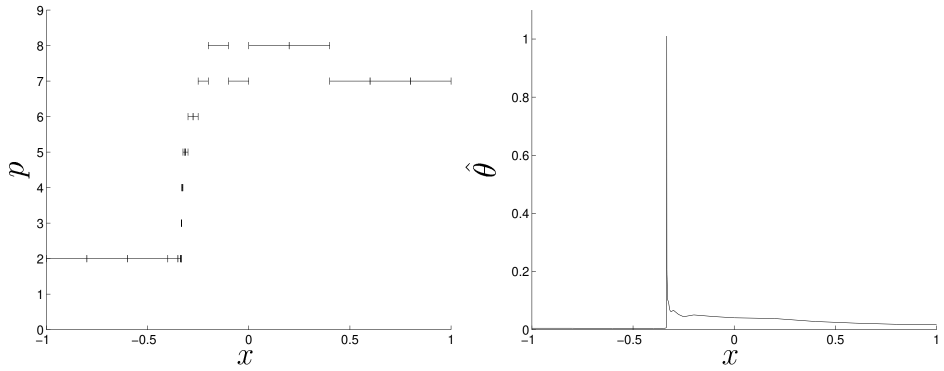


Fig. 3. *Example 1.* (a) Computational mesh after 7 adaptive refinements with 62 degrees of freedom; (b) Computed values of $\hat{\theta}$.

$-1 < \beta < 1$; thereby, $u \in H^{\alpha+1/2-\varepsilon}(-1,1)$, $\varepsilon > 0$, for any $\beta \in (-1,1)$. Here, we set $\alpha = 7/2$ and $\beta = -1/3$. Throughout this section we set the refinement parameter $\gamma = 1/2$ and the steering parameter $\theta = 1/2$ in steps 4 and 5(a), respectively, of Algorithm 1.

In Fig. 2, we compare the performance of the proposed hp–adaptive finite element algorithm using the analyticity estimation (5(a)) and the Sobolev regularity index estimation (5(b)), for the computation of the energy norm of the error in the computed finite element solution. In each case we have plotted the energy norm of the error against the number of degrees of freedom in the finite element space V on a linear–log scale. Using either strategy, we observe that, after an initial transient, the convergence line becomes roughly a straight line, thereby indicating exponential convergence of the error measured in the energy norm. Additionally, we note that for this simple example, the new strategy proposed in this article leads to an improvement of the error, for a given number of degrees of freedom, in comparison to the corresponding



(a) (b)

Fig. 4. *Example 1.* (a) Computational mesh after 14 adaptive refinements with 126 degrees of freedom; (b) Computed values of $\hat{\theta}$.

quantity computed using the Sobolev regularity index algorithm.

In Figs. 3 & 4 we show the *hp*–mesh and corresponding elemental values of $\hat{\theta}$ after 7 and 14 adaptive refinements, respectively, employing the analyticity indicator. In each case, we observe that the algorithm clearly identifies the region in the computational mesh where the singularity in the underlying analytical solution is located. Indeed, in this region $\hat{\theta}$ is very close to one and decays to almost zero as the distance to the singularity increases. The corresponding *hp*–mesh is *h*–refined into the region containing the singularity, while the polynomial degrees increase as we move away from this area.

3.2 Mixed elliptic–hyperbolic problem

To demonstrate the flexibility of the proposed *hp*–refinement technique, in this section we consider the symmetric interior–penalty discontinuous Galerkin finite element approximation to a linear advection–diffusion problem in two spatial dimensions. Here, we select the computational domain $I = (0,1)^2$, the diffusion matrix $a = \varepsilon(x)I$, where $\varepsilon = \frac{\delta}{2}(1 - \tanh((r - 1/4)(r + 1/4)/\zeta))$, $r^2 = (x - 1/2)^2 + (y - 1/2)^2$, $\delta = 0.05$ and $\zeta = 0.01$. Furthermore, we let the velocity vector $\mathbf{b} = (2y^2 - 4x + 1, 1 + y)$, and forcing function $f = 0$. The Dirichlet boundary conditions are prescribed by: $u(x,y) = 1$ for $x = 0$, $0 < y \leq 1$; $u(x,y) = \sin^2(\pi x)$ for $0 \leq x \leq 1$, $y = 0$; $u(x,y) = \mathrm{e}^{-50y^4}$ for $x = 1$, $0 < y \leq 1$. From the computational point of view, the underlying partial differential equation is elliptic in the circular region defined by $r < 1/4$, and is hyperbolic elsewhere; see [15], for further details.

Instead of seeking to control the energy norm of the error, we exploit a 'goal–oriented' *a posteriori* error bound in ensure the efficient estimation of a given linear function $J(\cdot)$ of the analytical solution u. To this end, we suppose that

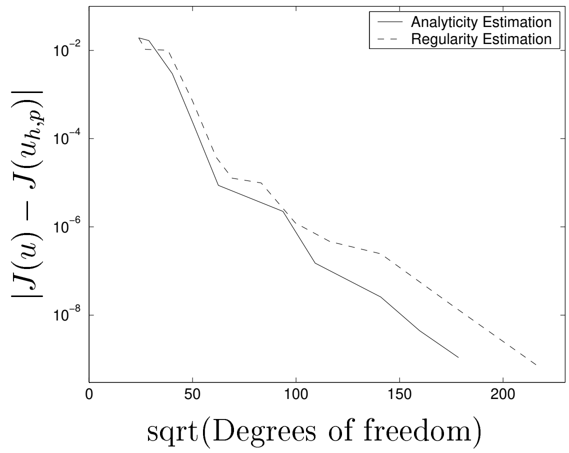


Fig. 5. *Example 2.* Comparison between employing analyticity estimation and Sobolev regularity index estimation.

the aim of the computation is to calculate the value of u at the point of interest $x = (0.43, 0.9)$, *i.e.*, $J(u) = u(0.43, 0.9)$; cf. [15]. Here, the extension of the analyticity estimation and the Sobolev regularity index estimation procedures is based on the application of these techniques in each coordinate direction on the reference element, assuming that a quadrilateral finite element mesh has been employed; see [17] for further details. Finally, in this example, we employ the fixed fraction refinement strategy in step 4 of Algorithm 1, with refinement and derefinement fractions set to 20% and 10%, respectively; we also set $\theta = 1/10$ in step 5(a).

In Fig. 5 we present a comparison between the two hp–adaptive algorithms outlined in this article. As in the previous section, we observe that, after an initial transient, both strategies deliver exponential rates of convergence. Additionally, we note that the analyticity estimation strategy leads to an improvement of the error in the computed target functional of interest, when compared to the same quantity computed using the Sobolev regularity index estimation technique. For brevity, the hp–mesh distribution has been omitted; this is very similar in structure to the mesh shown in [15].

4 Concluding remarks

In this article we have introduced an automatic hp–adaptive finite element algorithm based on estimating the size of the elemental Bernstein ellipse from the local Legendre series expansion of the numerical solution. The numerical experiments presented clearly demonstrate the flexibility of the proposed approach; indeed, in each case, exponential convergence of the computed numerical solution to the underlying analytical solution has been shown.

References

[1] S. Adjerid, M. Aiffa, J.E. Flaherty. Computational methods for singularly perturbed systems. In J. Cronin and R.E. O'Malley, eds., *Singular Perturbation Concepts of Differential Equations*, AMS, Providence, 1998.

[2] M. Ainsworth and J.T. Oden. *A Posteriori Error Estimation in Finite Element Analysis*. Series in Computational and Applied Mathematics. Elsevier, 1996.

[3] M. Ainsworth and B. Senior. An adaptive refinement strategy for hp–finite element computations. *Appl. Numer. Math.*, 26:165–178, 1998.

[4] J.E. Akin. *Finite Elements for Analysis and Design*. Academic Press, 1994.

[5] I. Babuška and M. Suri. The hp-Version of the finite element method with quasiuniform meshes. *M^2AN Modél. Math. Anal. Numér.*, 21:199–238, 1987.

[6] R. Becker and R. Rannacher. An optimal control approach to a-posteriori error estimation in finite element methods. In A. Iserles, ed., *Acta Numerica*, 2001.

[7] C. Bernardi. Indicateurs d'erreur en éléments spectraux. *RAIRO Anal. Numér.*, 30(1):1–38, 1996.

[8] C. Bernardi, N. Fiétier and R.G. Owens. An error indicator for mortar element solutions to the Stokes problem. *IMA J. Num. Anal.*, 21:857–886, 2001.

[9] K.S. Bey, A. Patra and J.T. Oden. hp–Version discontinuous Galerkin methods for hyperbolic conservation laws: a parallel adaptive strategy. *Internat. J. Numer. Methods Engrg.*, 38:3889–3908, 1995.

[10] P.J. Davis. *Interpolation and Approximation*. Blaisdell Publishing Co., 1963.

[11] L. Demkowicz, W. Rachowicz and Ph. Devloo. A fully automatic hp–adaptivity. *J. Sci. Comp.*, 17(1-4):117–142, 2002.

[12] S.A. Douglass. *Introduction to Mathematical Analysis*. Addison Wesley, 1996.

[13] K. Eriksson, D. Estep, P. Hansbo and C. Johnson. Introduction to adaptive methods for differential equations. In A. Iserles, ed., *Acta Numerica*, pp. 105–158, 1995.

[14] W. Gui and I. Babuška. The h, p and h–p versions of the finite element method in 1 Dimension. Part III. The adaptive h–p version. *Numer. Math.*, 49:659–683, 1986.

[15] K. Harriman, P. Houston, B. Senior and E. Süli. hp–Version discontinuous Galerkin methods with interior penalty for partial differential equations with nonnegative characteristic form. In C.-W. Shu, T. Tang, and S.-Y. Cheng, eds., *Recent Advances in Scientific Computing and Partial Differential Equations.* Contemporary Mathematics Vol. 330, pp. 89–119, AMS, 2003.

[16] V. Heuveline and R. Rannacher. Duality–based adaptivity in the hp–finite element method. *J. Numer. Math.*, 1(2), 95–113, 2003.

[17] P. Houston, B. Senior and E. Süli. Sobolev regularity estimation for hp–adaptive finite element methods. In F. Brezzi, A. Buffa, S. Corsaro and A. Murli, eds., *Numerical Mathematics and Advanced Applications*, Springer–Verlag, 619–644, 2003.

[18] P. Houston and E. Süli. hp–Adaptive discontinuous Galerkin finite element methods for hyperbolic problems. *SIAM J. Sci. Comp.*, 23(4):1225-1251, 2001.

[19] C. Mavriplis. Adaptive mesh strategies for the spectral element method. *Comput. Methods. Appl. Mech. Engrg.*, 116:77–86, 1994.

[20] J.M. Melenk and B.I. Wohlmuth. On residual–based a posteriori error estimation in hp–FEM. *Advances in Comp. Math.*, 15:311–331, 2001.

[21] J.T. Oden and A. Patra. A parallel adaptive strategy for hp finite elements. *Comput. Methods. Appl. Mech. Engrg.*, 121:449–470 1995.

[22] J.T. Oden, A. Patra and Y.S. Feng. An hp–adaptive strategy. In A.K. Noor, ed., *Adaptive, Multilevel, and Hierarchical Computational Strategies*, AMD-157, ASME, New York, pages 23–26, 1992.

[23] W. Rachowicz, L. Demkowicz and J.T. Oden. Toward a universal h–p adaptive finite element strategy, Part 3. Design of h–p meshes. *Comput. Methods. Appl. Mech. Engrg.*, 77:181–212, 1989.

[24] Ch. Schwab. *p– and hp–Finite Element Methods. Theory and Applications to Solid and Fluid Mechanics.* Oxford University Press, Oxford, 1998.

[25] Ch. Schwab and M. Suri. The p and $h - p$ version of the finite element method for problems with boundary layers. *Math. Comp.*, 65:1403–1429, 1996.

[26] P. Solin and L. Demkowicz. Goal–oriented hp–adaptivity for elliptic problems. *Comput. Meth. Appl. Mech. Engrg.*, (to appear).

[27] E. Süli and P. Houston. Adaptive finite element approximation of hyperbolic problems. In T. Barth and H. Deconinck, eds., *Error Estimation and Adaptive Discretization Methods in Computational Fluid Dynamics.* Lecture Notes in Computational Science and Engineering, 25, 269–344, Springer–Verlag, 2002.

[28] E. Süli, P. Houston and Ch. Schwab. hp–Finite element methods for hyperbolic problems. In: J.R. Whiteman, ed., *The Mathematics of Finite Elements and Applications X.* Elsevier, 143–162, 2000.

[29] E. Süli, P. Houston and B. Senior. hp–Discontinuous Galerkin finite element methods for nonlinear hyperbolic problems. *Int. J. Numer. Meth. Fluids*, 40(1-2):153-169, 2002.

[30] B. Szabó and I. Babuška. *Finite Element Analysis.* J. Wiley & Sons, New York, 1991.

[31] J. Valenciano and R.G. Owens. An h–p adaptive spectral element method for Stokes flow. *Appl. Numer. Math.*, 33:365–371, 2000.

[32] R. Verfürth. *A Review of a Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques.* B.G. Teubner, Stuttgart, 1996.