Mixed *hp*-discontinuous Galerkin finite element methods for the Stokes problem in polygons

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1 Introduction

Over the last few years, several mixed discontinuous Galerkin finite element methods (DGFEM) have been proposed for the discretization of incompressible fluid flow problems; see, e.g., [4, 6, 7, 9, 12, 16] and the references therein. The main motivations that led to these schemes are that mixed DGFEM provide robust and high-order accurate approximations, particularly in transport-dominated regimes, and that they are considerably flexible in the choice of velocity-pressure combinations, without excessive numerical stabilization. For example, no extra stabilization is required to use optimally matched combinations where the approximation degree for the pressure is of one order lower than that of the velocity; this result was first established in [11] in the context of linear elasticity of nearly incompressible materials.

The work in [13] presented a unifying framework for the analysis of mixed hp-DGFEM for Stokes flow. For discontinuous $\mathcal{Q}^p - \mathcal{Q}^{p-1}$ elements on hexahedral meshes, the results there ensure (slightly suboptimal) error bounds for the p-version of the DGFEM, where convergence is obtained by increasing the polynomial approximation order p on a fixed mesh. However, these bounds result in algebraic rates of convergence and are restricted to piecewise smooth solutions – an assumption that is unrealistic in general polygons, due to the presence of corner singularities.

In this note, we report on the recent results in [14] that extend the approach of [13] to mixed hp-DGFEM for Stokes flow in general polygons. In particular, we show that, on geometrically refined meshes, the hp-DGFEM yields exponential rates of convergence for problems with piecewise analytic input data; see [18] for similar results in the context of diffusion problems. We further present numerical results that confirm the exponential convergence rates as predicted in [14].

2 The Stokes problem in polygons

In this section, we introduce the Stokes problem and use the results of [10] to describe the regularity of its solution for piecewise analytic input data.

2.1 The Stokes problem

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain with outward unit normal vector $\mathbf{n}$ on the boundary $\partial\Omega$. Then, for a given forcing term $\mathbf{f} \in H^{-1}(\Omega)^2$ and a Dirichlet datum $\mathbf{g} \in H^{\frac{1}{2}}(\partial\Omega)^2$ satisfying the compatibility condition $\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n}\, ds = 0$, the Stokes problem is to find a velocity field $\mathbf{u} \in H^1(\Omega)^2$ and a pressure $p \in L^2_0(\Omega) := L^2(\Omega)/\mathbb{R}$ such that

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g} \quad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

This system is uniquely solvable; see, e.g., [5, 8] for details.

2.2 Analytic regularity in polygons

In [10] the regularity of the solution $(\mathbf{u}, p)$ to the Stokes equations with piecewise analytic data $\mathbf{f}$ and $\mathbf{g}$ was described in terms of the countably normed Sobolev spaces introduced by Babuška and Guo for closely related diffusion and elasticity problems (see [2, 3, 15], and the references cited therein). To define these spaces, let $\{A_i\}_{i=1}^M$ denote the vertices of the domain Ω. To each vertex A_i we assign a weight $\beta_i \geq 0$ and store these numbers in the M-tuple $\beta = (\beta_1, \ldots, \beta_M)$. We define $\beta \pm j := (\beta_1 \pm j, \ldots, \beta_M \pm j)$ and use the shorthand notation $C_1 > \beta > C_2$ to mean $C_1 > \beta_i > C_2$ for $i = 1, \ldots, M$. Writing $r_i(\mathbf{x}) = \min\{1, |\mathbf{x} - A_i|\}$, we define the weight function $\Phi_\beta(\mathbf{x}) := \prod_{i=1}^M r_i(\mathbf{x})^{\beta_i}$, and introduce the semi-norms

$$|u|^2_{H^{k,l}_\beta(\Omega)} := \sum_{|\alpha| \geq l}^{k} \|\Phi_{\beta+|\alpha|-l} D^\alpha u\|^2_{L^2(\Omega)}, \qquad k \geq l \geq 0.$$

We denote by $H^{k,l}_\beta(\Omega)$ the completion of $C^\infty(\overline{\Omega})$ with respect to the norm

$$\begin{aligned} \|u\|^2_{H^{k,l}_\beta(\Omega)} &:= \|u\|^2_{H^{l-1}(\Omega)} + |u|^2_{H^{k,l}_\beta(\Omega)}, \qquad l \geq 1, \\ \|u\|^2_{H^{k,0}_\beta(\Omega)} &:= \sum_{|\alpha|=0}^{k} \|\Phi_{\beta+|\alpha|} D^\alpha u\|^2_{L^2(\Omega)}. \end{aligned}$$

Here, we denote by $\|\cdot\|_{L^2(\Omega)}$ the usual L^2-norm. Similarly, $\|\cdot\|_{H^l(\Omega)}$ is the norm on the standard Sobolev space $H^l(\Omega)$.

Definition 1. *For an M-tuple $\beta = (\beta_1, \ldots, \beta_M)$ and $l \geq 0$, the countably normed space $B^l_\beta(\Omega)$ consists of all functions u for which $u \in H^{k,l}_\beta(\Omega)$ for $k \geq l$ and*

$$\|\Phi_{\beta+k-l} D^\alpha u\|_{L^2(\Omega)} \leq C d^{(k-l)} (k-l)!, \qquad |\alpha| = k \geq l,$$

for constants $C > 0$, $d \geq 1$ independent of k. Moreover, for $l \geq 1$, the space $B^{l-\frac{1}{2}}_\beta(\partial\Omega)$ is the space of traces of functions in $B^l_\beta(\Omega)$.

Functions in $B^l_\beta(\Omega)$ (or their traces) are referred to as *piecewise analytic functions*. Indeed, they are analytic in any interior domain $\Omega_{\text{int}} \subset \Omega$ with $\overline{\Omega}_{\text{int}} \subset \overline{\Omega} \setminus \{A_i\}_{i=1}^M$, but develop singularities at the corners $\{A_i\}_{i=1}^M$. Globally, we have $B^l_\beta(\Omega) \subset H^{l-1}(\Omega)$, but $B^l_\beta(\Omega) \not\subset H^l(\Omega)$. The following regularity result was proved in [10].

Theorem 1. *There exists a weight vector $0 \le \beta_{\min} < 1$ depending on the opening angles of Ω at the vertices $\{A_i\}_{i=1}^M$ such that for weight vectors β with $\beta_{\min} < \beta < 1$ and piecewise analytic data $(\mathbf{f}, \mathbf{g}) \in B^0_\beta(\Omega)^2 \times B^{\frac{3}{2}}_\beta(\partial\Omega)^2$, the solution $(\mathbf{u}, p)$ of the Stokes system (1) satisfies $(\mathbf{u}, p) \in B^2_\beta(\Omega)^2 \times B^1_\beta(\Omega)$.*

In the rest of the paper, we assume that $(\mathbf{f}, \mathbf{g}) \in B^0_\beta(\Omega)^2 \times B^{\frac{3}{2}}_\beta(\partial\Omega)^2$ for a weight vector β with $\beta_{\min} < \beta < 1$, in order to ensure the piecewise analyticity of the solution $(\mathbf{u}, p)$, as stated in Theorem 1.

3 Discontinuous Galerkin methods

In this section, we introduce discontinuous Galerkin methods for the Stokes problem (1) and review their well-posedness, using the recent results in [13].

3.1 Meshes

Throughout, we assume that the domain Ω can be subdivided into shape-regular affine meshes $\mathcal{T}_h = \{K\}$ consisting of parallelograms K. For each $K \in \mathcal{T}_h$, we denote by $\mathbf{n}_K$ the outward unit normal vector to the boundary ∂K, and by h_K the elemental diameter. Further, we assign to each element $K \in \mathcal{T}_h$ an approximation order $k_K \ge 1$. The local quantities h_K and k_K are stored in the vectors $\mathbf{h} = \{h_K\}_{K\in\mathcal{T}_h}$ and $\mathbf{k} = \{k_K\}_{K\in\mathcal{T}_h}$, respectively. An interior edge of $\mathcal{T}_h$ is the (non-empty) one-dimensional interior of $\partial K^+ \cap \partial K^-$, where K^+ and K^- are two adjacent elements of $\mathcal{T}_h$. Similarly, a boundary edge of $\mathcal{T}_h$ is the (non-empty) one-dimensional interior of $\partial K \cap \partial\Omega$ which consists of entire edges of ∂K. We denote by $\mathcal{E}_\mathcal{I}$ the union of all interior edges of $\mathcal{T}_h$, by $\mathcal{E}_\mathcal{D}$ the union of all boundary edges, and set $\mathcal{E} = \mathcal{E}_\mathcal{I} \cup \mathcal{E}_\mathcal{D}$. We allow for meshes with 1-irregular hanging nodes, and further assume that there is a constant $\kappa > 0$ such that $\kappa k_K \le k_{K'} \le \kappa^{-1} k_K$, whenever K and K' share a common edge.

3.2 Averages and jumps

Next, we define average and jump operators. To this end, let K^+ and K^- be two adjacent elements of $\mathcal{T}_h$ and $\mathbf{x}$ be an arbitrary point on the interior edge $e = \partial K^+ \cap \partial K^- \subset \mathcal{E}_\mathcal{I}$. Moreover, let q, $\mathbf{v}$, and $\underline{\tau}$ be scalar-, vector-, and matrix-valued functions, respectively, that are smooth inside each element $K^\pm$. By $(q^\pm, \mathbf{v}^\pm, \underline{\tau}^\pm)$ we denote the traces of $(q, \mathbf{v}, \underline{\tau})$ on e taken from within the interior of $K^\pm$, respectively. Then, we define the averages at $\mathbf{x} \in e$ by:

$$\{\!\{q\}\!\} = (q^+ + q^-)/2, \qquad \{\!\{\mathbf{v}\}\!\} = (\mathbf{v}^+ + \mathbf{v}^-)/2, \qquad \{\!\{\underline{\tau}\}\!\} = (\underline{\tau}^+ + \underline{\tau}^-)/2.$$

Similarly, the jumps at $\mathbf{x} \in e$ are given by

$$\begin{aligned} [\![q]\!] &= q^+\,\mathbf{n}_{K^+} + q^-\,\mathbf{n}_{K^-}, & [\![\mathbf{v}]\!] &= \mathbf{v}^+\cdot\mathbf{n}_{K^+} + \mathbf{v}^-\cdot\mathbf{n}_{K^-}, \\ \underline{[\![\mathbf{v}]\!]} &= \mathbf{v}^+\otimes\mathbf{n}_{K^+} + \mathbf{v}^-\otimes\mathbf{n}_{K^-}, & [\![\underline{\tau}]\!] &= \underline{\tau}^+\mathbf{n}_{K^+} + \underline{\tau}^-\mathbf{n}_{K^-}. \end{aligned}$$

On boundary edges $e \subset \mathcal{E}_\mathcal{D}$, we set $\{\!\{q\}\!\} = q$, $\{\!\{\mathbf{v}\}\!\} = \mathbf{v}$, $\{\!\{\underline{\tau}\}\!\} = \underline{\tau}$, as well as $[\![q]\!] = q\mathbf{n}$, $[\![\mathbf{v}]\!] = \mathbf{v}\cdot\mathbf{n}$, $\underline{[\![\mathbf{v}]\!]} = \mathbf{v}\otimes\mathbf{n}$, and $[\![\underline{\tau}]\!] = \underline{\tau}\mathbf{n}$.

3.3 Mixed *hp*-DGFEM

Given a mesh $\mathcal{T}_h$ and a degree vector $\mathbf{k} = \{k_K\}$, $k_K \geq 1$, we wish to approximate the Stokes problem (1) by finite element functions $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$, where

$$\mathbf{V}_h = \{ \mathbf{v} \in L^2(\Omega)^2 : \mathbf{v}|_K \in \mathcal{Q}^{k_K}(K)^2,\ K \in \mathcal{T}_h \},$$
$$Q_h = \{ q \in L^2_0(\Omega) : q|_K \in \mathcal{Q}^{k_K-1}(K),\ K \in \mathcal{T}_h \}.$$

Here, $\mathcal{Q}^k(K)$ is the space of polynomials of degree at most k in each variable on K. Thereby, we consider the following mixed method: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\begin{cases} A_h(\mathbf{u}_h, \mathbf{v}) + B_h(\mathbf{v}, p_h) = F_h(\mathbf{v}), \\ -B_h(\mathbf{u}_h, q) = G_h(q) \end{cases} \tag{2}$$

for all $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$. The forms A_h and B_h are discontinuous Galerkin forms that discretize the Laplacian and the incompressibility constraint, respectively, with corresponding right-hand sides F_h and G_h. These forms are given by

$$\begin{aligned} A_h(\mathbf{u}, \mathbf{v}) &= \int_\Omega \nabla_h \mathbf{u} : \nabla_h \mathbf{v}\, d\mathbf{x} - \int_{\mathcal{E}} \left(\{\!\{\nabla_h \mathbf{v}\}\!\} : \underline{[\![\mathbf{u}]\!]} + \{\!\{\nabla_h \mathbf{u}\}\!\} : \underline{[\![\mathbf{v}]\!]} \right) ds \\ &\quad + \int_{\mathcal{E}} \mathtt{c} \underline{[\![\mathbf{u}]\!]} : \underline{[\![\mathbf{v}]\!]}\, ds, \\ B_h(\mathbf{v}, q) &= -\int_\Omega q\, \nabla_h \cdot \mathbf{v}\, d\mathbf{x} + \int_{\mathcal{E}} \{\!\{q\}\!\} [\![\mathbf{v}]\!]\, ds, \\ F_h(\mathbf{v}) &= \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} - \int_{\mathcal{E}_\mathcal{D}} (\mathbf{g} \otimes \mathbf{n}) : \nabla_h \mathbf{v}\, ds + \int_{\mathcal{E}_\mathcal{D}} \mathtt{c}\mathbf{g} \cdot \mathbf{v}\, ds, \\ G_h(q) &= -\int_{\mathcal{E}_\mathcal{D}} q\, \mathbf{g} \cdot \mathbf{n}\, ds. \end{aligned}$$

Here, ∇_h and $\nabla_h \cdot$ denote the discrete gradient and divergence operator, respectively, taken element-wise. The function $\mathtt{c} \in L^\infty(\mathcal{E})$ is the so-called discontinuity stabilization function that is chosen as follows. Define the functions $\mathtt{h} \in L^\infty(\mathcal{E})$ and $\mathtt{k} \in L^\infty(\mathcal{E})$ by

$$\mathtt{h}(\mathbf{x}) := \begin{cases} \min\{h_K, h_{K'}\}, & \mathbf{x} \in e = \partial K \cap \partial K' \subset \mathcal{E}_\mathcal{I}, \\ h_K, & \mathbf{x} \in e = \partial K \cap \partial \Omega \subset \mathcal{E}_\mathcal{D}, \end{cases}$$

$$\mathtt{k}(\mathbf{x}) := \begin{cases} \max\{k_K, k_{K'}\}, & \mathbf{x} \in e = \partial K \cap \partial K' \subset \mathcal{E}_\mathcal{I}, \\ k_K, & \mathbf{x} \in e = \partial K \cap \partial \Omega \subset \mathcal{E}_\mathcal{D}. \end{cases}$$

Then we set

$$\mathtt{c} = \gamma \mathtt{h}^{-1} \mathtt{k}^2,$$

with a parameter $\gamma > 0$ that is independent of $\mathtt{h}$ and $\mathtt{k}$.

Remark 1. The form A_h corresponds to the so-called symmetric interior penalty (IP) discretization of the Laplace operator; see [1] and [13] where the presentation and analysis of several different DG methods were unified for diffusion problems and the Stokes system, respectively. All the results presented in this paper hold true verbatim for all the mixed DG methods investigated in [13].

Remark 2. For piecewise analytic data $\mathbf{f}$ and $\mathbf{g}$ as in Theorem 1, the forms F_h and G_h are well-defined.

3.4 Well-posedness

Well-posedness of the discrete system (2) was established in [13]. Indeed, by endowing $\mathbf{V}_h$ with the broken norm

$$\|\mathbf{v}\|_h^2 := \|\nabla_h \mathbf{v}\|_{L^2(\Omega)}^2 + \int_{\mathcal{E}} \mathtt{h}^{-1}\mathtt{k}^2 |\underline{[\![\mathbf{v}]\!]}|^2 \, ds,$$

the forms A_h and B_h are continuous on $\mathbf{V}_h \times \mathbf{V}_h$ and $\mathbf{V}_h \times Q_h$, respectively, with continuity constants $C > 0$ independent of $\mathbf{h}$ and $\mathbf{k}$. Furthermore, there exists a parameter $\gamma_{\min} > 0$ independent of $\mathbf{h}$ and $\mathbf{k}$ such that for any $\gamma \geq \gamma_{\min}$ there exists a coercivity constant $C > 0$ independent of $\mathbf{h}$ and $\mathbf{k}$ with

$$A_h(\mathbf{v}, \mathbf{v}) \geq C\|\mathbf{v}\|_h^2 \qquad \forall \mathbf{v} \in \mathbf{V}_h.$$

Finally, the following discrete inf-sup condition holds true:

$$\inf_{0 \neq q \in Q_h} \sup_{\mathbf{0} \neq \mathbf{v} \in \mathbf{V}_h} \frac{B_h(\mathbf{v}, q)}{\|\mathbf{v}\|_h \|q\|_{L^2(\Omega)}} \geq C|\mathbf{k}|^{-1} > 0,$$

with a constant $C > 0$ that is independent of $\mathbf{h}$ and $\mathbf{k}$. Here, $|\mathbf{k}| := \max_{K \in \mathcal{T}_h}\{k_K\}$.

The above properties of the forms A_h and B_h, combined with the continuity of the forms F_h and G_h, guarantee the well-posedness of the formulation (2) for $\gamma \geq \gamma_{\min}$.

4 Exponential rates of convergence

In this section, we present the main result of [14], namely exponential rates of convergence for mixed hp-DGFEM on geometrically refined meshes.

4.1 Geometric meshes

We first define geometric meshes on $\hat{Q} = (0, 1)^2$.

Definition 2. *Fix $n \in \mathbb{N}_0$ and $\sigma \in (0, 1)$. On $\hat{Q}$, the geometric mesh $\Delta_{n,\sigma}$ with $n+1$ layers and grading factor σ is created recursively as follows: If $n = 0$, $\Delta_{0,\sigma} = \{\hat{Q}\}$. Given $\Delta_{n,\sigma}$ for $n \geq 0$, $\Delta_{n+1,\sigma}$ is generated by subdividing the square K with $0 \in \overline{K}$ into four smaller rectangles by dividing its sides in a $\sigma : (1 - \sigma)$ ratio.*

An example of a geometric mesh $\Delta_{n,\sigma}$ on $\hat{Q}$ is shown in Figure 1. We denote the elements in the basic geometric mesh by $\{K_{ij}\}$, as indicated. We say that the elements K_{1j}, K_{2j} and K_{3j} constitute layer j for $j \geq 2$. The element at the origin is denoted as K_{11}.

Definition 3. *A geometric mesh $\mathcal{T}_{n,\sigma}$ on the polygon $\Omega \subset \mathbb{R}^2$ is obtained by mapping the geometric meshes $\Delta_{n,\sigma}$ on $\hat{Q}$ affinely to a vicinity of each convex corner of Ω. At reentrant corners three suitably scaled copies of $\Delta_{n,\sigma}$ are used (as shown in Figure 2). The remainder of Ω is subdivided with a fixed affine and quasi-uniform partition.*

In Figure 2 this local geometric refinement is illustrated. For ease of exposition, we only consider mesh patches that are identically refined with the same parameters σ and n, although different grading factors and numbers of layers may be used for the different corner patches.

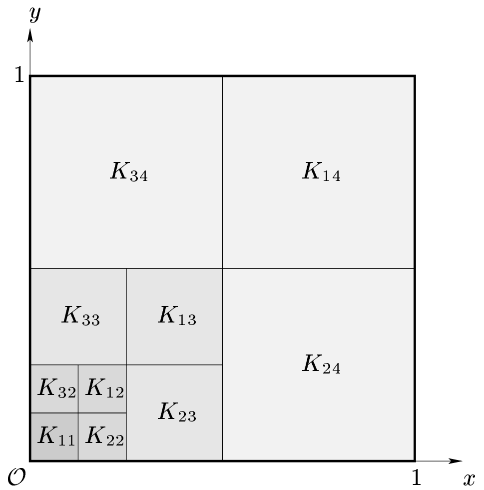

Fig. 1. The basic geometric mesh $\Delta_{n,\sigma}$ with $n = 3$ and $\sigma = 0.5$.

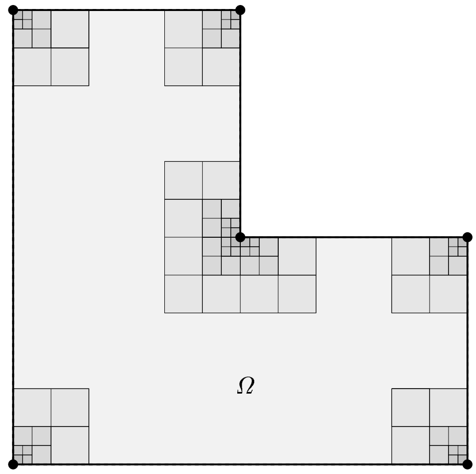

Fig. 2. Local geometric refinement towards the vertices of Ω. In all corners, $n = 3$ and $\sigma = 0.5$.

Definition 4. *A polynomial degree distribution* $\mathbf{k}$ *on a geometric mesh* $\mathcal{T}_{n,\sigma}$ *is called linear with slope* $\mu > 0$ *if the elemental polynomial degrees are layerwise constant in the geometric patches and given by* $k_j := \max(2, \lfloor \mu j \rfloor)$ *in layer* j, $j = 1, \ldots, n+1$. *In the interior of the domain the elemental polynomial degree is set to* $\max(2, \lfloor \mu(n+1) \rfloor)$.

4.2 Exponential convergence

Our main result is the exponential convergence of the mixed *hp*-DGFEM for problems with piecewise analytic data. Its detailed proof can be found in [14].

Theorem 2. *Assume that the analytical solution* $(\mathbf{u}, p)$ *of the Stokes problem* (1) *is piecewise analytic as stated in Theorem 1. Further, let* $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ *denote the DGFEM approximation defined in* (2) *with* $\gamma \geq \gamma_{\min}$ *obtained on geometric meshes* $\mathcal{T}_{n,\sigma}$. *Then there exists a parameter* $\mu_0 = \mu_0(\sigma, \beta) > 0$, *such that for linear degree vectors* $\mathbf{k}$ *with slope* $\mu \geq \mu_0$, *there holds*

$$\|\mathbf{u} - \mathbf{u}_h\|_h + \|p - p_h\|_{L^2(\Omega)} \leq C \exp(-b N^{1/3}),$$

with constants $C, b > 0$ *independent of* $N = \dim(\mathbf{V}_h) \approx \dim(Q_h)$.

Remark 3. If the polynomial degree is chosen to be constant throughout the mesh, i.e., $k_K = k$ for all $K \in \mathcal{T}_h$, exponential convergence is still obtained by choosing k proportional to the number of layers n.

5 Numerical experiment

The goal of this section is to numerically confirm the exponential convergence result stated in Theorem 2. To this end, let Ω be the L-shaped domain shown in Figure 3.

As in [17, p. 113], we select the right-hand side $\mathbf{f} = \mathbf{0}$ and take the Dirichlet boundary datum $\mathbf{g}$ in such a way that, in the polar coordinates (r, φ), the exact solution $(\mathbf{u}, p)$ to the Stokes problem (1) is given by

$$\mathbf{u}(r,\varphi) = r^\lambda \begin{pmatrix} (1+\lambda)\sin(\varphi)\Psi(\varphi) + \cos(\varphi)\Psi'(\varphi) \\ \sin(\varphi)\Psi'(\varphi) - (1+\lambda)\cos(\varphi)\Psi(\varphi) \end{pmatrix},$$
$$p = -r^{\lambda-1}[(1+\lambda)^2\Psi'(\varphi) + \Psi'''(\varphi)]/(1-\lambda),$$

where

$$\Psi(\varphi) = \sin((1+\lambda)\varphi)\cos(\lambda\omega)/(1+\lambda) - \cos((1+\lambda)\varphi)$$
$$- \sin((1-\lambda)\varphi)\cos(\lambda\omega)/(1-\lambda) + \cos((1-\lambda)\varphi).$$

Furthermore, $\omega = 3\pi/2$, and the exponent λ is the smallest positive solution of the equation $\sin(\lambda\omega) + \lambda\sin(\omega) = 0$; thereby, $\lambda \approx 0.54448373678246$.

We emphasize that the solution $(\mathbf{u}, p)$ is in fact the strongest corner singularity of the Stokes operator in the domain Ω; it is piecewise analytic, that is analytic in $\overline{\Omega} \setminus \{\mathcal{O}\}$, but both $\nabla\mathbf{u}$ and p are strongly singular at the origin. Indeed, here $\mathbf{u} \notin H^2(\Omega)^2$ and $p \notin H^1(\Omega)$; thus, this example reflects the typical (singular) behavior that solutions of the Stokes problem exhibit in the vicinity of reentrant corners.

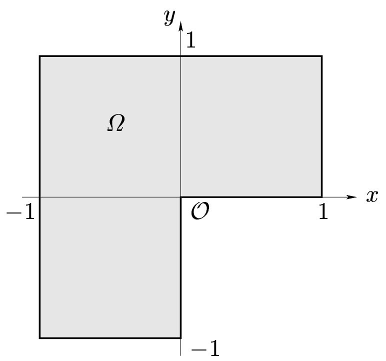

Fig. 3. L-shaped domain Ω.

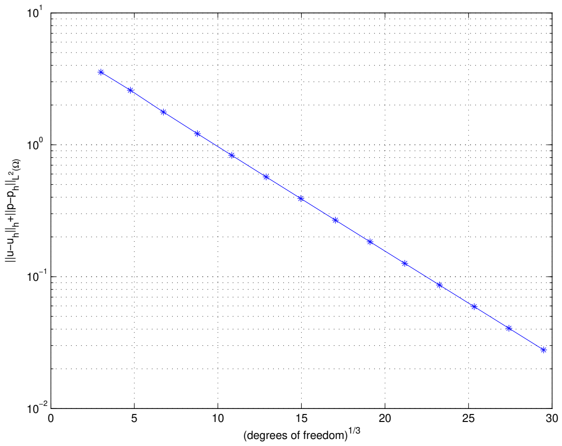

Fig. 4. Performance of the mixed hp-DGFEM.

Figure 4 shows the performance of the mixed hp-DGFEM for the above problem, on meshes that are geometrically refined towards the origin and for polynomial degree distributions that are linearly increasing away from the origin. In our computations we used the grading factor $\sigma = 0.5$ and the linear slope $\mu = 1$. The interior penalty parameter γ was chosen as $\gamma = 10$. The exponential convergence rate, according to Theorem 2, is clearly visible. In addition, we observe that the asymptotic regime is already achieved even with a small number of degrees of freedom.

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