Flatness, preorders and general metric spaces

Vincent Schmitt

September 4, 2003

Abstract

This paper studies a general notion of flatness in the enriched context: $\mathcal{P}$-flatness where the parameter $\mathcal{P}$ stands for a class of presheaves. One obtains a completion of a category A by considering the category $Flat_{\mathcal{P}}(A)$ of $\mathcal{P}$-flat presheaves over A. This completion is related to the free cocompletion under a class of colimits defined by Kelly. For a category A, for $\mathcal{P} = \mathcal{P}_0$ the class of all presheaves, $Flat_{\mathcal{P}_0}(A)$ is the Cauchy-completion of A. Two classes $\mathcal{P}_1$ and $\mathcal{P}_2$ of interest for general metric spaces are considered. The $\mathcal{P}_1$- and $\mathcal{P}_2$- flatness are investigated and the associated completions are characterized for general metric spaces (enrichments over $\bar{\mathbb{R}}_+$) and preorders (enrichments over **Bool**). We get this way two non-symmetric completions for metric spaces and retrieve the ideal completion for preorders.

1 Introduction

In [Law73] Lawvere showed amongst other results that enriched category theory was a suitable unifying framework for metric spaces and partial orders. He proved in particular the following. Preorders and their morphisms as well as general metric spaces with non-increasing maps occur as categories and functors enriched over closed monoidal categories. The base category is **Bool** for preorders and $\bar{\mathbb{R}}_+$ for general metric spaces. A categorical completion of enrichments may be defined so that for the base category $\mathcal{V} = \bar{\mathbb{R}}_+$ it amounts to the completion à la Cauchy of metric spaces whereas for $\mathcal{V} = $ **Bool** it corresponds to the Dedekind-Mac Neille completion of preorders. Lawvere's categorical completion was therefore just named Cauchy-completion.

Following the spirit of Lawvere's work, one may wonder what more theory common to metric spaces and preorders may be developed at the categorical level? The present paper tackles the following problem. It is known that partial orders admits various completions:

- the Dedekind-Mac Neille completion,
- the downward completion,
- the "algebraic" or "ideal" completion,
- ...

The terminology may vary for the last completions, but it is clear what they are once said that

- the Dedekind-Mac Neille completion is defined in terms of maximal cuts;
- the downward completion is in terms of downward closed subsets;
- the algebraic one is in terms of non-empty directed down-sets (sometimes called "ideals" but we shall avoid this confusing terminology).

Quite natural questions are whether all these completions may be described in terms of enrichments and if so what they correspond to for metric spaces.

The answer to this requires a general notion of flatness. Flatness in the enriched context is already treated in [Kel82-2], [BQR98]. In the last paper the definition "filtered weights" relies on left Kan extensions preserving certain limits, which is similar to the flatness defined in this paper. Nevertheless both these works focus on the case when the base $\mathcal{V}$ is locally presentable and we shall avoid here such a restriction on $\mathcal{V}$. Also Street showed in [Str83] that the weights of absolute colimits form an important class of presheaves related to the Cauchy-completion. We shall see that this class may be defined in terms of flatness. We propose the following definition. Given a class $\mathcal{P}$ of indexes, a presheaf $F : A^{op} \to \mathcal{V}$ is called "$\mathcal{P}$-flat" if its left Kan extension along Y preserves all the limits in $[A, \mathcal{V}]$ with indexes in $\mathcal{P}$.

We have established the following results. Let us write $Flat_{\mathcal{P}}$ for the class of $\mathcal{P}$-flat presheaves. For any category A, the full subcategory $Flat_{\mathcal{P}}(A)$ of $[A^{op}, \mathcal{V}]$ with objects $\mathcal{P}$-flat presheaves is its free $Flat_{\mathcal{P}}$-cocompletion in the sense of [Kel82]. Calling simply $Flat_{\mathcal{P}}(A)$ the "$\mathcal{P}$-completion" of A, one obtains therefore a family of completions for categories with parameter a family $\mathcal{P}$ of presheaves. First the free cocompletion of categories is just the $\mathcal{P}$-completion for $\mathcal{P}$ the empty class of presheaves. On the other hand the Cauchy-completion is shown to be the $\mathcal{P}_0$-completion for $\mathcal{P}_0$ the whole class of presheaves. Focusing on metric spaces we found two more notions of flatness of particular interest. We define the families

- $\mathcal{P}_1$ of presheaves over the empty category or the unit category I (with one point $*$, and $I(*, *) = I$)
- $\mathcal{P}_2$ of presheaves on categories with finite number of objects.

In the context $\mathcal{V} = \bar{I\!R}_+$, one may express the $\mathcal{P}_1$- and $\mathcal{P}_2$- completions in terms of filters on the metric spaces. This generalizes the fact that minimal Cauchy filters on a general metric space are in one-to-one correspondence with left adjoint modules on the associated category. We call the filters corresponding to the $\mathcal{P}_1$-flat and $\mathcal{P}_2$-flat presheaves respectively *weakly flat* and *flat*. To sum up, let us say that:

- with the right notion of morphisms, weakly flat filters, respectively flat filters, occur as "non-empty colimits", respectively "non-empty filtered colimits" of the so-called forward Cauchy sequences. These sequences were introduced in the literature as a generalization of Cauchy sequences in non-symmetric spaces [Sun95].
- Cauchy filters are flat, and when the pseudo metric is symmetric, flat filters are Cauchy.

Eventually, we show that one can forget category theory and describe the $\mathcal{P}_1$- and $\mathcal{P}_2$- completions of non-symmetric metric spaces in pure topological/metric terms. The $\mathcal{P}_2$-completion of a symmetric space amounts to its Cauchy-completion but the $\mathcal{P}_2$-completion of a non-symmetric space certainly generally differs from its bi-completion [FL82],[Fla92] and [Sch03]. In the case $\mathcal{V} = \mathbf{Bool}$, it appears that the $\mathcal{P}_1$-completion yields a completion defined in terms of non-empty downward subsets whereas the $\mathcal{P}_2$-completion is the algebraic completion. For the applications we tried to use as much as possible categorical techniques. To this respect the only result that seems not related to category theory is the characterization of weakly flat/flat filters in terms of forward Cauchy sequences.

This work relies much on the indexed limits/colimits computation à la Kelly. We adopt the notation and pick up many results from [Kel82]. We also use also a little of the 2-categorical theory of enriched modules, our references for it are [StWa78], [BCSW83], and more recently [DaSt97]. The author has been also much inspired by [BvBR98] and [Vic]. Both of these works study metric spaces and partial orders as enrichments and define completions by considering ordinary colimits in the presheaf categories.

The paper is organized as follows. Section 2 treats the notion of flatness in the enriched context. Sections 3 and 4 are devoted to applications, respectively to general metric spaces($\mathcal{V} = \bar{I\!\!R}_+$) and to general preorders ($\mathcal{V} = \mathbf{Bool}$).

2 Flatness

This section treats briefly flatness in the enriched context. A generic notion of $\mathcal{P}$-flat presheaf where $\mathcal{P}$ stands for a class of indexes is investigated. A completion in terms of $\mathcal{P}$-flat presheaves, the $\mathcal{P}$-completion, is defined. It is shown to coincide with the free $Flat_{\mathcal{P}}$-cocompletion in the sense of [Kel82] where $Flat_{\mathcal{P}}$ denotes the class of $\mathcal{P}$-flat presheaves. If $\mathcal{P} = \emptyset$ the $\emptyset$-completion is just the free-cocompletion of categories whereas for $\mathcal{P}_0$ the class of all presheaves the $\mathcal{P}_0$-completion amounts to the Cauchy-completion. Actually $\mathcal{P}_0$-flat presheaves are exactly the presheaves which are left adjoint modules. We define then two more notions of flatness associated with classes $\mathcal{P}_1$ and $\mathcal{P}_2$ of presheaves. Their relevance will appear with the applications in the next sections.

We shall consider in this section a symmetric monoidal complete closed $\mathcal{V}$. For a matter of consistency all the $\mathcal{V}$-categories considered are by default small, i.e. they have small sets of objects. We shall precise "large" when a category may not be small. Large $\mathcal{V}$-categories that we shall use are the category $\mathcal{V}$ itself, the presheaf categories $[A^{op}, \mathcal{V}]$ for small A's and the categories PA of accessible presheaves $A^{op} \to \mathcal{V}$ for large $\mathcal{V}$-categories A. Indexes of limits and colimits will be also considered small.

Given a class of presheaves ϕ, a ϕ-limit (respectively. ϕ-colimit) is a limit (respectively. colimit) with index in ϕ. A functor is ϕ-continuous (respectively. ϕ-cocontinuous) if and only if it preserves ϕ-limits (respectively. ϕ-colimits).

Definition 2.1 (P -flatness) *Given a family $\mathcal{P}$ of indexes, a presheaf $F : A^{op} \to \mathcal{V}$ is said $\mathcal{P}$-*flat *when its left Kan extension along Y, $- * F : [A, \mathcal{V}] \to \mathcal{V}$ preserves all $\mathcal{P}$-limits. $Flat_{\mathcal{P}}$ will denote the family of all $\mathcal{P}$-flat presheaves, and for any $\mathcal{V}$-category A, $Flat_{\mathcal{P}}(A)$ will denote the full subcategory of $[A^{op}, \mathcal{V}]$ with objects $\mathcal{P}$-flat presheaves.*

For any family $\mathcal{P}$ of indexes, representables are $\mathcal{P}$-flat since for any $A(-, a)$, $Lan_Y(A(-, a))$ is the evaluation in a that is cocontinuous.

Since limits and colimits in functor categories are pointwise, we remind that given functors $F : A^{op} \to \mathcal{V}$, $P : K \to \mathcal{V}$ and $G : A \otimes K \to \mathcal{V}$ equivalent to $G' : A \to [K, \mathcal{V}]$ and also to $G'' : K \to [A, \mathcal{V}]$, one has $(F * G')k \cong F * (G''k)$ and $\{P, G''\}a \cong \{P, G'a\}$. Further on we shall use quite freely these isomorphisms.

Lemma 2.2 *For any $\mathcal{P}$-flat $G : A^{op} \to \mathcal{V}$ and any functor $H : A \to [C^{op}, \mathcal{V}]$ with values $\mathcal{P}$-flat functors, the colimit $G * H$ is again $\mathcal{P}$-flat.*

PROOF:Consider $F : K \to \mathcal{V}$ in $\mathcal{P}$ and $L : K \to [C, \mathcal{V}]$. One has the successive isomorphisms:

$$\begin{aligned} Lan_Y(G * H)(\{F, L\}) &\cong \{F, L\} * (G * H) \\ &\cong (G * H) * \{F, L\} \\ &\cong G * (H - *\{F, L\}) \text{ ([Kel82], (3.23) “continuity of a colimit in its index”)} \\ &\cong G * (\{F, H - *L-\}) \text{ (since for all } a,\ Ha * - \text{ preserves } \mathcal{P}\text{-limits)} \\ &\cong \{F, G * (H - *L-)\} \text{ (since } G * - \text{ preserves } \mathcal{P}\text{-limits)} \end{aligned}$$

$$\begin{aligned}
&\cong \{F, (G * H) * L-\}) \text{ ([Kel82], (3.23))} \\
&\cong \{F, L - *(G * H)\}) \\
&\cong \{F, Lan_Y(G * H) \circ L\})
\end{aligned}$$

The resulting isomorphism $Lan_Y(G * H)(\{F, L\}) \cong \{F, Lan_Y(G * H) \circ L\})$ corresponds actually to the preservation of $\{F, L\}$ by $Lan_Y(G * H)$.

Note to the referee: this part of the proof may be omitted
To check this last point, one may consider the following natural isomorphisms:

$$[C, \mathcal{V}](\gamma, \{F, L\}) \cong^{\phi^1_\gamma} [K, \mathcal{V}](F, [C, \mathcal{V}](\gamma, L-));$$

$$[A, \mathcal{V}](\tau, H? * \{F, L\}) \cong^{\phi^2_\tau} [K, \mathcal{V}](F, [A, \mathcal{V}](\tau, H? * L-))$$

that exhibits $H - *\{F, L\}$ as the limit $\{F, H - *L-\}$;

$$[v, G * (H? * \{F, L\})] \cong^{\phi^3_v} [K, \mathcal{V}](F, [v, G * (H? * L-)])$$

that exhibits $G * (H - *\{F, L\}) \cong G * \{F, H - *L-\}$ as the limit $\{F, G * (H - *L-)\}$.

Now the commutation of square (I) on the diagram below corresponds to the preservation of $\{F, L\}$ by $H? * - : [C, \mathcal{V}] \to [A, \mathcal{V}]$. Also the commutation of square (II) is the preservation of $\{F, H - *L-\} \cong H - *\{F, L\}$ by $G * - : [A, \mathcal{V}] \to \mathcal{V}$. So eventually the outer square commutes which is the preservation of $\{F, L\}$ by $G * (H - *-)$.

$$\begin{tikzcd}
{[C, \mathcal{V}](\gamma, \{F, L\})} \arrow[r, phantom, "\cong^{\phi^1_\gamma}"] \arrow[d, "H?*-"'] \arrow[dr, phantom, "(I)"] & {[K, \mathcal{V}](F, [C, \mathcal{V}](\gamma, L-))} \arrow[d, "{[K,\mathcal{V}](F, H?*-)}"] \\
{[A, \mathcal{V}](H? * \gamma, H? * \{F, L\})} \arrow[r, phantom, "\cong^{\phi^2_{H?*\gamma}}"] \arrow[d, "G*-"'] \arrow[dr, phantom, "(II)"] & {[K, \mathcal{V}](F, [A, \mathcal{V}](H? * \gamma, H? * L-))} \arrow[d, "{[K,\mathcal{V}](F, G*-)}"] \\
{[G * (H? * \gamma), G * (H? * \{F, L\})]} \arrow[r, phantom, "\cong^{\phi^3_{G*(H?*\gamma)}}"] & {[K, \mathcal{V}](F, [G * (H? * \gamma), G * (H? * L-)])}
\end{tikzcd}$$

Since the isomorphisms $\gamma * (G * H) \cong (G * H) * \gamma \cong G * (H - *\gamma)$ are natural in γ, $- * (G * H)$ also preserves $\{F, L\}$. ∎

The above lemma has a few consequences that we shall see now. First, since by Yoneda for any presheaf $F : K \to \mathcal{V}$, $F \cong F * Y$, one has

Proposition 2.3 *For any family $\mathcal{P}$ of indexes, F is $\mathcal{P}$-flat if and only if it is a $Flat_\mathcal{P}$-colimit of representables.*

This together with the following result 2.4 from [Kel82] (see also [AK88]) that relates the closure of categories under ϕ-colimits to their free ϕ-cocompletion will yield a universal completion for categories in terms of $\mathcal{P}$-flat presheaves (2.5).

Remember from [Kel82] that given a family ϕ of indexes, and a category A, the *closure of A under ϕ-colimits*, say $\bar{A}$, is defined as the smallest full (replete) subcategory of $[A^{op}, \mathcal{V}]$, containing the representables and closed under the formation of ϕ-colimits in $[A^{op}, \mathcal{V}]$, which means that for any $G : K \to [A^{op}, \mathcal{V}]$ taking values in $\bar{A}$ and any $F \in \phi$, $F * G$ is in $\bar{A}$. $\bar{A}$ is a full subcategory of the $\mathcal{V}$-category $[A^{op}, \mathcal{V}]$ and may be not small.

Theorem 2.4 *For any family of indexes ϕ, for any A, the closure $\bar{A}$ of A in $[A^{op}, \mathcal{V}]$ under ϕ-colimits constitutes its* free ϕ-cocompletion. *This means that for any categories A:*

- *$\bar{A}$ is ϕ-cocomplete;*
- *For any possibly large category B one has an equivalence $Lan_K : [A,B]' \cong \phi$-$Cocts[\bar{A},B]$ where:*
  - *K is the full and faithful inclusion $A \to \bar{A}$ (sending any $a \in A$ to $A(-,a)$);*
  - *$[A,B]'$ stands for the full subcategory of $[A,B]$ of functors admitting a left Kan extension along K;*
  - *ϕ-$Cocts[\bar{A},B]$ is the full subcategory of $[\bar{A},B]$ of ϕ-cocontinuous functors;*
  - *Lan_K stands for the "left Kan extension functor", it has inverse the restriction to ϕ-$Cocts[\bar{A},B]$ of $[K,1] : [\bar{A},B] \to [A,B]$. In particular if B is ϕ-cocomplete then $[A,B]' = [A,B]$.*

Thus by 2.3,

Theorem 2.5 *For any family $\mathcal{P}$ of indexes and any category A, $Flat_{\mathcal{P}}(A)$ is the free $Flat_{\mathcal{P}}$-cocompletion of A.*

We shall therefore simplify the terminology and call $Flat_{\mathcal{P}}(A)$ the *$\mathcal{P}$-completion* of A, for any category A, and any family of indexes $\mathcal{P}$. For $\mathcal{P} = \emptyset$ the empty class of presheaves all the presheaves are $\emptyset$-flat thus the $\emptyset$-completion is just the free cocompletion. On the other hand let $\mathcal{P}_0$ denote the whole class of presheaves, then the $\mathcal{P}_0$-completion is the Cauchy-completion. This is a straightforward consequence of

Theorem 2.6 *For a presheaf $F : A^{op} \to \mathcal{V}$ the following assertions are equivalent:*

- (1) *F is $\mathcal{P}_0$-flat;*
- (2) *F as a module $I \to A$ is a left adjoint.*

Before to establish 2.6, we need

Proposition 2.7 *Let $F : A^{op} \to \mathcal{V}$, $P : K \to \mathcal{V}$ and $G : A \otimes K \to \mathcal{V}$ equivalent to $G' : A \to [K,\mathcal{V}]$ and also to $G'' : K \to [A,\mathcal{V}]$. Then $F * - : [A,\mathcal{V}] \to \mathcal{V}$ preserves the limit $\{P, G''\}$ if and only if $\{P,-\} : [K,\mathcal{V}] \to \mathcal{V}$ preserves the colimit $F * G'$.*

PROOF:Let

$$P \xrightarrow{\eta} [A,\mathcal{V}](\{P,G''\}, G''-)$$

be the unit of $\{P, G''\}$ and

$$F \xrightarrow{\lambda} [K,\mathcal{V}](G'-, F * G')$$

be the unit of $F * G'$. We need to show that

$$(1) \quad P \xrightarrow{\eta} [A,\mathcal{V}](\{P,G''\}, G''-) \xrightarrow{F*-} [F * \{P,G''\}, (F * G''-)]$$

exhibits $F * \{P, G''\}$ as $\{P, F * G'\}$ if and only if

$$(2) \quad F \xrightarrow{\lambda} [K,\mathcal{V}](F * G', G'-) \xrightarrow{\{P,-\}} [\{P, F * G'\}, \{P, G'-\}]$$

exhibits $\{P, F * G'\}$ as $F * \{P, G''\}$.

First note that given $x \in \mathcal{V}$, any natural in k,

$$(1')\ Pk \to_k [x, (F * G')k]$$

corresponds via Yoneda to a natural in v, $[v,x] \to_v [K,\mathcal{V}](P,[v,(F*G')-])$. Since

$$[K,\mathcal{V}](P,[v,(F*G')-] \cong_v [v,\{P,F*G'\}],$$

it corresponds also to an arrow

$$(1'')\ \ x \to \{P,F*G'\}.$$

Also that $(1')$ exhibits x as the limit $\{P,F*G'\}$ is equivalent to the fact that $(1'')$ is iso. Analogously any natural in a,

$$(2')\ \ Fa \to [\{P,G''\}a,x]$$

corresponds to an arrow

$$(2'')\ \ F*\{P,G''\} \to x,$$

and $(2')$ exhibits x as the colimit $F*\{P,G''\}$ if and only if $(2'')$ is iso.

Now the result follows from the fact the arrow (1) above corresponds by the bijection $(1')-(1'')$ to the same arrow as (2) by $(2')-(2'')$. To check this last point, use the following sequences of isomorphisms

$$\begin{array}{ll} & [K,\mathcal{V}](P,[F*\{P,G''\},(F*G')-]) \\ \cong & [K,\mathcal{V}](P,[A^{op},\mathcal{V}](F,[\{P,G''\}-,(F*G')-])) \\ \cong & [K\otimes A^{op},\mathcal{V}](F\otimes P,[\{P,G''\},F*G']) \\ \cong & [A^{op},\mathcal{V}](F,[K,\mathcal{V}](P,[\{P,G''\}-,(F*G')-])) \\ \cong & [A^{op},\mathcal{V}](F,[\{P,G''\}-,\{P,F*G'\}]. \end{array}$$

Through this sequence of isomorphisms the natural in k

$$Pk \xrightarrow{\eta} [A,\mathcal{V}](\{P,G''\},G''k) \xrightarrow{F*-} [F*\{P,G''\},(F*G')k]$$

corresponds to the natural in a, k

$$Fa\otimes Pk \xrightarrow{\lambda_{k,a}\otimes\eta_{k,a}} [G(k,a),F*G''k]\otimes[\{P,G'a\},G(k,a)] \xrightarrow{\mu} [\{P,G'a\},F*G''k]$$

where μ is the composition in $\mathcal{V}$, the latter one corresponds to the natural in a,

$$Fa \xrightarrow{\lambda} [K,\mathcal{V}](G'a,F*G') \xrightarrow{\{P,-\}} [\{P,G'a\},\{P,F*G'\}]\ .$$

∎

According to 2.7,

2.8 *a presheaf $F: A^{op}\to\mathcal{V}$ is $\mathcal{P}_0$-flat if and only if for any $G: A\to[K,\mathcal{V}]$ the colimit $F*G$ is preserved by any representable $[K,\mathcal{V}]\to\mathcal{V}$.*

To prove 2.6 we will need to use a bit of the 2-categorical machinery developed in [StWa78], in particular the description of indexed colimits in terms of right liftings. It is proved in [Str83]that

Theorem 2.9 *A module $\theta:\ A \longrightarrow\!\!\!\circ\!\!\!\rightarrow B$ is left adjoint if and only if any colimit indexed by θ is absolute.*

So 2.9 and 2.8 give immediately (2) $\Rightarrow$ (1) in 2.6. Now a minor adaptation of the proof presented in [Str83] will show (1) $\Rightarrow$ (2) in 2.6.

Proposition 2.10 *A presheaf $F: A^{op}\to\mathcal{V}$ is a left adjoint as a module $I \longrightarrow\!\!\!\circ\!\!\!\rightarrow A$ if the colimit $F\cong F*Y$ is preserved by any representable $[A^{op},\mathcal{V}]\to\mathcal{V}$.*

PROOF:That $F \cong F * Y$ amounts to saying that there is a right lifting of $\mathcal{V}$-modules as below:

$$(*)\qquad \begin{tikzcd} & {[A^{op}, \mathcal{V}]} \arrow[dl] \arrow[dr, "Y^*"] \arrow[d, phantom, "\Rightarrow"] & \\ I \arrow[rr, "F"'] & {} & A \end{tikzcd}$$

where

- the unlabeled diagonal is the right adjoint module δ^* given by the functor $\delta : I \to [A^{op}, \mathcal{V}]$ that sends the one point to the presheaf F;
- the horizontal arrow, denoted F, is the module $I \to A$ corresponding to the presheaf F.

Recall that any left adjoint module respects right liftings. So by pasting $Y_* : A \to [A^{op}, \mathcal{V}]$ to the 2-cell $(*)$ and since $Y^* \circ Y_* = 1$ one obtains a right lifting

$$(**)\qquad \begin{tikzcd} & A \arrow[dl] \arrow[dr, "1"] \arrow[d, phantom, "\Rightarrow"] & \\ I \arrow[rr, "F"'] & {} & A \end{tikzcd}$$

That the left diagonal constitutes a right adjoint to the module F, is equivalent to say that this right lifting is absolute. Consider any module $\theta : B \to [A^{op}, \mathcal{V}]$. It may be decomposed in a product $Y^* h_*$ for the functor $Y : [A^{op}, \mathcal{V}] \to P[A^{op}, V]$ and a functor $h : B \to [A^{op}, \mathcal{V}]$ (sending any b to $\theta(-, b)$). By assumption the colimit $F \cong F * Y$ is preserved by any representable presheaf $[A^{op}, \mathcal{V}] \to \mathcal{V}$, thus because colimits in $P[A^{op}, \mathcal{V}]$ are "pointwise" ([AK88]), $Y : [A^{op}, \mathcal{V}] \to P[A^{op}, \mathcal{V}]$ also preserves the colimits indexed by F. That is to say that Y^* respects the right lifting $(*)$. $\theta = h_* Y^*$ also respects $(*)$ (since h_* is left adjoint). So $(*)$ is absolute as well as $(**)$. ∎

Let us mention another consequence of 2.7.

Definition 2.11 (Q -coflatness) *Given a family $\mathcal{Q}$ of indexes, a presheaf $P : K \to \mathcal{V}$ is said* $\mathcal{Q}$-coflat *when $\{P, -\} : [K, \mathcal{V}] \to \mathcal{V}$ preserves $\mathcal{Q}$-colimits. Let $Coflat_{\mathcal{Q}}$ denote the family of all $\mathcal{Q}$-coflat presheaves.*

Consider the class $CPSh$ of classes of presheaves. Then the classes of presheaves are partially ordered by inclusion and one has a Galois connection

$$Flat_- \dashv Coflat_- : CPSh \rightharpoonup CPSh^{op}.$$

We shall investigate in this paper two more notions of flatness. Let us define

Definition 2.12 *$\mathcal{P}_1$ is the class of indexes of the form $F : K \to \mathcal{V}$ where K is the empty $\mathcal{V}$-category or $K = I$. $\mathcal{P}_2$ is the class of indexes $F : K \to \mathcal{V}$ with $Obj(K)$ finite.*

We shall call *conical finite limit* a conical limit indexed by a finite ordinary category. From now on we write A_0 for the underlying ordinary category of a $\mathcal{V}$-category A. A minor adaptation of the proof of theorem [Kel82] (3.73) as in [Kel82-2] (4.3), shows that

Proposition 2.13 *A $\mathcal{V}$-category A is $\mathcal{P}_2$-complete if and only if it has all conical finite limits and cotensors. Given a $\mathcal{P}_2$-complete A, a $\mathcal{V}$-functor $P : A \to B$ is $\mathcal{P}_2$-exact if only if it preserves conical finite limits and cotensors.*

PROOF:(Sketch of) It suffices to reuse the argument developed in the sketch of proof of [Kel82-2] (4.3). Remark that if A has conical finite limits and cotensors, the indexed limit $\{F, G\}$ of any $F : K \to \mathcal{V}$ and $G : K \to A$ with $Obj(K)$ finite may be computed as the equalizer in A_0 of

$$\prod_{k\in K} Fk \pitchfork Gk \rightrightarrows \prod_{k,k'\in K} K(k,k') \pitchfork (Fk \pitchfork Gk') \ .$$

Actually all the ordinary limits involved in this equalizer, i.e. the two products and the equalizer itself are finite and thus conical. Also revisiting the sketched proof of theorem (3.73) in [Kel82], one gets that any functor $H : A \to B$ preserving conical finite limits and cotensors, H will preserve the above conical equalizer which image in B is then the limit $\{F, HG\}$. ∎

Proposition 2.14 *Given a $\mathcal{P}$-flat presheaf $F : A \to \mathcal{V}$, and any functor $G : A \to B$, the left Kan extension of F along G is $\mathcal{P}$-flat.*

PROOF:Given a $\mathcal{P}$-flat F, it is a $Flat_{\mathcal{P}}$-colimit of representables. The image by Lan_G of any representable is again representable (For any $a \in A$, $Lan_G(A(a,-))(b) \cong_b B(G-,b) * A(a,-) \cong_b B(Ga,b)$). Also the left Kan extension functor $Lan_G : [A,\mathcal{V}] \to [B,\mathcal{V}]$ is cocontinuous as shown below 2.15, so $Lan_G(F)$ is also a $Flat_{\mathcal{P}}$-colimit of representables and thus $\mathcal{P}$-flat according to 2.3. ∎

Lemma 2.15 *Given any functor $G : A \to B$, the left Kan extension functor $Lan_G : [A,\mathcal{V}] \to [B,\mathcal{V}]$ is cocontinuous.*

Given $J : K^{op} \to \mathcal{V}$ and $H : K \to [A,\mathcal{V}]$, one has the following pointwise computation in $b \in B$,

$$\begin{aligned} Lan_G(J*H)(b) &\cong \tilde{G}b * (J*H), \\ &\cong (J*H) * \tilde{G}b \\ &\cong J * (H - * \tilde{G}b) \\ &\cong J * (\tilde{G}b * H-) \\ &\cong J * (Lan_G(H-)(b)) \\ &\cong (J * Lan_G(H-))(b). \end{aligned}$$

Actually the resulting natural isomorphism exhibits $Lan_G(J*H)$ as the colimit $J * Lan_G(H-)$. $Lan_G : [A,\mathcal{V}] \to [B,\mathcal{V}]$ preserves $J*H$ if and only if for any $b \in B$, $E_b \circ Lan_G : [A,\mathcal{V}] \to \mathcal{V}$ does. But $E_b \circ Lan_G \cong Lan_G(-)(b) \cong \tilde{G}(b) * - = - * \tilde{G}(b)$ that is known cocontinuous. ∎

The rest of the paper treats notions of flatness for enrichments over particular bases namely $\mathcal{V} = \mathbf{Bool}$ and $\mathcal{V} = \bar{\mathbb{R}}_+$. An important point to make is that for both these cases the base $\mathcal{V}$ is small and thus is necessarily a preorder (see [Bor94] prop. 2.7.1 p.59). In the case of a small $\mathcal{V}$, for any small $\mathcal{V}$-category A, the presheaf category $[A,\mathcal{V}]$ remains small and so does $Flat_{\mathcal{P}}(A)$ for any family $\mathcal{P}$ of presheaves. Still in this case, if A is $\mathcal{P}$-complete then it is a retract of $Flat_{\mathcal{P}}(A)$ (i.e. the inclusion $A \hookrightarrow Flat_{\mathcal{P}}(A)$ is a split monic) but it is generally NOT isomorphic to A.

3 The case $\mathcal{V} = \bar{\mathbb{R}}_+$.

This section treats flatness in the context of general metric spaces. First we come back quickly in 3.1 on Lawvere's Cauchy-completion of general metric spaces. In 3.2, the existing correspondence between Cauchy filters and left adjoint modules is extended: the ordinary category of $Flat_{\mathcal{P}_1}$-modules is reflective in a category of particular filters, the so called *weakly-flat* ones, with reverse inclusion ordering. By considering the category of fractions induced by this full reflection one defines a notion of morphisms of weakly flat filters that yields an enriched equivalence with the categories of $Flat_{\mathcal{P}_1}$-modules. This equivalence restricts to the category of $Flat_{\mathcal{P}_2}$-modules on one side and on the other side to a full subcategory of filters, the so-called *flat* ones. Also in the symmetric case flat filters are Cauchy and one retrieves via the latter equivalence the well known one-to-one correspondence between Cauchy filters and left adjoint modules. Weakly flat and flat filters are then related to forward Cauchy sequences. These sequences were introduced in the literature as a generalization of Cauchy sequences [Sun95]. They are relevant as with the right notion of morphisms, both weakly flat and flat filters occur as canonical colimits of functors with values these forward Cauchy sequences. In 3.3, the $\mathcal{P}_1$- and $\mathcal{P}_2$-cocompletions of general metric spaces are defined and "internally" described in pure metric/topological terms by means of the previous filters. A few examples of these completions follow.

3.1 Lawvere's completion

Let us recall a few results that are from [Law73] or belong to folklore.

$\bar{I\!R}_+$ stands for the monoidal closed category with:

- objects: positive reals and $+\infty$;
- arrows: the reverse ordering, $x \to y$ if and only if $x \geq y$;
- tensor: the addition (with $+\infty + x = x + +\infty = +\infty$);
- unit: 0.

For any pair x, y of objects in $\bar{I\!R}_+$, the exponential object $[x, y]$ is $max\{y - x, 0\}$.

A $\bar{I\!R}_+$*-category* A corresponds to a *general metric space*. It consists in a set of *objects* or *elements*, $Obj(A)$ (sometimes just denoted A) together with a map $A(-,-) : Obj(A) \times Obj(A) \to \bar{I\!R}_+$ that satisfies:

- for all $x, y, z \in Obj(A)$, $A(y, z) + A(x, y) \geq A(x, z)$;
- for all $x \in Obj(A)$, $0 \geq A(x, x)$.

A $\bar{I\!R}_+$*-functor* $F : A \to B$ corresponds to a non-expansive map $F : Obj(A) \to Obj(B)$, i.e. for all $x, y \in Obj(A)$, $A(x, y) \geq B(F(x), F(y))$. A $\bar{I\!R}_+$*-natural transformation* $F \Rightarrow G : A \to B$ corresponds to the fact that for all $x \in Obj(A)$, $0 \geq B(F(x), G(x))$. A $\bar{I\!R}_+$-module $M : I \multimap A$ - or left module on A - is a map $Obj(A) \to \bar{I\!R}_+$ such that for all $x, y \in A$, $M(y) + A(x, y) \geq M(x)$. Dually a $\bar{I\!R}_+$-module $N : A \multimap I$ - or right module on A - is a map $Obj(A) \to \bar{I\!R}_+$ such that for all $x, y \in A$, $A(x, y) + N(x) \geq N(y)$. The presheaf category $[A^{op}, \bar{I\!R}_+]$ has homsets given by $[A^{op}, \bar{I\!R}_+](M, N) = \bigvee_{x \in A}[M(x), N(x)]$. Its underlying category is a partial order with arrows given by the pointwise reverse ordering $M \Rightarrow N$ if and only if $\forall x \in A$, $M(x) \geq N(x)$. The composition of left and right modules is as follows. Given $I \overset{M}{\multimap} A \overset{N}{\multimap} I$, the composite $N * M$ is $\bigwedge_{x \in A} M(x) + N(x)$. For such M and N, M is left adjoint to N if and only if:

- (1) $0 \geq N * M$;
- (2) for all $x, y \in A$, $N(y) + M(x) \geq A(x, y)$.

The key point for the Cauchy-completion of general metric spaces is that for a general metric space A there is a one-to-one correspondence between left adjoint modules on A and minimal Cauchy filters on A. From this observation mainly, one gets that the full subcategory of $[A^{op}, \bar{I\!R}_+]$ with objects left adjoint modules is isomorphic the completion "à la Cauchy" of A, that is its Cauchy-completion if A is a metric space or more generally its bi-completion if the space is not symmetric (see [FL82] and [Fla92] or [Sch03] for the connection with Lawvere's work). As this is the starting point of our investigation, we recall briefly this correspondence.

Let A stand for a general metric space.

Definition 3.1 *A filter $\mathcal{F}$ on A is* Cauchy *if and only if for any $\epsilon > 0$, there exists an $f \in \mathcal{F}$ such that for any elements x, y of f, $A(x,y) \leq \epsilon$ or equivalently when:*

$$\bigwedge_{f\in\mathcal{F}} \bigvee_{x,y\in f} A(x,y) = 0.$$

Definition 3.2 *For any left adjoint module M on A, with right adjoint $\tilde{M}$ one defines $\Gamma^s(M)$ as the subset of $\wp(A)$: $\{\Gamma^s(M)(\epsilon) \mid \epsilon \in]0, +\infty]\}$, where $\Gamma^s(M)(\epsilon)$ denotes the set $\{x \in A \mid M(x) + \tilde{M}(x) \leq \epsilon\}$.*

For any left adjoint module M on A, $\Gamma^s(M)$ is a Cauchy basis. The filter that it generates, that we denote $\mathcal{F}^s(M)$, is a minimal Cauchy filter. The map $M \mapsto \mathcal{F}^s(M)$ defines a bijection between left adjoint modules $I \longrightarrow\!\!\!\!\!\circ\;\; A$ and minimal Cauchy filters on A. Actually one may check the following points (see for example [Sch03]). To any Cauchy filter $\mathcal{F}$ one may associate a left adjoint module $M^l(\mathcal{F})$ defined by

$$x \mapsto \bigwedge_{f\in\mathcal{F}} \bigvee_{y\in f} A(x,y) = \bigvee_{f\in\mathcal{F}} \bigwedge_{y\in f} A(x,y).$$

$M^l(\mathcal{F})$ has right adjoint $M^r(\mathcal{F})$ given by the map

$$x \mapsto \bigwedge_{f\in\mathcal{F}} \bigvee_{y\in f} A(y,x) = \bigvee_{f\in\mathcal{F}} \bigwedge_{y\in f} A(y,x).$$

For any left adjoint module M on A, $M^l(\mathcal{F}^s(M)) = M$ and for any Cauchy filter $\mathcal{F}$ on A, $\mathcal{F}^s(M^l(\mathcal{F}))$ is the only minimal Cauchy filter contained in $\mathcal{F}$.

Note to the referee - to be omitted
For what it is worth. It is well known that any Cauchy filter contains only one minimal Cauchy filter. But I don't know from the literature - apart from [Sch03] - any explicit proof that for any Cauchy filter $\mathcal{F}$ on A, $\mathcal{F}^s(M^l(\mathcal{F}))$ is the only minimal Cauchy filter contained in $\mathcal{F}$. So here are two key points to retrieve quickly that result once you suppose that for all left adjoint module M, $M = M^l \circ \mathcal{F}^s(M)$.
(1) For any $\mathcal{F}$ Cauchy, one may check that $\mathcal{F} \supseteq \mathcal{F}^s(M^l(\mathcal{F}))$ using the definition $M^l(\mathcal{F})(x) = \bigwedge_{f\in\mathcal{F}} \bigvee_{y\in f} A(x,y)$.
(2) Also for Cauchy filters $\mathcal{F}_1$ and $\mathcal{F}_2$, if $\mathcal{F}_1 \supseteq \mathcal{F}_2$ then $M^l(\mathcal{F}_2) \Rightarrow M^l(\mathcal{F}_1)$ by using the definition $M^l(\mathcal{F})(x) = \bigwedge_{f\in\mathcal{F}} \bigvee_{y\in f} A(x,y)$ and $M^l(\mathcal{F}_1) \Rightarrow M^l(\mathcal{F}_2)$ by using the definition $\bigvee_{f\in\mathcal{F}} \bigwedge_{y\in f} A(x,y)$, so eventually $M^l(\mathcal{F}_1) = M^l(\mathcal{F}_2)$.

3.2 Modules and Filters

A natural question is whether the previous correspondence left adjoint modules / Cauchy filters may be extended to a class of $\mathcal{P}$-flat modules. We shall show that this is the case for $\mathcal{P} = \mathcal{P}_1$ and $\mathcal{P}_2$.

Let A denote from now on a general metric space.

Let us give an explicit definition of those $\mathcal{P}_1$-flat and $\mathcal{P}_2$-flat modules. We shall recall first a few technical points. For the assertions 3.3, 3.4 and 3.5 below, $\mathcal{V}$ denotes a complete monoidal closed $\mathcal{V}$. Remember that cotensors are defined pointwise in functor categories. In particular

3.3 *Any presheaf $\mathcal{V}$-category $[A,\mathcal{V}]$ is cotensored: for any presheaf N, $v \pitchfork N$ is the composite* $A \xrightarrow{N} \mathcal{V} \xrightarrow{[v,-]} \mathcal{V}$.

Also for functor between cocomplete categories the preservation of conical colimits amounts to the preservation of ordinary colimits. Precisely one may check:

3.4 *Given a $\mathcal{V}$-functor $T : A \to B$ with underlying ordinary functor $T_0 : A_0 \to B_0$ and an ordinary functor $P : J \to A_0$, if the conical limits of P and of T_0P exist and T_0 preserves the ordinary limit of P, then T preserves the conical limit of P.*

Eventually the preservation of limits/colimits is simple in the case $\mathcal{V} = \bar{\mathbb{R}}_+$ since

3.5 *If the base category $\mathcal{V}$ is a preorder, then given a presheaf $F : A^{op} \to \mathcal{V}$ and a functor $G : A \to B$ such that $F * G$ exists and $H : B \to C$ then H preserves $F * G$ if and only if $F * (GH)$ exists and $H(F * G) \cong F * (GH)$.*

According to the three previous point one gets

3.6 *Let $M :\ I \longrightarrow\!\!\circ\!\!\!\!\rightarrow A$ be a left module.*

- *$- * M : [A, \bar{\mathbb{R}}_+] \to \bar{\mathbb{R}}_+$ preserves the unique conical limit with index with empty domain if and only if the underlying ordinary functor preserves the terminal object i.e. $0 * M = 0$ if and only if*

$$(1) \quad \bigwedge_{x\in A} M(x) = 0.$$

- *$- * M$ preserves conical finite limits if and only*
  (2) For any finite family of right modules $N_i :\ A \longrightarrow\!\!\circ\!\!\!\!\rightarrow I$, $i \in I$,

$$\bigwedge_{x\in A} (M(x) + \bigvee_{i\in I} N_i(x)) = \bigvee_{i\in I} (\bigwedge_{x\in A} M(x) + N_i(x));$$

- *$- * M$ preserves cotensors if and only if*
  (3) For any $v \in \bar{\mathbb{R}}_+$ and any right module $N :\ A \longrightarrow\!\!\circ\!\!\!\!\rightarrow I$,

$$\bigwedge_{x\in A} (M(x) + [v, N(x)]) = [v, \bigwedge_{x\in A} (M(x) + N(x))].$$

So $\mathcal{P}_1$-flat modules are those satisfying (1) and (3) above, and $\mathcal{P}_2$-flat modules are those satisfying (2) and (3).

It is convenient to introduce now the following notations.

Definition 3.7 *Given a filter $\mathcal{F}$ on A and a map $f : Obj(A) \to Obj(\bar{\mathbb{R}}_+)$, $lim^+_{x\in\mathcal{F}} f(x)$ or simply $lim^+_{\mathcal{F}} f$ denotes $\bigwedge_{f\in\mathcal{F}} \bigvee_{x\in f} f(x)$. Also $lim^-_{x\in\mathcal{F}} f(x)$ or $lim^-_{\mathcal{F}} f$ will stand for $\bigvee_{f\in\mathcal{F}} \bigwedge_{x\in f} f(x)$.*

From the correspondence Cauchy filters/left adjoint modules, we know two operators that associate filters to modules.

Definition 3.8 *Given any filter $\mathcal{F}$ on A, we define the following $\bar{\mathbb{R}}_+$-valued maps on objects of A:*

$$M^-(\mathcal{F}) : x \mapsto lim^-_{\mathcal{F}} A(x,-) = \textstyle\bigvee_{f\in\mathcal{F}} \bigwedge_{y\in f} A(x,y),$$
$$M^+(\mathcal{F}) : x \mapsto lim^+_{\mathcal{F}} A(x,-) = \textstyle\bigwedge_{f\in\mathcal{F}} \bigvee_{y\in f} A(x,y).$$

For any filter $\mathcal{F}$ on A, one has $M^-(\mathcal{F}) \leq M^+(\mathcal{F})$, and if $\mathcal{F}$ is Cauchy then $M^-(\mathcal{F}) = M^+(\mathcal{F})$.

3.9 *Given any filter $\mathcal{F}$ on A, the map $M^+(\mathcal{F})$ defines a module $I \longrightarrow\!\!\!\circ\!\!\!\rightarrow A$.*

PROOF:One has to show that for all $x, y \in A$, $M^+(\mathcal{F})(x) + A(y,x) \geq M^+(\mathcal{F})(y)$. For all $x, y \in A$,

$$\begin{aligned}
M^+(\mathcal{F})(x) + A(y,x) &= (\bigwedge_{f\in\mathcal{F}} \bigvee_{z\in f} A(x,z)) + A(y,x) \\
&= \bigwedge_{f\in\mathcal{F}}((\bigvee_{z\in f} A(x,z)) + A(y,x)) \\
&\geq \bigwedge_{f\in\mathcal{F}} \bigvee_{z\in f}(A(x,z) + A(y,x)) \\
&\geq \bigwedge_{f\in\mathcal{F}} \bigvee_{z\in f} A(y,z) \\
&= M^+(\mathcal{F})(y).
\end{aligned}$$

∎

3.10 *Given any filter $\mathcal{F}$ on A, the map $x \mapsto M^-(\mathcal{F})(x)$ defines a module $I \longrightarrow\!\!\!\circ\!\!\!\rightarrow A$.*

PROOF:One has to show that for all $x, y \in A$, $M^-(\mathcal{F})(x) + A(y,x) \geq M^-(\mathcal{F})(y)$. For all $x, y \in A$,

$$\begin{aligned}
M^-(\mathcal{F})(x) + A(y,x) &= (\bigvee_{f\in\mathcal{F}} \bigwedge_{z\in f} A(x,z)) + A(y,x) \\
&\geq \bigvee_{f\in\mathcal{F}}((\bigwedge_{z\in f} A(x,z)) + A(y,x)) \\
&= \bigvee_{f\in\mathcal{F}} \bigwedge_{z\in f}(A(x,z) + A(y,x)) \\
&\geq \bigvee_{f\in\mathcal{F}} \bigwedge_{z\in f} A(y,z) \\
&= M^-(\mathcal{F})(y).
\end{aligned}$$

∎

Let us define

Definition 3.11 *A filter $\mathcal{F}$ on A is* weakly flat *if and only if*

$$lim^+_{\mathcal{F}} M^-(\mathcal{F}) = 0.$$

The previous definition may be interpreted as a generalization to non-symmetric spaces of the idea that the diameter of the elements of the filter may be chosen arbitrary small. Let us rephrase this definition. A filter $\mathcal{F}$ on A is weakly flat if and only if for any $\epsilon > 0$, there exists an $f \in \mathcal{F}$ such that for any element x of f, for any $g \in \mathcal{F}$, there exists $y \in g$ such that $A(x,y) \leq \epsilon$.

We shall introduce also the following filters whose relevance will appear later.

Definition 3.12 *A filter $\mathcal{F}$ on A is* flat *if and only if for any $\epsilon > 0$, there exists an $f \in \mathcal{F}$ such that for any finite family of elements $(x_i)_{i\in I}$ of f, for any $g \in \mathcal{F}$, there exists $y \in g$ such that $A(x_i, y) \leq \epsilon$.*

A few remarks are in order.

One has the inclusion of classes of filters:

$$\text{Cauchy} \Rightarrow \text{flat} \Rightarrow \text{weakly flat}.$$

If the space A is symmetric, that is when $A(x,y) = A(y,x)$, then any flat filters on A is also Cauchy. We shall see later 3.33 a few consequences of this fact.

Also one might think to consider the filters $\mathcal{F}$ satisfying

3.13
$$lim^+_{\mathcal{F}} M^+(\mathcal{F}) = 0.$$

These filters are actually useless for the study of non-symmetric spaces as for the obvious example non-symmetric space the base $\bar{I\!R}_+$ itself, they do not correspond to "oriented neighborhoods". Consider any real x and define its neighborhood filter as generated by the family $\{y \mid [y,x] \leq \epsilon\}$, where $\epsilon > 0$. This filter does not satisfy 3.13. For the same reason it will occur that the class of filters defined by 3.13 cannot generally correspond to any class of modules containing the representable presheaves. We have therefore focussed on the weakly flat filters and the associated operator M^-.

The operator sending modules to the associated filters seems obvious. With $\wp(X)$ denoting the powerset of X with inclusion ordering.

Definition 3.14 *For any module M, Let $\Gamma(M)$ denote the subset of $\wp(A)$, $\{\Gamma(M)(\epsilon) \mid \epsilon \in]0,+\infty]\}$, where $\Gamma(M)(\epsilon)$ denotes the set $\{x \in A \mid M(x) \leq \epsilon\}$. Let also $\mathcal{F}(M)$ denote the upper closure in $\wp\wp(A)$ of $\Gamma(M)$.*

We are ready to establish correspondences between various ($\bar{I\!R}_+$- !) categories of modules and filters as well as a few other "•" points. Let $WFil(A)$ and $FFil(A)$ will stand for the ordinary categories with objects respectively weakly flat filters, and flat filters on A, both with reverse inclusion ordering. We are going to prove

Theorem 3.15 *The map $\mathcal{F} \mapsto M^-(\mathcal{F})$ determines reflectors $WFil(A) \to Flat_{\mathcal{P}_1}(A)_0$ and $FFil(A) \to Flat_{\mathcal{P}_2}(A)_0$. Their respective right adjoints $Flat_{\mathcal{P}_1}(A)_0 \hookrightarrow WFil(A)$ and $Flat_{\mathcal{P}_2}(A)_0 \hookrightarrow FFil(A)$ send any module M to the filter $\mathcal{F}(M)$ with basis $\Gamma(M)$. They are full.*

Moreover the inclusions $Flat_{\mathcal{P}_2}(A)_0 \hookrightarrow Flat_{\mathcal{P}_1}(A)_0$ and $FFil(A) \hookrightarrow WFil(A)$ are maps between the above adjunctions. This reflection will yield a notion of morphisms between weakly flat filters more general than the inclusion ordering. We shall later consider the associated category of fractions $WFil^*(A)$ that is equivalent to $Flat_{\mathcal{P}_1}(A)_0$. In this category weakly flat filters and flat filters will be defined then in terms of colimits of the so-called *forward Cauchy sequences* (3.35).

- *$Flat_{\mathcal{P}_1}$-modules and weakly flat filters.*

We shall establish the reflection $M^- \dashv \mathcal{F} : WFil(A) \leftrightarrow Flat_{\mathcal{P}_1}(A)_0$ as well as a couple of results regarding weakly flat filters. This full reflection results from 3.16, 3.17, 3.18, 3.19, 3.23, 3.26, 3.27 and 3.28 below.

3.16 *For any modules M_1 and M_2 on A, if $M_1 \Rightarrow M_2$ then $\mathcal{F}(M_1) \supseteq \mathcal{F}(M_2)$.*

3.17 *For any filters $\mathcal{F}_1$ and $\mathcal{F}_2$ on A, if $\mathcal{F}_1 \supseteq \mathcal{F}_2$ then $M^-(\mathcal{F}_1) \Rightarrow M^-(\mathcal{F}_2)$.*

Proposition 3.18 *Let M be a left module on A then for all x, $M(x) \leq \bigvee_{\epsilon>0} \bigwedge_{y|M(y)\leq\epsilon} A(x,y)$, i.e. $M \Leftarrow M^- \circ \mathcal{F}(M)$.*

PROOF:Let $x \in A$. For all $y \in A$, $M(x) \leq M(y) + A(x,y)$ thus for all $y \in A$, such that $M(y) \leq \alpha$, $M(x) \leq A(x,y) + \alpha$ and $M(x) \leq \bigwedge_{y|M(y)\leq\alpha} A(x,y) + \alpha$. Consider $\epsilon > 0$. The map $\alpha \mapsto \bigwedge_{y|M(y)\leq\alpha} A(x,y)$ reverses the order so $\bigwedge_{y|M(y)\leq\epsilon} A(x,y) = \bigvee_{\alpha\geq\epsilon} \bigwedge_{y|M(y)\leq\alpha} A(x,y)$ and $(*)$ $M(x) \leq \bigvee_{\alpha\geq\epsilon} \bigwedge_{y|M(y)\leq\alpha} A(x,y) + \epsilon$. Also for any $\alpha \leq \epsilon$,

$$\begin{aligned} M(x) &\leq (\textstyle\bigwedge_{y|M(y)\leq\alpha} A(x,y)) + \alpha \\ &\leq (\textstyle\bigwedge_{y|M(y)\leq\alpha} A(x,y)) + \epsilon \end{aligned}$$

and thus $(**)$ $M(x) \leq \bigvee_{\alpha \leq \epsilon} \bigwedge_{y|M(y)\leq\alpha} A(x,y)+\epsilon$. $(*)$ and $(**)$ give $M(x) \leq \bigvee_{\alpha>0} \bigwedge_{y|M(y)\leq\alpha} A(x,y)+\epsilon$.
∎

Proposition 3.19 *A filter $\mathcal{F}$ on A is weakly flat if and only if $\mathcal{F} \supseteq \mathcal{F} \circ M^{-}(\mathcal{F})$.*

PROOF:One has the successive equivalences.

$\mathcal{F}$ is weakly flat
if and only if
$\bigwedge_{f\in\mathcal{F}} \bigvee_{x\in f} M^{-}(\mathcal{F}) = 0$
if and only if
for all $\epsilon > 0$, there exists $f \in \mathcal{F}$ such that for all $x \in f$, $M^{-}(\mathcal{F})(x) \leq \epsilon$,
if and only if
for all $\epsilon > 0$, there exists $f \in \mathcal{F}$ such that $f \subseteq \Gamma(M^{-}(\mathcal{F}))(\epsilon)$
if and only if
$\mathcal{F} \supseteq \mathcal{F} \circ M^{-}(\mathcal{F})$.

∎

Let us note at this stage

Proposition 3.20 *For any weakly flat filter $\mathcal{F}$ on A and any left module M on A, $\mathcal{F} \supseteq \mathcal{F}(M)$ if and only if $M^{-}(\mathcal{F}) \Rightarrow M$.*

PROOF:If $\mathcal{F} \supseteq \mathcal{F}(M)$ then $M^{-}(\mathcal{F}) \Rightarrow M^{-} \circ \mathcal{F}(M) \Rightarrow M$ by 3.17 and 3.18. Conversely if $M^{-}(\mathcal{F}) \Rightarrow M$ then $\mathcal{F} \supseteq \mathcal{F} \circ M^{-}(\mathcal{F}) \supseteq \mathcal{F}(M)$ by 3.16 and 3.19. ∎

Proposition 3.21 *For any right module $N:\ A \multimap I$, and any filter $\mathcal{F}$ in A,*

$$N * M^{-}(\mathcal{F}) \geq lim^{-}_{\mathcal{F}} N$$

If $\mathcal{F}$ is moreover weakly flat then the previous inequality becomes an equality.

PROOF:For any module N and any filter $\mathcal{F}$ as above,

$$\begin{aligned}
N * M^{-}(\mathcal{F}) &= \textstyle\bigwedge_{x\in A}(M^{-}(\mathcal{F})(x) + N(x)) \\
&= \textstyle\bigwedge_{x\in A}(\bigvee_{f\in\mathcal{F}} \bigwedge_{y\in f} A(x,y)) + N(x)) \\
&\geq \textstyle\bigwedge_{x\in A} \bigvee_{f\in\mathcal{F}}((\bigwedge_{y\in f} A(x,y)) + N(x)) \\
&= \textstyle\bigwedge_{x\in A} \bigvee_{f\in\mathcal{F}} \bigwedge_{y\in f}(A(x,y) + N(x)) \\
&\geq \textstyle\bigvee_{f\in\mathcal{F}}(\bigwedge_{y\in f} N(y)).
\end{aligned}$$

Let us suppose moreover that $\mathcal{F}$ is weakly flat. Let $\epsilon > 0$. One may choose $f_\epsilon \in \mathcal{F}$ such that when $x \in f_\epsilon$, $M^{-}(\mathcal{F})(x) \leq \epsilon$. Thus

$$\begin{aligned}
N * M^{-}(\mathcal{F}) &= \textstyle\bigwedge_{x\in A}(M^{-}(\mathcal{F})(x) + N(x)) \\
&\leq M^{-}(\mathcal{F})(x) + N(x), \text{ for any } x \in f_\epsilon \\
&\leq \epsilon + N(x), \text{ for any } x \in f_\epsilon.
\end{aligned}$$

Thus

$$\begin{aligned}
N * M^{-}(\mathcal{F}) &\leq \textstyle\bigwedge_{x\in f_\epsilon}(\epsilon + N(x)) \\
&= \epsilon + \textstyle\bigwedge_{x\in f_\epsilon} N(x) \\
&\leq \epsilon + \textstyle\bigvee_{f\in\mathcal{F}} \bigwedge_{x\in f} N(x).
\end{aligned}$$

■

3.22 *For any module $M : I \multimap A$ if $- * M : [A, \bar{I\!R}_+] \to \bar{I\!R}_+$ preserves the terminal object (i.e. $\bigwedge_{x \in A} M(x) = 0$) then $\mathcal{F}(M)$ is a filter on A with basis the family $\Gamma(M)$.*

PROOF:Let us see first that the set of subsets of the form $\Gamma(M)(\epsilon)$ for $\epsilon > 0$, is a filter basis on A. $\Gamma(M)$ is trivially cofiltered subset of $\wp(A)$ ordered by inclusion. Since $\bigwedge_{x \in A} M(x) = 0$, for any $\epsilon > 0$ there is one x with $M(x) \leq \epsilon$, i.e. $\Gamma(M)(\epsilon) \neq \emptyset$. ■

As a consequence of 3.22 and 3.24 below, one gets

Corollary 3.23 *For any module $M : I \multimap A$ if M is $\mathcal{P}_1$-flat then $\mathcal{F}(M)$ is weakly flat.*

Lemma 3.24 *If $M : I \multimap A$ is $\mathcal{P}_1$-flat then for any $\epsilon > 0$ and any x with $M(x) < \epsilon$ and any $\alpha > 0$, there is a y such $M(y) \leq \alpha$ and $A(x, y) \leq \epsilon$.*

PROOF:$- * M$ preserves cotensors and the (conical) terminal object. Consider $\epsilon > 0$ and x with $M(x) < \epsilon$. Then

$$\begin{aligned} 0 &= [M(x), M(x)] \\ &= [M(x), A(x, -) * M] \\ &= (M(x) \pitchfork A(x, -)) * M \\ &= \bigwedge\nolimits_{y \in A}(M(y) + [M(x), A(x, y)]). \end{aligned}$$

So for any $\delta > 0$, there is an y such that $M(y) + [M(x), A(x, y)] \leq \delta$. This y satisfies $M(y) \leq \delta$, and $A(x, y) \leq M(x) + \delta$. Now given any $\alpha > 0$, considering $\delta = min\{\alpha, \epsilon - M(x)\}$ one may find a y as required. ■

$\bar{I\!R}_+$ has a very peculiar property that we are going to use.

3.25 *For any v in $\bar{I\!R}_+$ and any non empty family $(a_i)_{i \in I}$ in $\bar{I\!R}_+$, $[v, \bigwedge_{i \in I} a_i] = \bigwedge_{i \in I}[v, a_i]$.*

PROOF:Since $[v, -]$ preserves the usual ordering on $\bar{I\!R}_+$, $[v, \bigwedge_{i \in I} a_i] \leq \bigwedge_{i \in I}[v, a_i]$ (even if I is empty). Conversely, fix $\epsilon > 0$. Since I is not empty, there exists $j \in I$ such that $\epsilon + \bigwedge_{i \in I} a_i \geq a_j$. Also $[v, \bigwedge_{i \in I} a_i] \geq [v, \bigwedge_{i \in I} a_i]$, so $v + [v, \bigwedge_{i \in I} a_i] \geq \bigwedge_{i \in I} a_i$. For a j as above, $\epsilon + v + [v, \bigwedge_{i \in I} a_i] \geq a_j$ and $\epsilon + [v, \bigwedge_{i \in I} a_i] \geq [v, a_j] \geq \bigwedge_{i \in I}[v, a_i]$. ■

3.26 *If $\mathcal{F}$ is a weakly flat filter on A then $- * M^-(\mathcal{F})$ preserves cotensors.*

PROOF:Given $v \in \bar{I\!R}_+$ and $N : A \multimap I$ we have to show $(v \pitchfork N) * M^-(\mathcal{F}) = [v, N * M^-(\mathcal{F})]$. According to 3.21, $(v \pitchfork N) * M^-(\mathcal{F}) = \bigvee_{f \in \mathcal{F}} \bigwedge_{x \in f}[v, N(x)]$ and $[v, N * M^-(\mathcal{F})] = [v, \bigvee_{f \in \mathcal{F}} \bigwedge_{x \in f} N(x)] = \bigvee_{f \in \mathcal{F}}[v, \bigwedge_{x \in f} N(x)]$. Since all the $f \in \mathcal{F}$ are non empty the result follows then from 3.25. ■

Proposition 3.27 *If $\mathcal{F}$ is a weakly flat filter on A then $- * M^-(\mathcal{F})$ preserves the terminal object.*

PROOF:We have to show that $\bigwedge_{x \in A} M^-(\mathcal{F})(x) = 0$. For any $\epsilon > 0$, since $lim^+_{\mathcal{F}} M^-(\mathcal{F}) = 0$ one may find an $f \in \mathcal{F}$ such that for any $x \in f$, $M^-(\mathcal{F})(x) \leq \epsilon$. Since that f is not empty then $\bigwedge_{x \in A} M^-(\mathcal{F})(x) \leq \epsilon$ ■

Proposition 3.28 *Let M be a $\mathcal{P}_1$-flat left module on A then for all x, $M(x) \geq \bigvee_{\epsilon > 0} \bigwedge_{y | M(y) \leq \epsilon} A(x, y) = M^- \circ \mathcal{F}(M)$.*

PROOF:Let $x \in A$ and ϵ be such that $M(x) < \epsilon$. According to 3.24, for all α, there exists y such that $M(y) \leq \alpha$ and $A(x, y) \leq \epsilon$. Thus for all α, $\bigwedge_{y|M(y)\leq\alpha} A(x, y) \leq \epsilon$, i.e $\bigvee_{\alpha>0} \bigwedge_{y|M(y)\leq\alpha} A(x, y) \leq \epsilon$. ■

- *$Flat_{\mathcal{P}_2}$-modules and flat filters.*

Now we establish the reflection $\mathcal{F} : WFil(A) \hookleftarrow Flat_{\mathcal{P}_1}(A)_0$. This results from 3.29 and 3.31 below.

3.29 *For any $\mathcal{P}_2$-flat module $M : I \longrightarrow\!\!\circ\!\!\!\rightarrow A$, $\mathcal{F}(M)$ is a flat filter.*

PROOF:If $- * M$ preserves conical finite limits then it preserves in particular the terminal object and according to 3.22,$\mathcal{F}(M)$ is a filter on A. The fact that the filter basis $\Gamma(M)$ generates a flat filter is a consequence of the following lemma. ■

Lemma 3.30 *If M is $\mathcal{P}_2$-flat then for any $\epsilon > 0$ and and any finite family $(x_i)_{i\in I}$ such that for all i, $M(x_i) < \epsilon$ and any $\alpha > 0$, there is a y such $M(y) \leq \alpha$ and for all $i \in I$, $A(x_i, y) \leq \epsilon$.*

PROOF:$- * M$ preserves conical finite limits and cotensors. Consider $\epsilon > 0$ and a finite family of x_i's such that $M(x_i) < \epsilon$. Let us write $\epsilon' = \bigvee_{i\in I} M(x_i)$. Then

$$\begin{aligned} 0 &= [\epsilon', \textstyle\bigvee_{i\in I} M(x_i)] \\ &= [\epsilon', \textstyle\bigvee_{i\in I}(A(x_i, -) * M)] \\ &= [\epsilon', (\textstyle\bigvee_{i\in I} A(x_i, -)) * M] \\ &= (\epsilon' \pitchfork (\textstyle\bigvee_{i\in I} A(x_i, -)) * M \\ &= \textstyle\bigwedge_{y\in A}(M(y) + [\epsilon', \bigvee_{i\in I} A(x_i, y)]). \end{aligned}$$

So for any $\delta > 0$, there is an y such that $M(y) + [\epsilon', \bigvee_{i\in I} A(x_i, y)] \leq \delta$. This y satisfies $M(y) \leq \delta$, and for all i, $A(x_i, y) \leq \epsilon' + \delta$. Now given any $\alpha > 0$, by considering $\delta = min\{\alpha, \epsilon - \epsilon'\}$ one may find a y as required. ■

3.31 *If the filter $\mathcal{F}$ is flat then $- * M^-(\mathcal{F})$ preserves conical finite limits, i.e. for any finite family $(N_i)_{i\in I}$ of right modules on A*

$$\bigwedge_{x\in A} (M^-(\mathcal{F})(x) + \bigvee_{i\in I} N_i(x)) = \bigvee_{i\in I} \bigwedge_{x\in A} (M^-(\mathcal{F})(x) + N_i(x)).$$

PROOF:We shall only prove $\bigwedge_{x\in A}(M^-(\mathcal{F})(x) + \bigvee_{i\in I} N_i(x)) \leq \bigvee_{i\in I} \bigwedge_{x\in A}(M^-(\mathcal{F})(x) + N_i(x))$ since the reverse inequality is trivial.

Let $\epsilon > 0$.
If there is a filter $\mathcal{F}$ on A then A is not empty and for each $i \in I$, there is an $x_i \in A$ such that

$$N_i * M^-(\mathcal{F}) + \epsilon = \bigwedge_{x\in A} (M^-(\mathcal{F})(x) + N_i(x)) + \epsilon \geq M^-(\mathcal{F})(x_i) + N_i(x_i).$$

Let $f \in \mathcal{F}$. Given a family of x_i's as above, for each i, $M^-(\mathcal{F})(x_i) \geq \bigwedge_{y\in f} A(x_i, y)$, thus there is an $y_i \in f$ such that $M^-(\mathcal{F})(x_i) + \epsilon \geq A(x_i, y_i)$ and

$$\begin{aligned} 2 \cdot \epsilon + N_i * M^-(\mathcal{F}) &\geq A(x_i, y_i) + N_i(x_i) \\ &\geq N_i(y_i). \end{aligned}$$

Since $\mathcal{F}$ is flat, we can choose f so that for the $y_i \in f$ as above, for all $g \in \mathcal{F}$, there exists $z \in g$ such that for all i, $A(y_i, z) \leq \epsilon$. Thus for all $g \in \mathcal{F}$, there exists $z \in g$ such that for all i,

$$
\begin{aligned}
3\cdot\epsilon + N_i * M^-(\mathcal{F}) &\geq A(y_i, z) + N_i(y_i) \text{ for some suitable } y_i\text{'s},\\
&\geq N_i(z).
\end{aligned}
$$

Because ϵ is arbitrary we have shown so far that for any $g \in \mathcal{F}$,

$$\bigvee_{i\in I}(N_i * M^-(\mathcal{F})) \geq \bigwedge_{z\in g}\bigvee_{i\in I} N_i(z).$$

So

$$
\begin{aligned}
\textstyle\bigvee_{i\in I}(N_i * M^-(\mathcal{F})) &\geq \textstyle\bigvee_{g\in\mathcal{F}}\bigwedge_{z\in g}\bigvee_{i\in I} N_i(z)\\
&= (\textstyle\bigvee_{i\in I} N_i) * M^-(\mathcal{F}), \text{ according to 3.21.}
\end{aligned}
$$

∎

- *The right morphisms for weakly flat filters.*

For any weakly flat filter $\mathcal{F}$ on A, that $\mathcal{F} = \mathcal{F}\circ M^-(\mathcal{F})$ is to say that for all $f \in \mathcal{F}$ there exists $\epsilon > 0$ such that $\Gamma(M^-(\mathcal{F}))(\epsilon) \subseteq f$. Call a weakly flat filter *closed* when it satisfies the latter condition. $CWFil(A)$ denoting the full subcategory of $WFil(A)$ of closed weakly flat filters, the inclusion $CWFil(A) \hookrightarrow WFil(A)$ has left adjoint $\mathcal{F}\circ M^-$ according to 3.15. From this one gets a notion of morphisms of weakly flat filters more relevant than the inclusion as it will yield the characterization of weakly flat/flat filters in terms of forward Cauchy sequences 3.35.

Definition 3.32 *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be weakly flat filters on A. We write $\mathcal{F}_1 \to \mathcal{F}_2$ if and only if for all $\epsilon > 0$, there exists $f \in \mathcal{F}_1$ such that for all $x \in f$, for all $g \in \mathcal{F}_2$, there exists $y \in g$ such that $A(x, y) \leq \epsilon$.*

Note that $\mathcal{F}_1 \to \mathcal{F}_2$ is by definition $\mathcal{F}_1 \supseteq \mathcal{F}\circ M^-(\mathcal{F}_2)$. Thus weakly flat filters on A with the above relation $\to$ define a preorder denoted $WFil^*(A)$ equivalent to $Flat_{\mathcal{P}_1}(A)_0$ ($WFil^*(A)$ is the category of fractions induced by the reflector $WFil(A) \to Flat_{\mathcal{P}_1}(A)_0$ - see [Bor94] prop 5.3.1 p.190). Equivalence classes in $WFil^*(A)$ are in one to one correspondence with closed weakly flat filters. In the same way, $CFFil(A)$ will denote the full sucategory of $WFil(A)$ with objects closed flat filters and $FFil^*(A)$ will denote the full subcategory of $WFil^*(A)$ with objects flat filters. $FFil^*(A)$ is equivalent to $Flat_{\mathcal{P}_2}(A)_0$.

Closed weakly flat filters play a similar role for non-symmetric spaces as the minimal Cauchy filters do for symmetric spaces. Note that

3.33 *If A is symmetric,*

- (1) *flat filters on A are Cauchy;*
- (2) *For any Cauchy filter $\mathcal{F}$, $M^l(\mathcal{F}) = M^r(\mathcal{F})$.*
- (3) *Any left adjoint module on A has the same underlying map as its right adjoint;*
- (4) *For any left adjoint module M on A, $\mathcal{F}(M) = \mathcal{F}^s(M)$;*
- (5) *$\mathcal{P}_2$-flat modules are left adjoint;*
- (6) *Closed Cauchy filters are exactly the minimal Cauchy filters;*
- (7) *$CFFil(A)$ is discrete.*

PROOF:(1) is already known.
(2) trivial.
(3) For any left module M with right adjoint $\tilde{M}$, according to (2) their underlying maps satisfy $M = M^l \circ \mathcal{F}^s(M) = M^r \circ \mathcal{F}^s(M) = \tilde{M}$.
(4) straightforward from (3).
(5) One has the successive equivalences

> M is $\mathcal{P}_2$-flat
> if and only if $M = M^-(\mathcal{F})$ for a flat filter (3.15)
> if and only if $M = M^-(\mathcal{F})$ for a Cauchy filter, (1)
> if and only if M is left adjoint.

(6) A Cauchy filter $\mathcal{F}$ is closed if and only $\mathcal{F} = \mathcal{F} \circ M^-(\mathcal{F}) = \mathcal{F}^s \circ M^-(\mathcal{F})$ if and only if it is minimal as a Cauchy filter.
(7) Actually the underlying subcategory $\mathcal{C}$ of the full subcategory of presheaves $[A^{op}, \bar{I\!R}_+]$ with objects left adjoint modules is - in this particular case - discrete. And $CFFil(A)$ is equivalent to $\mathcal{C}$ according to (6). Let us see that $\mathcal{C}$ is discrete. For any left adjoint module M on A, M has the same underlying map as its right adjoint $\tilde{M}$. Now consider another left adjoint module N on A, with right adjoint $\tilde{N}$. Then $M \Rightarrow N$ if and only if $\forall x \in A, M(x) \geq N(x)$ if and only if $\forall x \in A, \tilde{M}(x) \geq \tilde{N}(x)$ if and only if $\tilde{M} \Rightarrow \tilde{N}$. But also if $M \Rightarrow N$ then $1 \Rightarrow \tilde{M}N$ since $M \dashv \tilde{M}$ and $\tilde{N} \Rightarrow \tilde{M}$ since $N \dashv \tilde{N}$. So $M \Rightarrow N$ if and only if $M = N$. ∎

- *Forward Cauchy sequences.*

Given a sequence $(x_n)_{n \in I\!N}$ on A, the associated filter, still denoted (x_n), has basis the family of sets $\{x_p \mid p \geq n\}$. We say that

Definition 3.34 (x_n) *is:*
- weakly flat*, respectively* flat*, if the associated filter is so;*
- forward Cauchy *if and only if* $\forall \epsilon > 0, \exists N \in I\!N, \forall m \geq n \geq N, A(x_n, x_m) \leq \epsilon$.

Any forward Cauchy sequence is obviously flat. The relevance of forward Cauchy sequences for non-symmetric spaces appears with following result:

Theorem 3.35 *$WFil^*(A)$ has the colimits of functors with non-empty domain. The family of forward Cauchy sequences is dense in $WFil^*(A)$. Flat filters are exactly the colimits in $WFil^*(A)$ of functors taking values forward Cauchy sequences and with non-empty filtered domain.*

This is proved below. We shall explicit the colimits in $WFil^*(A)$ of functors with non-empty domain in 3.39. According to 3.39 and 3.40, any weakly flat filter $\mathcal{F}$ is a canonical colimit in $WFil^*(A)$ of a functor with non-empty domain and taking values forward Cauchy sequences, moreover this colimit is filtered when $\mathcal{F}$ is flat according 3.41.

To simplify our notation, for any $f \subseteq A$, any $\epsilon > 0$ and, any $F \subseteq \wp\wp(A)$, we let:
- $P(f, \epsilon, F)$ denote the property: "for all x in f, for all $g \in F$ there exists $y \in g$ such that $A(x, y) \leq \epsilon$";
- $Q(f, \epsilon, F)$ denote the property: "for any finite family $x_1, ..., x_n \in f$, for all $g \in F$ there exists $y \in g$ such that for all $i \in \{1, ..., n\}$, $A(x_i, y) \leq \epsilon$".

So to say that a filter $\mathcal{F}$ on A is weakly flat (respectively. flat) is to say that for all $\epsilon > 0$, there exists $f \in \mathcal{F}$ such that $P(f, \epsilon, \mathcal{F})$ (respectively. $Q(f, \epsilon, \mathcal{F})$). Also for weakly flat filters $\mathcal{F}_1, \mathcal{F}_2$,

3.36 *$\mathcal{F}_1 \to \mathcal{F}_2$ if and only if for all $\epsilon > 0$, $\exists f \in \mathcal{F}_1, P(f, \epsilon, \mathcal{F}_2)$.*

When $\mathcal{F}_2$ is moreover flat, one has that

3.37 *If $\mathcal{F}_1 \to \mathcal{F}_2$ then for all $\epsilon > 0$, $\exists f \in \mathcal{F}_1, Q(f, \epsilon, \mathcal{F}_2)$.*

This is a consequence of the following lemma.

Lemma 3.38 *Given $f \subseteq A$, $\epsilon > 0$ and a flat filter $\mathcal{F}$, if $P(f, \epsilon, \mathcal{F})$ then for all $\alpha > 0$, $Q(f, \epsilon+\alpha, \mathcal{F})$.*

PROOF:Let $\alpha > 0$. Since $\mathcal{F}$ is flat, there is a $g_\alpha \in \mathcal{F}$ such that $Q(g_\alpha, \alpha, \mathcal{F})$. Consider a finite family $x_1, ..., x_n$ in f. Since $P(f, \epsilon, \mathcal{F})$, there exist y_1,..., y_n and in g_α such that $A(x_i, y_i) \leq \epsilon$ for all i. Since $Q(g_\alpha, \alpha, \mathcal{F})$, for any $g \in \mathcal{F}$, one may find t in g such that $A(y_i, t) \leq \alpha$ for all i, so that $A(x_i, t) \leq \epsilon + \alpha$ for all i. ∎

Let $Fil(A)$ stand for the category of filters on A with reverse inclusion ordering. $WFil(A)$ and $FFil(A)$ are full subcategories of $Fil(A)$. Moreover,

Proposition 3.39 *In $Fil(A)$, any non empty family I has a least upper bound, moreover if I is a family of weakly flat filters, its upper bound is weakly flat.*

PROOF:Let $\mathcal{F} = \bigcap_{i \in I} \mathcal{F}_i$, where I is a non-empty set. $\mathcal{F}$ is a filter. Suppose first that all the $\mathcal{F}_i$'s are weakly flat. Given $\epsilon > 0$, for any i there exists $f_i \in \mathcal{F}_i$ such that $P(f_i, \epsilon, \mathcal{F}_i)$. For all these f_i, $P(f_i, \epsilon, \mathcal{F})$ since $\mathcal{F} \subseteq \mathcal{F}_i$. Thus $f = \bigcup_{i \in I} f_i$ that belongs to $\mathcal{F}$ satisfies $P(f, \epsilon, \mathcal{F})$.

Since $CWFil(A)$ is full and reflective in $WFil(A)$, if i denotes here the inclusion $CWFil(A) \hookrightarrow WFil(A)$, any functor $F : J \to CWFil(A)$ will have a colimit if the composite $i \circ F$ has a colimit. So from 3.39, $CWFil(A)$ has colimits of functors with non-empty domains, and since $WFil^*(A) \cong CWFil(A)$, $WFil^*(A)$ has these colimits as well (the colimit in $CWFil(A)$ of any non-empty family $(\mathcal{F}_i)_{i \in I}$ is $\mathcal{F} \circ M^-(\bigcap_{i \in I} \mathcal{F}_i)$). Remark that one can prove directly that the ordinary category $Flat_{\mathcal{P}_1}(A)_0$ has colimits of functors with non-empty domain: for any non-empty family M_i of $\mathcal{P}_1$-flat left modules on A, the pointwise $\bigwedge_{i \in I} M_i$ is $\mathcal{P}_1$-flat. Nevertheless this straightforward proof relies on the non-categorical argument 3.25.
Note to the referee - that part may be omitted
Given a non-empty family of $\mathcal{P}_1$-flat $(M_i)_{i \in I}$, a right module N and $v \in \bar{\mathbb{R}}_+$

$$\begin{aligned}
(\textstyle\bigwedge_{i \in I} M_i) * (v \pitchfork N) &= \textstyle\bigwedge_{x \in A}((\bigwedge_{i \in I} M_i(x)) + [v, N(x)]) \\
&= \textstyle\bigwedge_{x \in A} \bigwedge_{i \in I}(M_i(x) + [v, N(x)]) \\
&= \textstyle\bigwedge_{i \in I} \bigwedge_{x \in A}(M_i(x) + [v, N(x)]) \\
&= \textstyle\bigwedge_{i \in I}(M_i * (v \pitchfork N)) \\
&= \textstyle\bigwedge_{i \in I}[v, (M_i * N)], \text{ since each } M_i * - \text{ preserves cotensors} \\
&= [v, \textstyle\bigwedge_{i \in I}(M_i * N)], \text{ since } I \text{ is not empty 3.25}, \\
&= [v, \textstyle\bigwedge_{i \in I} \bigwedge_{x \in A}(M_i(x) + N(x))], \\
&= [v, \textstyle\bigwedge_{x \in A} \bigwedge_{i \in I}(M_i(x) + N(x))], \\
&= [v, \textstyle\bigwedge_{x \in A}((\bigwedge_{i \in I} M_i(x)) + N(x))], \\
&= [v, (\textstyle\bigwedge_{i \in I} M_i) * N].
\end{aligned}$$

Proposition 3.40 *Given a weakly flat filter $\mathcal{F}$ on A and a left module M such that $M^-(\mathcal{F}) \not\Rightarrow M$ (i.e. $M^-(\mathcal{F}) \not\geq M$), there is a forward Cauchy sequence (y_n) such that $(y_n) \to \mathcal{F}$ and $M^-(y_n) \not\Rightarrow M$.*

PROOF:By hypothesis there exists $x \in A$ such that $\bigvee_{f \in \mathcal{F}} \bigwedge_{y \in f} A(x, y) < M(x)$. Consider such an x. There exists $\alpha > 0$ such that for any $f \in \mathcal{F}$, there exists $y \in f$ such that $A(x, y) + \alpha < M(x)$. Note then that for such a y, $A(x, y)$ is necessarely finite.

Since $\mathcal{F}$ is weakly flat, one can define a sequence (f_n) of elements of $\mathcal{F}$ such that for all $n \in \mathbb{N}$, $f_{n+1} \subseteq f_n$ and $P(f_n, \alpha \cdot 2^{-2-n}, \mathcal{F})$. (f_n) is defined inductively as follows.

Choose first f_0 such that $P(f_0, \alpha \cdot 2^{-2}, \mathcal{F})$.

If f_n is defined then one can find $g \in \mathcal{F}$ such that $P(g, \alpha \cdot 2^{-2-(n+1)}, \mathcal{F})$ and let $f_{n+1} = f_n \cap g$.

Then one can build a sequence (y_n) where for all integer n, $y_n \in f_n$, y_0 is such that $A(x, y_0)+\alpha < M(x)$, and for all integer n, $y_n \in f_n$, $A(y_n, y_{n+1}) \leq \alpha \cdot 2^{-2-n}$.

Actually this ensures that:
(1) (y_n) is forward Cauchy;
(2) $(y_n) \to \mathcal{F}$;
(3) $M^-(y_n) \not\to M$.

(1) holds since for all $n \leq p \in I\!N$,

$$\begin{aligned} A(y_n, y_p) &\leq A(y_n, y_{n+1}) + \ldots + A(y_{p-1}, y_p) \\ &\leq \alpha \cdot (2^{-2-n} + 2^{-2-(n+1)} + \ldots) \\ &= \alpha \cdot 2^{-1-n}. \end{aligned}$$

(2) holds since (y_n) is forward Cauchy, for any $n \in I\!N$, $\{y_p / p \geq n\} \subseteq f_n$ and $P(f_n, \alpha \cdot 2^{-2-n}, \mathcal{F})$.

(3) holds since for all $n \in I\!N$,

$$\begin{aligned} A(x, y_n) &\leq A(x, y_0) + A(y_0, y_1) + \ldots + A(y_{n-1}, y_n) \\ &\leq A(x, y_o) + \alpha/2 \end{aligned}$$

so $A(x, y_n) + \alpha/2 < M(x)$. Thus $M^-(y_n)(x) < M(x)$. ∎
Note that according to 3.40 any weakly flat filter $\mathcal{F}$ dominates at least one forward Cauchy sequence as $lim_{\mathcal{F}} M^-(\mathcal{F}) = 0$ and thus $M^-(\mathcal{F}) \not\to +\infty$ with $+\infty$ the constant module with value $+\infty$.

Proposition 3.41 *If (x_n) and (y_n) are weakly flat sequences and $\mathcal{F}$ is a flat filter such that $(x_n) \to \mathcal{F} \leftarrow (y_n)$, then there exists a forward Cauchy sequence (z_n) such that $(x_n) \to (z_n) \leftarrow (y_n)$ and $(z_n) \to \mathcal{F}$.*

PROOF:Since $(x_n) \to \mathcal{F}$ and $\mathcal{F}$ is weakly flat, one can define a sequence of integers $(N_i)_{i \in I\!N}$ such that $P(\{x_n \mid n \geq N_i\}, 2^{-i}, \mathcal{F})$.

Define for all integers i, X_i has the finite set $\{x_n / N_i \leq n < N_{i+1}\}$. Note then that $P(X_i, 2^{-i}, \mathcal{F})$.

Analogously define a sequence $(M_i)_{i \in I\!N}$ such that $P(\{y_n \mid n \geq M_i\}, 2^{-i}, \mathcal{F})$, and the sets $Y_i = \{y_n / M_i \leq n < M_{i+1}\}$.

One may also find for any integer i, a $f_i \in \mathcal{F}$ such that $P(f_i, 2^{-i}, \mathcal{F})$.

We are going to build by recurrence a sequence (z_n) such that, for all integer i:
(1) for all $x \in X_i$, $A(x, z_{i+1}) \leq 2^{-i+1}$;
(2) for all $y \in Y_i$, $A(y, z_{i+1}) \leq 2^{-i+1}$;
(3) $z_i \in f_i$;
(4) $A(z_i, z_{i+1}) \leq 2^{-i+1}$.

Choose first $z_0 \in f_0$.

Suppose that $z_i \in f_i$. Since $P(X_i, 2^{-i}, \mathcal{F})$, $P(Y_i, 2^{-i}, \mathcal{F})$ and $P(f_i, 2^{-i}, \mathcal{F})$ then $P(X_i \cup Y_i \cup f_i, 2^{-i}, \mathcal{F})$. And since $\mathcal{F}$ is flat, according to 3.38, $Q(X_i \cup Y_i \cup f_i, 2^{-i+1}, \mathcal{F})$. Because X_i and Y_i are finite one may find $z_{i+1} \in f_{i+1}$ satisfying the point (1), (2) and (4) below.

According to (4), (z_n) is forward Cauchy. Also from (1), respectively (2), one deduces $(x_n) \to (z_n)$, respectively $(y_n) \to (z_n)$. According to (3), for any $p \geq n \in I\!N$, $M^-(\mathcal{F})(z_p) \leq 2^{-n}$ thus $(z_n) \supseteq \mathcal{F} \circ M^-(\mathcal{F})$. ∎

3.3 Non symmetric completions of general metric spaces

Let A denote a general metric space. We are going to explicit in topological/metric terms the $\mathcal{P}_1$- and $\mathcal{P}_2$-completions of A.

The ordinary category A_0 is a preorder with $x \to y$ if and only if $0 \geq A(x, y)$.

$Flat_{\mathcal{P}_1}(A)$ and $Flat_{\mathcal{P}_2}(A)$ are the small $\bar{I\!R}_+$-categories with objects respectively the $\mathcal{P}_1$-flat presheaves and $\mathcal{P}_2$-flat presheaves on A, and with homs given by

$$Hom(F, G) = [A^{op}, \bar{I\!R}_+](F, G) = \bigvee_{a \in A} [Fa, Ga].$$

A embeds fully and faithfully into $Flat_{\mathcal{P}_1}(A)$ (respectively into $Flat_{\mathcal{P}_2}(A)$) by $a \mapsto Ya = A(-, a)$. Alternatively, according to 3.15, one has the following metric/topological description. $Flat_{\mathcal{P}_1}(A)$, respectively $Flat_{\mathcal{P}_2}(A)$, is isomorphic to the general metric space with points closed weakly flat filters on A, respectively closed flat filters, and with pseudo distance d defined for any $\mathcal{F}_1$, $\mathcal{F}_2$ by

$$d(\mathcal{F}_1, \mathcal{F}_2) = [A^{op}, \bar{I\!R}_+](M^-(\mathcal{F}_1), M^-(\mathcal{F}_2)).$$

Actually we shall give an expression of this distance that do not refer to presheaves. First let us show

Proposition 3.42 *For any left module M on A and any filter $\mathcal{F}$,*

$$[A^{op}, \bar{I\!R}_+](M^-(\mathcal{F}), M) \leq lim^+_{\mathcal{F}} M.$$

If $\mathcal{F}$ is weakly flat then the inequality above becomes an equality.

PROOF:To simplify notations, let LHS and RHS denote respectively $\bigvee_{x \in A}[M^-(\mathcal{F})(x), M(x)]$ and $\bigwedge_{f \in \mathcal{F}} \bigvee_{z \in f} M(z)$.

According to the definition of $M^-(\mathcal{F})$, for all $x \in A$, for all $f \in \mathcal{F}$, $M^-(\mathcal{F})(x) \geq \bigwedge_{z \in f} A(x, z)$. So for any $x \in A$, $f \in \mathcal{F}$ and any $\epsilon > 0$, there exists a $z \in f$ such that $A(x, z) \leq M^-(\mathcal{F})(x) + \epsilon$. For such a z, $M(x) \leq M(z) + A(x, z)$ and $M(x) \leq M(z) + M^-(\mathcal{F})(x) + \epsilon$. So

$\forall x \in A, \forall f \in \mathcal{F}, \forall \epsilon > 0, \exists z \in f$, $[M^-(\mathcal{F})(x), M(x)] \leq M(z) + \epsilon$,
thus $\forall x \in A, \forall f \in \mathcal{F}, \forall \epsilon > 0$, $[M^-(\mathcal{F})(x), M(x)] \leq (\bigvee_{z \in f} M(z)) + \epsilon$,
thus $\forall f \in \mathcal{F}, \forall \epsilon > 0$, $LHS \leq (\bigvee_{z \in f} M(z)) + \epsilon$,
thus $\forall f \in \mathcal{F}$, $LHS \leq \bigvee_{z \in f} M(z)$,
thus $LHS \leq RHS$.

Suppose now that $\mathcal{F}$ is weakly flat. Consider $\epsilon > 0$. One may find an $f_\epsilon \in \mathcal{F}$ such that for all $z \in f_\epsilon$, $M^-(\mathcal{F})(z) \leq \epsilon$. So,

for any $z \in f_\epsilon$, $M(z) \leq [M^-(\mathcal{F})(z), M(z)] + \epsilon$,
thus $\bigvee_{z \in f_\epsilon} M(z) \leq (\bigvee_{x \in A}[M^-(\mathcal{F})(x), M(x)]) + \epsilon$,
and $RHS \leq LHS + \epsilon$.

∎

According to the latter property, one gets

3.43 *For any weakly flat filters $\mathcal{F}_1$ and $\mathcal{F}_2$,*

$$[A^{op}, \bar{I\!R}_+](M^-(\mathcal{F}_1), M^-(\mathcal{F}_2)) = lim^+_{x\in\mathcal{F}_1} lim^-_{y\in\mathcal{F}_2} A(x,y).$$

The limits in 3.43 "commute" when the first argument is Cauchy:

3.44 *For any Cauchy filter $\mathcal{F}_1$ and any weakly flat filter $\mathcal{F}_2$,*

$$[A^{op}, \bar{I\!R}_+](M^-(\mathcal{F}_1), M^-(\mathcal{F}_2)) = lim^-_{y\in\mathcal{F}_2} lim^+_{x\in\mathcal{F}_1} A(x,y).$$

To see this we shall establish the following categorical result

Proposition 3.45 *Given a complete monoidal closed $\mathcal{V}$, if a $\mathcal{V}$-module $M: I \multimap A$ has a right adjoint $\tilde{M}: A \multimap I$ then for any left $\mathcal{V}$-module N: $[A^{op}, \mathcal{V}](M,N) \cong \tilde{M} * N$.*

PROOF:Given any presheaves $M, N : A^{op} \to \mathcal{V}$, $[A^{op}, \mathcal{V}](M,N)$ is the right lifting of N through M. Suppose that moreover M has a right adjoint $\tilde{M}$. If $\epsilon : M * \tilde{M} \to 1 : A \multimap A$ denotes the counit of the adjunction $M \dashv \tilde{M}$ then it is easy to see that $M * (\tilde{M} * N) \cong (M * \tilde{M}) * N \xrightarrow{\epsilon * N} N$ exhibits $\tilde{M} * N$ as the right lifting of N through M. ∎

PROOF:(of 3.44) Now remember that if $\mathcal{F}$ is a Cauchy filter on A then $M^-(\mathcal{F}) = M^+(\mathcal{F}) = M^l(\mathcal{F})$ and this left module on A has right adjoint the module $M^r(\mathcal{F})$ defined by the map $x \mapsto lim^+_{y\in\mathcal{F}} A(y,x) = lim^-_{y\in f} A(y,x)$. So according to 3.45 and 3.21, for any Cauchy filter $\mathcal{F}_1$ and any weakly flat filter $\mathcal{F}_2$,

$$\begin{aligned}[A^{op}, \bar{I\!R}_+](M^-(\mathcal{F}_1), M^-(\mathcal{F}_2)) &= M^r(\mathcal{F}_1) * M^-(\mathcal{F}_2) \\ &= lim^-_{y\in\mathcal{F}_2} M^r(\mathcal{F}_1)(y) \\ &= lim^-_{y\in\mathcal{F}_2} lim^+_{x\in\mathcal{F}_1} A(x,y).\end{aligned}$$

∎

One has a notion of non-symmetric convergence in A. The *neighborhood filter* of $x \in A$, denoted $V_A(x)$, is the filter generated by the family of subsets $\{y \mid A(y,x) \leq \epsilon\}$ with $\epsilon > 0$. Which is to say that $V_A(x)$ is $\mathcal{F}(A(-,x))$. Given a filter $\mathcal{F}$ on A and $x \in A$, we say that $\mathcal{F}$ *converges* to x, that we write $\mathcal{F} \to x$, if and only if $\mathcal{F} \supseteq V_A(x)$. If $\mathcal{F}$ is weakly flat then by 3.20 $\mathcal{F}$ converges to x if and only if $M^-(\mathcal{F}) \Rightarrow A(-,x)$. By Yoneda, this is also equivalent to say that for any $a \in A$,

$$A(x,a) \geq [A^{op}, \bar{I\!R}_+](M^-(\mathcal{F}), A(-,a)),$$

or according to 3.42 that

$$A(x,a) \geq lim^+_{\mathcal{F}} A(-,a).$$

It remains to explicit in topological terms the notion of colimits indexed by flat presheaves.

Definition 3.46 *A filter $\mathcal{F}$ on A has* representative x_0 *if and only if for all $a \in A$,*

$$A(x_0,a) = lim^+_{\mathcal{F}} A(-,a).$$

Which is exactly to say that x_0 is the colimit $M^-(\mathcal{F}) * 1$. In particular if a representative of $\mathcal{F}$ exists then it is unique up to isomorphism. In this case we denote it $rep(\mathcal{F})$. Note that $rep(\mathcal{F})$ when it exists is necessarely the greatest lower bound in A_0 amongst objects such that $\mathcal{F}$ converges to.

Given a filter $\mathcal{F}$ on A and a map $G : A \to B$ the direct image of $\mathcal{F}$ denoted $G(\mathcal{F})$ is the filter on B generated by the subsets $G(f)$ for $f \in \mathcal{F}$. It is easy to check for $\mathcal{F}$ and G as above that if G is non-increasing and $\mathcal{F}$ is weakly flat, respectively flat, then $G(\mathcal{F})$ is again weakly flat, respectively flat. Moreover,

Proposition 3.47 *Given a weakly flat filter $\mathcal{F}$ on A, and a functor $G : A \to B$, $M^-(G(\mathcal{F})) : B^{op} \to \bar{I\!R}_+$ is the (pointwise) left Kan extension of $M^-(\mathcal{F}) : A^{op} \to \bar{I\!R}_+$ along G^{op}.*

PROOF:One has the pointwise computation (see [Kel82], (4.17), p.115),

$$\begin{aligned} Lan_{G^{op}}(M^-(\mathcal{F}))(b) &= B(b, G-) * M^-(\mathcal{F}) \\ &= \bigvee_{f\in\mathcal{F}} \bigwedge_{x\in f} B(b, Gx), \text{ according to 3.21,} \\ &= \bigvee_{g\in G(\mathcal{F})} \bigwedge_{y\in g} B(b, y) \\ &= M^-(G(\mathcal{F}))(b). \end{aligned}$$

∎

Note that we could already infer from 2.14 that for any filter $\mathcal{F}$ and any functor $G : A \to B$, if $M^-(\mathcal{F})$ is $\mathcal{P}_1$-flat, respectively $\mathcal{P}_2$-flat, then $Lan_{G^{op}}(M^-(\mathcal{F}))$ is also $\mathcal{P}_1$-flat, respectively $\mathcal{P}_2$-flat.

As a consequence of 3.47,

Corollary 3.48 *Given a weakly flat filter $\mathcal{F}$ on A, a functor $G : A \to B$ and a presheaf $M : B^{op} \to \bar{I\!R}_+$,*

$$[B^{op}, \bar{I\!R}_+](M^-(G(\mathcal{F})), M) = [A^{op}, \bar{I\!R}_+](M^-(\mathcal{F}), MG^{op})$$

And thus,

Corollary 3.49 *Given a weakly flat filter $\mathcal{F}$ on A and a non expansive map $G : A \to B$, for an object in B, to be the colimit $M^-(\mathcal{F}) * G$ is equivalent to be representative of the weakly flat filter $G(\mathcal{F})$.*

note to the referee, to be omitted in the final version:

$$\begin{aligned} B(M^-(\mathcal{F}) * G, b) &\cong [A^{op}, \bar{I\!R}_+](M^-(\mathcal{F}), B(G-, b)) \\ &\cong [B^{op}, \bar{I\!R}_+](M^-(G(\mathcal{F})), B(-, b)). \end{aligned}$$

We shall call a general metric space A $\mathcal{P}_1$*-complete*, respectively $\mathcal{P}_2$*-complete* if the corresponding category A is. So according to 3.49, A is $\mathcal{P}_1$-complete, respectively $\mathcal{P}_2$-complete if and only if any weakly flat, respectively flat, filter on A admits a representative.

Now given a weakly flat filter $\mathcal{F}$ on K, and non-increasing maps $G : K \to A$, and $H : A \to B$, H (as a functor) preserves the colimit $M^-(\mathcal{F}) * G$ if and only H (as a non-expansive map) preserves the representative of $G(\mathcal{F})$, i.e.

$$H(rep(G(\mathcal{F}))) = rep(H \circ G(\mathcal{F})).$$

To sum up: the $\bar{I\!R}_+$-functors preserving $Flat_{\mathcal{P}_1}$-colimits (respectively $Flat_{\mathcal{P}_2}$-colimits) are exactly the non-expansive maps preserving the representatives of weakly flat filters (respectively those of flat filters).

A direct translation of 2.4 gives for any general metric space A two completions.

Theorem 3.50 *For any general metric space A, there exists a $\mathcal{P}_1$-complete (respectively $\mathcal{P}_2$-complete) metric space $\bar{A}$ together with a map $i_A : A \to \bar{A}$ such that for any non-expansive $f : A \to B$ with codomain B $\mathcal{P}_1$-complete (respectively $\mathcal{P}_2$-complete) there exists a unique $\bar{f} : \bar{A} \to B$ preserving representatives (respectively representatives of flat filters) and such that $\bar{f} \circ i_a = f$.*

For an $f : A \to B$ as above, if one considers the completion $\bar{A}$ as a space of filters on A, then the extension $\bar{f}$ sends any filter $\mathcal{F}$ to the representative of its direct image by f in B. To check this just come back to the categorical formulation. From [Kel82] Theorem 4.97, $\bar{f}$ is the left Kan extension of f along i_A and sends any M in $Flat_{\mathcal{P}}(A)$ to $M * f$, ($\mathcal{P} = \mathcal{P}_1, \mathcal{P}_2$). Translate then with 3.49.

The rest of this section investigates examples of these two completions.

Recall from [Kel82] (3.74)

Proposition 3.51 *Any monoidal closed $\mathcal{V}$ that is complete as an ordinary category is complete and cocomplete as a $\mathcal{V}$-category, i.e. $\mathcal{V}$ admits all limits and colimits indexed by small $\mathcal{V}$-categories.*

So

Corollary 3.52 *$\bar{\mathbb{R}}_+$ is $\mathcal{P}_1$-complete and $\mathcal{P}_2$-complete.*

We shall also show that

Proposition 3.53 *The $\mathcal{P}_1$- and $\mathcal{P}_2$-completion of $\bar{\mathbb{R}}_+$ are both isomorphic to $\bar{\mathbb{R}}_+$.*

This results from 3.54, 3.56, 3.57 and 3.58 below.

3.54 *Weakly flat filters on $\bar{\mathbb{R}}_+$ are flat.*

This results from the following lemma.

Lemma 3.55 *Let $\mathcal{F}$ be a weakly flat filter on $\bar{\mathbb{R}}_+$. Let $\epsilon > 0$, and $f \in \mathcal{F}$ such that $P(f, \epsilon, \mathcal{F})$ then $Q(f, \epsilon, \mathcal{F})$.*

PROOF:Consider any finite family x_1, ..., x_n in f. Let $g \in \mathcal{F}$. There exist y_1,...,y_n in g such that $[x_i, y_i] \le \epsilon$, i.e. $y_i \le x_i + \epsilon$, for $i = 1, ..., n$. Choosing the least of those y_i's, say z one has $[x_i, z] \le \epsilon$ for all i's. ∎

Let us identify now the weakly flat filters on $\bar{\mathbb{R}}_+$. For a filter $\mathcal{F}$ on $\bar{\mathbb{R}}_+$, we write $\liminf(\mathcal{F})$ for $lim^-_{\mathcal{F}} id = \bigvee_{f \in \mathcal{F}} \bigwedge_{x \in f} x$.

3.56 *If $\liminf(\mathcal{F}) \neq \infty$ then $\mathcal{F}$ is weakly flat.*

PROOF:Let $\epsilon > 0$. Since $\liminf(\mathcal{F})$ is finite, one may consider an $f \in \mathcal{F}$ such that $\bigwedge f$ $(= \bigwedge_{x \in f} x)$ $\ge \liminf(\mathcal{F}) - \epsilon$ and thus for any $x \in f$, $\liminf(\mathcal{F}) \le x + \epsilon$. Now given any $g \in \mathcal{F}$, there exists $y \in g$ such that $y \le \liminf(\mathcal{F}) + \epsilon$. For this y, for any $x \in f$, $y \le x + 2 \cdot \epsilon$, i.e $[x, y] \le 2 \cdot \epsilon$. ∎

3.57 *If $\liminf(\mathcal{F}) = \infty$ then there are two cases either $\mathcal{F}$ is the principal filter generated by ∞ or not. In the first case $\mathcal{F}$ is weakly flat, in the second case $\mathcal{F}$ is not weakly flat.*

The first case is trivial: $\mathcal{F}$ is a neighborhood filter. For the second case, consider $\epsilon > 0$ and $f \in \mathcal{F}$. Then $\bigwedge f \neq \infty$ and there exists $g \in \mathcal{F}$ such that $\bigwedge f + 2 \cdot \epsilon < \bigwedge g$. So one may find $x \in f$ such that for any $y \in g$, $x + \epsilon < y$, i.e. $[x, y] > \epsilon$. ∎

Eventually, for weakly flat filters $\mathcal{F}_1$ and $\mathcal{F}_2$ on $\bar{\mathbb{R}}_+$, one has the successive equations:

3.58
$$\begin{aligned}
[\bar{\mathbb{R}}_+, \bar{\mathbb{R}}_+](M^-(\mathcal{F}_1), M^-(\mathcal{F}_2)) &= lim^+_{x \in \mathcal{F}_1} lim^-_{y \in \mathcal{F}_2}[x, y], \\
&= lim^+_{x \in \mathcal{F}_1} (\textstyle\bigvee_{g \in \mathcal{F}_2} \bigwedge_{y \in g}[x, y]), \\
&= lim^+_{x \in \mathcal{F}_1} (\textstyle\bigvee_{g \in \mathcal{F}_2}[x, \bigwedge g]), && (1) \\
&= lim^+_{x \in \mathcal{F}_1} [x, \textstyle\bigvee_{g \in \mathcal{F}_2} \bigwedge g]), && (2)
\end{aligned}$$

$$
\begin{aligned}
&= lim^+_{x\in\mathcal{F}_1}[x, \liminf(\mathcal{F}_2)], \\
&= \bigwedge_{f\in\mathcal{F}_1}\bigvee_{x\in f}[x, \liminf(\mathcal{F}_2)], \\
&= \bigwedge_{f\in\mathcal{F}_1}[\bigwedge f, \liminf(\mathcal{F}_2)], && (3)\\
&= [\bigvee_{f\in\mathcal{F}_1}\bigwedge f, \liminf(\mathcal{F}_2)], && (4)\\
&= [\liminf(\mathcal{F}_1), \liminf(\mathcal{F}_2)].
\end{aligned}
$$

- (1) holds due to 3.25 since any $g \in \mathcal{F}_2$ is non empty;
- (2) holds since representables $[x, -]$ preserves limits;
- (3) holds since representables $[-, y]$ turns colimits into limits;
- actually (4) holds due to the characterization of weakly flat filters on $\bar{I\!R}_+$ 3.57 and another peculiar property of $\bar{I\!R}_+$, 3.59 above.

3.59 *Given any v in $\bar{I\!R}_+$, and any non-empty family $(a_i)_{i\in I}$ in $\bar{I\!R}_+$ that satisfies the condition that if $\bigvee_{i\in I} a_i = +\infty$ then there exists at least one $j \in I$ such that $a_j = +\infty$, then*

$$[\bigvee_{i\in I} a_i, v] = \bigwedge_{i\in I}[a_i, v].$$

PROOF:Since $[-, v]$ reverses the usual ordering on $\bar{I\!R}_+$, certainly $[\bigvee_{i\in I} a_i, v] \leq \bigwedge_{i\in I}[a_i, v]$ (even if I is empty). Conversely, let us fix $\epsilon > 0$. By assumption one may find $j \in I$ such that $\bigvee_{i\in I} a_i \leq a_j+\epsilon$. For such a j, $[\bigvee_{i\in I} a_i, v] \geq [a_j + \epsilon, v] = [\epsilon, [a_j, v]]$, so $[\bigvee_{i\in I} a_i, v] + \epsilon \geq [a_j, v] \geq \bigwedge_{i\in I}[a_i, v]$. ∎

Eventually we shall study the completions of symmetric spaces. For a symmetric general metric space A, its $\mathcal{P}_2$-completion is its Cauchy completion (3.33). But the $\mathcal{P}_1$-completion of A may not be symmetric as shown below.

Proposition 3.60 *The $\mathcal{P}_1$-completion of a symmetric A is the set of non-empty closed subsets in its Cauchy-completion $\bar{A}$ with pseudo distance d given by $d(X, Y) = \bigvee_{x\in X}\bigwedge_{y\in Y}\bar{A}(x, y)$.*

To prove this we shall establish first

Lemma 3.61 *For any filter $\mathcal{F}$ on A, any set X of filters such that $\mathcal{F}$ is the intersection of the filters in X - that we write $\mathcal{F} = \bigcap X$ - and any map $t : Obj(A) \to \bar{I\!R}_+$,*

$$lim^-_{\mathcal{F}}t = \bigwedge_{\varphi\in X}(lim^-_{\varphi}t).$$

PROOF:Let us note $m = \bigwedge_{\varphi\in X}(lim^-_{\varphi}t)$. For any $\varphi \in X$, $\varphi \supseteq \mathcal{F}$ so $lim^-_{\varphi}t \geq lim^-_{\mathcal{F}}t$ and $m \geq lim^-_{\mathcal{F}}t$. Conversely, let us consider any positive real $v \leq m$. Then for any $\varphi \in X$, $v \leq \bigvee_{g\in\varphi}\bigwedge_{x\in g} t(x)$. Fix $\epsilon > 0$. For any $\varphi \in X$, there exists $g_\varphi \in \varphi$ such that $v \leq \bigwedge_{x\in g_\varphi} t(x) + \epsilon$. Let $f = \bigcup_{\varphi\in X} g_\varphi$. Then $f \in \mathcal{F}$, $\bigwedge_{x\in f} t(x) = \bigwedge_{\varphi\in X}\bigwedge_{x\in g_\varphi} t(x)$ and $v \leq \bigwedge_{x\in f} t(x) + \epsilon \leq lim^-_{\mathcal{F}}t + \epsilon$. ∎

PROOF:(of 3.60) Let $\bar{A}$ denote the Cauchy completion of A, it is isomorphic to the metric space with objects closed Cauchy filters on A with distance given for all φ, ψ by $\bar{A}(\varphi, \psi) = M^r(\varphi) * M^-(\psi)$ $= lim^-_{\psi}M^r(\varphi)$ by 3.45/3.44 and 3.21. Since A is symmetric, $\bar{A}$ is symmetric, also forward Cauchy sequences in A are Cauchy.

Consider a closed weakly flat filter $\mathcal{F}$ on A. $\bar{\mathcal{F}}$ will denote the set of closed Cauchy filters containing $\mathcal{F}$. According to 3.33-(6) and 3.35, $\bar{\mathcal{F}}$ is not empty and $\mathcal{F}$ is the intersection of the filters in $\bar{\mathcal{F}}$.

We shall show now the following property:
$(*)$ For any subset X of $\bar{A}$ such that $\mathcal{F}$ is the intersection of the filters in X and any closed Cauchy filter φ,

$$[A^{op}, \bar{I\!R}_+](M^-(\varphi), M^-(\mathcal{F})) = \bigwedge_{\psi \in X} \bar{A}(\varphi, \psi).$$

Consider a Cauchy filter φ. Then

$$\begin{aligned}
[A^{op}, \bar{I\!R}_+](M^-(\varphi), M^-(\mathcal{F})) &= M^r(\varphi) * M^-(\mathcal{F}), \text{ since } M^-(\varphi) \text{ is left adjoint 3.45/3.44,} \\
&= lim^-_{\mathcal{F}} M^r(\varphi), \text{ according to 3.21,} \\
&= \textstyle\bigwedge_{\psi \in X} lim^-_{\psi} M^r(\varphi), \text{ according to 3.61,} \\
&= \textstyle\bigwedge_{\psi \in X} M^r(\varphi) * M^-(\psi) \\
&= \textstyle\bigwedge_{\psi \in X} \bar{A}(\varphi, \psi).
\end{aligned}$$

As a consequence of $(*)$, for any subset X of $\bar{A}$ such that $\mathcal{F} = \bigcap X$, the adherence $\bar{X}$ of X in $\bar{A}$ is $\bar{\mathcal{F}}$. Hence $\bar{\mathcal{F}}$ is the only closed subset X in the metric space $\bar{A}$, such that $\mathcal{F} = \bigcap X$.

Now given two closed weakly flat filters $\mathcal{F}_1$ and $\mathcal{F}_2$ on A,

$$\begin{aligned}
[A^{op}, \bar{I\!R}_+](M^-(\mathcal{F}_1), M^-(\mathcal{F}_2)) &= [A^{op}, \bar{I\!R}_+](\textstyle\bigwedge_{\varphi \in \bar{\mathcal{F}}_1} M^-(\varphi), M^-(\mathcal{F}_2)), \text{ according to 3.61,} \\
&= \textstyle\bigvee_{\varphi \in \bar{\mathcal{F}}_1} [A^{op}, \bar{I\!R}_+](M^-(\varphi), M^-(\mathcal{F}_2)), \\
&= \textstyle\bigvee_{\varphi \in \bar{\mathcal{F}}_1} \bigwedge_{\psi \in \bar{\mathcal{F}}_2} \bar{A}(\varphi, \psi), \text{ according to } (*) \text{ above.}
\end{aligned}$$

∎

Remark 3.62 *Consider an embedding $i_A : A \hookrightarrow \bar{A}$ of a general metric space A into its Cauchy-completion. For any subset X of A, the closure in $\bar{A}$ of the direct image $i_A(X)$ is the set of Cauchy filters adherent to X. Considering now the inverse image by i_A of this set one obtains the closure of X in A. Nevertheless the direct image by i_A does NOT yield in general a surjective correspondence from closed subsets of A to closed subsets of $\bar{A}$. A counter-example for this would be the real line with the usual metric less one point. The Cauchy completion just adds the missing point say x, $\{x\}$ is closed in the completion but its inverse image by the canonical embedding is empty.*

4 The case $\mathcal{V} =$ Bool.

Preorders as enrichments over the "boolean" category **Bool** and their Cauchy-completion were treated in [Law73]. We shall recall briefly these results. Then we characterize in this context the $\mathcal{P}_1$- and $\mathcal{P}_2$- flat presheaves and the $\mathcal{P}_1$-completion and show that the $\mathcal{P}_2$-completion coincide with the classic "algebraic" (or "ideal") completion.

Bool stands for the two-object category generated by the graph $0 \longrightarrow 1$. It is a partial order and it has a monoidal structure with tensor $\wedge$ (the logical "and") and unit 1. **Bool** is closed as for all $x, y, z \in$ **Bool**,

$$x \wedge y \leq z \Leftrightarrow x \leq (y \Rightarrow z)$$

where $\Rightarrow$ denotes the usual entailment relation.

Bool-categories are just preorders: for any **Bool**-category A, its associated preorder is defined by $x \to y$ if and only if $A(x, y) = 1$. Along the same line there are one-to-one correspondences between

- **Bool**-functors and order preserving maps;

- Right modules on a **Bool**-category A and downward closed subsets, or "downsets", on the preorder A;

- left modules and upper closed subsets, in a dual way.

Under the above correspondences the Lawvere Cauchy-completion for **Bool**-categories is the Dedekind-Mac Neille completion for preorders. Also the $\emptyset$-completion occurs as the so-called the *downward completion* that is defined for a preorder as the set of its downsets with inclusion ordering.

Let us focus on the $\mathcal{P}_1$- and $\mathcal{P}_2$-flatness. Further on A denotes a **Bool**-category that we might freely see as a preorder. Using 3.3, 3.4 and 3.5 again, one gets

4.1 *For any module* $M : I \longrightarrow\!\!\circ\!\!\!\!\!\rightarrow A$ *,*

- $- * M : [A, \mathbf{Bool}] \to \mathbf{Bool}$ *preserves the (conical) terminal object i.e.* $1 * M = 1$*, if and only if*
$$(1) \quad \bigvee_{x \in A} M(x) = 1;$$

- $- * M$ *preserves conical finite limits if and only*
(2) *For any finite family of right modules* $N_i : A \longrightarrow\!\!\circ\!\!\!\!\!\rightarrow I$ *,* i *ranging in* I*,*
$$\bigvee_{x \in A} (M(x) \wedge \bigwedge_{i \in I} N_i(x)) = \bigwedge_{i \in I} (\bigvee_{x \in A} M(x) \wedge N_i(x));$$

- $- * M$ *preserves cotensors if and only if*
(3) *For any* $v \in \mathbf{Bool}$ *and any right module* $N : A \longrightarrow\!\!\circ\!\!\!\!\!\rightarrow I$ *,*
$$\bigvee_{x \in A} (M(x) \wedge (v \Rightarrow N(x))) \quad = \quad (v \Rightarrow \bigvee_{x \in A} (M(x) \wedge N(x))).$$

Condition (1) above is equivalent to the fact that $\mathcal{I}_M$, the downset corresponding to M, is not empty. Condition (3) reduces for $v = 1$ to the trivial equation $N * M = N * M$. For $v = 0$, it reduces to $1 = \bigvee_{x \in A} M(x)$, that is (1) again. Eventually condition (2) is equivalent to the fact that $\mathcal{I}_M$ is directed i.e. any finite family in $\mathcal{I}_M$ has an upper bound in $\mathcal{I}_M$. So there are bijections between the following objects on A:

- $\mathcal{P}_1$-flat left modules and non-empty downsets,

- $\mathcal{P}_2$-flat left modules and non-empty directed downsets.

From these observations, one obtains straightforwardly that

- $Flat_{\mathcal{P}_1}(A)$ as a preorder is the set of non-empty downsets on A with inclusion ordering;

- $Flat_{\mathcal{P}_2}(A)$ as a preorder is $Alg(A)$, the *algebraic completion* (or *ideal completion*) of A, that is the set of its non-empty directed downsets with inclusion ordering.

Eventually given $M : A^{op} \to \mathbf{Bool}$ and $G : A \to B$, that $b \in B$ is the colimit $M * G$ is equivalent to the fact that b is the least upper bound in the preorder B of the downset generated by the direct image of $\mathcal{I}_M$ by G. So from 2.3 one gets two completions:

Theorem 4.2 *Given a preorder* A*,* $Flat_{\mathcal{P}_1}(A)$ *together with the order preserving map* $i_A : A \to Flat_{\mathcal{P}_1}(A)$ *sending* a *to the downset generated by* a*, are such that for any preorder* B *with least upper bounds, for any non-empty set and any order preserving map* $f : A \to B$*, there is a unique* $\bar{f} : Flat_{\mathcal{P}_1}(A) \to B$ *preserving least upper bounds of non empty sets and satisfying* $\bar{f} \circ i_A = f$*.*

And also

Theorem 4.3 *Given a preorder A, $Alg(A)$ together with the order preserving map $j_A : A \to Alg(A)$ sending a to the downset generated by a, are such that for any preorder B with least upper bounds for any non-empty directed set and any order preserving map $f : A \to B$, there is a unique $\bar{f} : Alg(A) \to B$ preserving least upper bounds of non-empty directed sets and satisfying $\bar{f} \circ j_A = f$.*

It is common in the literature to define directed subsets in a partial order as non-empty. Partial orders with all least upper bounds for non-empty directed sets are usually called *directed complete* partial orders or *dcpo*'s. Also monotone maps between dcpos that preserve least upper bounds of non-empty directed subsets are called *continuous*.

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