Energy Norm *A Posteriori* Error Estimation for Mixed Discontinuous Galerkin Approximations of the Maxwell Operator

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Abstract

In this paper we develop the *a posteriori* error estimation of mixed discontinuous Galerkin finite element approximations of the Maxwell operator. In particular, by employing suitable Helmholtz decompositions of the error, together with the conservation properties of the underlying method, computable upper bounds on the error, measured in terms of a natural (mesh-dependent) energy norm, are derived. Numerical experiments testing the performance of our *a posteriori* error bounds for problems with both smooth and singular analytical solutions are presented.

Key words: Discontinuous Galerkin methods, *A posteriori* error estimation, Mixed methods, Maxwell operator

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1 Introduction

In recent work we have been concerned with the development of mixed discontinuous Galerkin (DG, for short) finite element methods for the numerical approximation of the Maxwell operator in the time harmonic regime; here, the underlying system of partial differential equations consists of a curl-curl operator subject to a divergence-free constraint. Indeed, in the series of papers [5–7] we have introduced, analyzed and numerically tested nonconforming mixed finite element methods based on a discontinuous Galerkin approach, where the underlying analytical solution is approximated by completely discontinuous piecewise polynomials. Optimal *a priori* error bounds for both stabilized equal-order and non-stabilized mixed-order methods have been derived in an appropriate mesh-dependent DG energy norm. One of the key advantages of this approach in comparison with standard conforming finite element methods (see [8], and the references cited therein) lies in the greater flexibility available for designing the underlying finite element space. Indeed, within a DG approach, non-matching grids containing hanging nodes and non-uniform, even anisotropic, polynomial approximation degrees can easily be handled. Thereby, DG methods are naturally suited for the application within adaptive refinement algorithms where the mesh and/or polynomial degree are locally refined in order to yield reliable and efficient control of the discretization error.

In this article, we initiate the development of the *a posteriori* error estimation and adaptive mesh design for the non-stabilized mixed DG method introduced in [6]. In particular, computable upper bounds on the error, measured in terms of a natural (mesh-dependent) DG energy norm, will be derived. Inspired by the recent article [3], the proof of this error bound is based on employing suitable Helmholtz decompositions of the error, together with the conservation properties of the underlying DG method. The performance of this error bound within an adaptive local mesh refinement procedure will be demonstrated for problems with both smooth and singular analytical solutions.

2 Model problem

We first introduce some notation: given a bounded domain D in $\mathbb{R}^d$, $d \geq 1$, we write $H^t(D)$ to denote the usual Sobolev space of real-valued functions with regularity exponent t and norm $\|\cdot\|_{t,D}$. For $t = 0$, we write $L_2(D)$ in lieu of $H^0(D)$. The space $H^t(D)^d$ consists of vector fields whose components belong to $H^t(D)$; it is endowed with the usual product norm which we denote, for simplicity, also by $\|\cdot\|_{t,D}$. The dual spaces of $H^{1/2}(\partial D)$ and $H^{1/2}(\partial D)^d$ are denoted by $H^{-1/2}(\partial D)$ and $H^{-1/2}(\partial D)^d$, respectively, where ∂D denotes the boundary of D; in both cases, we write $\|\cdot\|_{-1/2,\partial D}$ to denote the corresponding dual norm. Additionally, we write $\langle\cdot,\cdot\rangle_{\partial D}$ to denote the duality pairings in $H^{-1/2}(\partial D)\times H^{1/2}(\partial D)$ and $H^{-1/2}(\partial D)^d\times H^{1/2}(\partial D)^d$, as well as for the inner products in $L_2(\partial D)$ and $L_2(\partial D)^d$. For $D\subset\mathbb{R}^3$, we write $H(\mathrm{curl};D)$ and $H(\mathrm{div};D)$ to denote the spaces of vector fields $\mathbf{u}\in L_2(D)^3$ with $\nabla\times\mathbf{u}\in L_2(D)^3$ and $\nabla\cdot\mathbf{u}\in L_2(D)$, respectively, endowed with their corresponding graph norms $\|\cdot\|_{\mathrm{curl}}$ and $\|\cdot\|_{\mathrm{div}}$. Finally, we denote by $H_0^1(D)$, $H_0(\mathrm{curl};D)$ and $H_0(\mathrm{div};D)$ the subspaces of $H^1(D)$, $H(\mathrm{curl};D)$ and $H(\mathrm{div};D)$, respectively, of functions with zero trace, tangential trace and normal trace, respectively, and by $H(\mathrm{curl}^0;D)$ and $H(\mathrm{div}^0;D)$ the subspaces of $H(\mathrm{curl};D)$ and $H(\mathrm{div};D)$, respectively, of functions with zero curl and divergence, respectively.

Given $\Omega\subset\mathbb{R}^3$, a simply connected bounded Lipschitz polyhedron with connected boundary Γ, we consider the following problem: find $(\mathbf{u},p)$ such that

$$\nabla\times(\mu^{-1}\nabla\times\mathbf{u})-\varepsilon\nabla p=\mathbf{f}\quad\text{in }\Omega,\tag{1}$$

$$\nabla\cdot(\varepsilon\mathbf{u})=0\quad\text{in }\Omega,\tag{2}$$

$$\mathbf{n}\times\mathbf{u}=\mathbf{0}\quad\text{and}\quad p=0\quad\text{on }\Gamma.\tag{3}$$

Here, $\mathbf{n}(\mathbf{x})$ denotes the unit outward normal vector to Γ at $\mathbf{x}\in\Gamma$. The coefficients $\mu=\mu(\mathbf{x})>0$ and $\varepsilon=\varepsilon(\mathbf{x})>0$ are the magnetic permeability and the electric permittivity of the medium, respectively; for simplicity, here we assume that both μ and ε are smooth functions. Throughout, we consider the physically most relevant case of a divergence free source field and assume that $\mathbf{f}\in H(\mathrm{div}^0;\Omega)$; i.e., $\mathbf{f}\in L_2(\Omega)^3$ and

$$\nabla\cdot\mathbf{f}=0\qquad\text{in }\Omega.\tag{4}$$

This assumption actually implies that $p\equiv0$. Nevertheless, it is necessary to discretize the Lagrange multiplier p in order to ensure the stability and well-posedness of the corresponding mixed approximation.

The following embedding results from [1] will play a crucial role in the forthcoming *a posteriori* analysis: under the above assumptions on the domain Ω, there exists a regularity exponent $s\in(1/2,1]$, depending only on Ω, such that

$$\begin{aligned}H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div};\Omega)&\hookrightarrow H^s(\Omega)^3,\\ H(\mathrm{curl};\Omega)\cap H_0(\mathrm{div};\Omega)&\hookrightarrow H^s(\Omega)^3.\end{aligned}\tag{5}$$

The maximal value of s for which the above embeddings hold is closely related to the regularity properties of the Laplacian in polyhedra and only depends on the opening angles at the corners and edges of the domain, cf. [1]. In particular, for a convex domain, we can choose $s=1$, cf. Remark 4 below.

The embeddings in (5) imply that, for smooth coefficients μ and ε, the analytical solution $\mathbf{u}$ possesses the following regularity properties: $\varepsilon\mathbf{u}\in H^s(\Omega)^3$ and $\mu^{-1}\nabla\times\mathbf{u}\in H^s(\Omega)^3$, with s denoting the exponent in (5).

3 Mixed discontinuous Galerkin finite element approximation

In this section we define the mixed DG approximation to (1)–(3) proposed and analyzed in the recent paper [6]. To this end, we first introduce the following notation: we consider shape regular meshes $\mathcal{T}_h$ that partition Ω into tetrahedra $\{K\}$. We denote by h_K the diameter of $K \in \mathcal{T}_h$ and set $h = \max_{K\in\mathcal{T}_h} h_K$.

We write $\mathcal{F}_h^{\mathcal{I}}$ and $\mathcal{F}_h^{\mathcal{B}}$ to denote the union of all *interior* and *boundary* faces, respectively, and set $\mathcal{F}_h = \mathcal{F}_h^{\mathcal{I}} \cup \mathcal{F}_h^{\mathcal{B}}$. We define the *local mesh size* function $\mathtt{h}$ on $\mathcal{F}_h$ by setting $\mathtt{h}(\mathbf{x}) = \min\{h_{K^+}, h_{K^-}\}$, if $\mathbf{x}$ is in the interior of $\partial K^+ \cap \partial K^-$, where K^+ and K^- are two adjacent elements; if $\mathbf{x}$ is in the interior of $\partial K \cap \Gamma$, we set $\mathtt{h}(\mathbf{x}) = h_K$. For $f \subset \mathcal{F}_h^{\mathcal{I}}$, shared by two elements K^+ and K^- with outward unit normals $\mathbf{n}^{\pm}$, respectively, we define, with obvious notation, the *jumps* across f by $[\![\mathbf{v}]\!]_T = \mathbf{n}^+ \times \mathbf{v}^+ + \mathbf{n}^- \times \mathbf{v}^-$, $[\![\mathbf{v}]\!]_N = \mathbf{v}^+ \cdot \mathbf{n}^+ + \mathbf{v}^- \cdot \mathbf{n}^-$, $[\![q]\!]_N = q^+\mathbf{n}^+ + q^-\mathbf{n}^-$, and the *averages* by $\{\!\{\mathbf{v}\}\!\} = (\mathbf{v}^+ + \mathbf{v}^-)/2$ and $\{\!\{q\}\!\} = (q^+ + q^-)/2$. On $f \subset \mathcal{F}_h^{\mathcal{B}}$, we set $[\![\mathbf{v}]\!]_T = \mathbf{n} \times \mathbf{v}$, $[\![q]\!]_N = q\,\mathbf{n}$ and $\{\!\{\mathbf{v}\}\!\} = \mathbf{v}$.

The generic discontinuous finite element space is given by $P^\ell(\mathcal{T}_h) := \{u \in L_2(\Omega) : u|_K \in \mathbb{P}^\ell(K) \;\; \forall K \in \mathcal{T}_h\}$, $\ell \geq 0$, where $\mathbb{P}^\ell(K)$ is the space of polynomials of degree less than or equal ℓ on K. For $\ell \geq 1$, we use the finite element spaces $\mathbf{V}_h = P^\ell(\mathcal{T}_h)^3$ and $Q_h = P^{\ell+1}(\mathcal{T}_h)$.

The mixed DG finite element approximation to (1)–(3) is then defined as follows: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$a_h(\mathbf{u}_h, \mathbf{v}) + b_h(\mathbf{v}, p_h) = f_h(\mathbf{v}) \qquad \forall\, \mathbf{v} \in \mathbf{V}_h, \tag{6}$$
$$b_h(\mathbf{u}_h, q) - c_h(p_h, q) = 0 \qquad \forall\, q \in Q_h. \tag{7}$$

Here, a_h, b_h, c_h and f_h are defined, respectively, by

$$\begin{aligned}
a_h(\mathbf{u}, \mathbf{v}) &= \int_\Omega \mu^{-1} \nabla_h \times \mathbf{u} \cdot \nabla_h \times \mathbf{v}\, d\mathbf{x} + \int_{\mathcal{F}_h} \mathtt{a}\, [\![\mathbf{u}]\!]_T \cdot [\![\mathbf{v}]\!]_T \, ds \\
&\quad - \int_{\mathcal{F}_h} \Big([\![\mathbf{u}]\!]_T \cdot \{\!\{\mu^{-1}\nabla_h \times \mathbf{v}\}\!\} + [\![\mathbf{v}]\!]_T \cdot \{\!\{\mu^{-1}\nabla_h \times \mathbf{u}\}\!\} \Big)\, ds, \\
b_h(\mathbf{v}, p) &= -\int_\Omega \varepsilon \mathbf{v} \cdot \nabla_h p \, d\mathbf{x} + \int_{\mathcal{F}_h} \{\!\{\varepsilon \mathbf{v}\}\!\} \cdot [\![p]\!]_N \, ds, \\
c_h(p, q) &= \int_{\mathcal{F}_h} \mathtt{c}\, [\![p]\!]_N \cdot [\![q]\!]_N \, ds, \\
f_h(\mathbf{v}) &= \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x},
\end{aligned}$$

where ∇_h denotes the elementwise “nabla” operator.

Remark 1 *In order to incorporate the inhomogeneous Dirichlet boundary conditions $\mathbf{n} \times \mathbf{u} = \mathbf{g}$ on Γ, $\mathbf{g}$ in $L_2(\Gamma)^3$, within the above formulation, it* *is sufficient to simply modify the definition of the functional f_h as follows:*

$$f_h(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} - \int_{\mathcal{F}_h^\mathcal{B}} \mathbf{g} \cdot \mu^{-1} \nabla_h \times \mathbf{v} \, ds + \int_{\mathcal{F}_h^\mathcal{B}} \mathtt{a} \, \mathbf{g} \cdot (\mathbf{n} \times \mathbf{v}) \, ds.$$

The interior penalty parameters $\mathtt{a}$ and $\mathtt{c}$ are defined, respectively, on each face f in $\mathcal{F}_h$, as follows:

$$\mathtt{a} = \alpha \mu^{-1} \mathtt{h}^{-1} \quad \text{and} \quad \mathtt{c} = \gamma \varepsilon \mathtt{h}^{-1}. \tag{8}$$

Here, α and γ are positive real constants, which are independent of $\mathtt{h}$ and the coefficients μ and ε. To ensure stability of the underlying DG method (6)–(7) the parameter α, which is associated to the interior penalty discretization of the curl-curl operator, must be chosen sufficiently large, cf. [5,6].

The stability and *a priori* error analysis of the mixed DG approximation to (1)–(3) was developed in the recent article [6]. To this end, writing $\|\cdot\|^2_{0,\mathcal{F}_h}$ to denote the quantity $\sum_{f \subset \mathcal{F}_h} \|\cdot\|^2_{0,f}$, we introduce the DG-norm

$$|\!|\!| (\mathbf{v}, q) |\!|\!|^2_{\mathrm{DG}} = \|\mathbf{v}\|^2_{\mathrm{curl},h} + \|q\|^2_{1,h},$$

where

$$\begin{aligned} \|\mathbf{v}\|^2_{\mathrm{curl},h} &= \|\varepsilon^{1/2}\mathbf{v}\|^2_{0,\Omega} + \|\mu^{-1/2}\nabla_h \times \mathbf{v}\|^2_{0,\Omega} + \|\mu^{-1/2}\mathtt{h}^{-1/2}[\![\mathbf{v}]\!]_T\|^2_{0,\mathcal{F}_h}, \\ \|q\|^2_{1,h} &= \|\varepsilon^{1/2}\nabla_h q\|^2_{0,\Omega} + \|\varepsilon^{1/2}\mathtt{h}^{-1/2}[\![q]\!]_N\|^2_{0,\mathcal{F}_h}. \end{aligned}$$

With this notation, we quote the following error bound.

Theorem 3.1 *Assuming (4) holds, so that $p \equiv 0$, and that the interior penalty parameters $\mathtt{a}$ and $\mathtt{c}$ are selected as in (8), where α is a sufficiently large real constant. Then, the mixed DG approximation $(\mathbf{u}_h, p_h)$ obtained by (6)–(7) satisfies the following* a priori *error bound*

$$|\!|\!| (\mathbf{u} - \mathbf{u}_h, p - p_h) |\!|\!|_{\mathrm{DG}} \leq C \, h^{\min\{s,\ell\}} \left[\|\varepsilon \mathbf{u}\|_{s,\Omega} + \|\mu^{-1} \nabla \times \mathbf{u}\|_{s,\Omega} \right],$$

where C is a positive constant independent of the mesh size h, and s is the regularity exponent defined in (5).

Remark 2 *In contrast to the continuous level where $p \equiv 0$, when assumption (4) holds, on the discrete level, we generally have $p_h \neq 0$. Thus, it is necessary to control the size of $\|p_h\|_{1,h}$ in order to control the error $\|\mathbf{u} - \mathbf{u}_h\|_{\mathrm{curl},h}$ which is of primary interest.*

Remark 3 *The* a priori *error bound in Theorem 3.1 can be extended to arbitrary source terms* $\mathbf{f} \in L_2(\Omega)^3$*; cf. [6] for details. Indeed, provided that* $p \in H^{s+1}(\Omega)$*, the following error estimate holds*

$$\|\!|\, (\mathbf{u} - \mathbf{u}_h, p - p_h) \,\|\!|_{\mathrm{DG}} \leq C\, h^{\min\{s,\ell\}} \Big[\|\varepsilon \mathbf{u}\|_{s,\Omega} + \|\mu^{-1} \nabla \times \mathbf{u}\|_{s,\Omega} + \|p\|_{s+1,\Omega} \Big],$$

where C *is a positive constant independent of the mesh size* h*.*

We conclude this section by establishing a crucial property of the DG scheme (6)–(7). To this end, we define, for any $K \in \mathcal{T}_h$ and piecewise smooth function $\mathbf{v}$, the flux $\tau_K(\mathbf{v}) = \mathbf{n}_K \times (\mu^{-1} \nabla \times \mathbf{v})$ on ∂K, and its discrete version

$$\widehat{\tau}_K(\mathbf{v}) = \begin{cases} \mathbf{n}_K \times (\{\!\{\mu^{-1} \nabla \times \mathbf{v}\}\!\} - \mathtt{a}\, [\![\mathbf{v}]\!]_T) & \text{on } \partial K \setminus \Gamma, \\ \mathbf{n}_K \times (\mu^{-1} \nabla \times \mathbf{v} - \mathtt{a}\, (\mathbf{n}_K \times \mathbf{v})) & \text{on } \partial K \cap \Gamma. \end{cases}$$

The expression $\widehat{\tau}_K(\mathbf{v})$ is referred to as the *numerical flux*, cf. [2], for example. With this notation, we have the following local conservation property of the DG method (6)–(7).

Lemma 3.1 *For any* $K \in \mathcal{T}_h$*, if* $\mathbf{w}$ *is constant on* K *and zero outside* K*, then the following equality holds:*

$$\langle \tau_K(\mathbf{u}) - \widehat{\tau}_K(\mathbf{u}_h), \mathbf{w} \rangle_{\partial K} = \frac{1}{2} \langle [\![p_h]\!]_N, \varepsilon \mathbf{w} \rangle_{\partial K} - \int_K \nabla p_h \cdot \varepsilon \mathbf{w}\, d\mathbf{x}.$$

Proof. From the continuous problem, we immediately have $\langle \tau_K(\mathbf{u}), \mathbf{w} \rangle_{\partial K} = \int_K \mathbf{f} \cdot \mathbf{w}\, d\mathbf{x}$. Further, from the definition of the method, τ_K and $\widehat{\tau}_K$, we obtain $\langle \widehat{\tau}_K(\mathbf{u}_h), \mathbf{w} \rangle_{\partial K} + \frac{1}{2} \langle [\![p_h]\!]_N, \varepsilon \mathbf{w} \rangle_{\partial K} - \int_K \nabla p_h \cdot \varepsilon \mathbf{w}\, d\mathbf{x} = \int_K \mathbf{f} \cdot \mathbf{w}\, d\mathbf{x}$; subtracting these two expression gives the desired result. □

4 *A posteriori* error estimation

In this section we establish an *a posteriori* estimator for the error measured in terms of the energy norm $\|\!| \cdot \|\!|_{\mathrm{DG}}$.

Theorem 4.1 *We assume that (4) holds, so that* $p \equiv 0$*. Let* $(\mathbf{u}_h, p_h)$ *be the mixed discontinuous Galerkin approximation obtained by (6)–(7). Then there is a constant* $C > 0$ *independent of the mesh size* h*, such that*

$$\|\!|\, (\mathbf{u} - \mathbf{u}_h, p - p_h) \,\|\!|_{\mathrm{DG}} \leq C \Big(\sum_{K \in \mathcal{T}_h} \eta_K^2 \Big)^{1/2},$$

where the elemental error indicator η_K is given by

$$\eta_K^2 = h_K^{2s}\|\mathbf{f} - \nabla \times (\mu^{-1}\nabla \times \mathbf{u}_h) + \varepsilon\nabla p_h\|_{0,K}^2 + h_K^{2s-1}\|\widehat{\tau}_K(\mathbf{u}_h) - \tau_K(\mathbf{u}_h)\|_{0,\partial K}^2$$
$$+h_K^{-1}\|\mu^{-1/2}[\![\mathbf{u}_h]\!]_T\|_{0,\partial K}^2 + h_K\|\varepsilon^{1/2}[\![\mathbf{u}_h]\!]_N\|_{0,\partial K\setminus\Gamma}^2 + h_K^2\|\nabla\cdot(\varepsilon\mathbf{u}_h)\|_{0,K}^2$$
$$+\|\varepsilon^{1/2}\nabla p_h\|_{0,K}^2 + h_K^{-1}\|\varepsilon^{1/2}[\![p_h]\!]_N\|_{0,\partial K}^2,$$

with s denoting the regularity parameter in (5).

Remark 4 *Recall that the value of $s \in (1/2, 1]$ only depends on the opening angles at the corners and edges of the domain and that we can choose $s = 1$ for a convex domain. These limited smoothness properties of the function spaces for Maxwell's equations are the reason that some of the terms in the estimator η_K are weighted differently with respect to the local mesh size than their counterparts arising for diffusion problems. Finally, we note that for the two-dimensional analogue of problem (1)–(2) considered in the numerical experiments presented in Section 5, the parameter s can be chosen as $s \approx \min(1, \pi/\omega)$, where ω is the maximum opening angle of the polygon Ω under consideration.*

Remark 5 *To deal with the case of inhomogeneous boundary conditions $\mathbf{n} \times \mathbf{u} = \mathbf{g}$ on Γ, the definition of the numerical flux $\widehat{\tau}_K$ must be suitably modified on elements K in $\mathcal{T}_h$ such that $\partial K \cap \Gamma \neq \emptyset$. Then, if $\mathbf{g}$ is the tangential trace of a function belonging to $\mathbf{V}_h \cap H^1(\Omega)^3$, the error indicator is simply modified according to the new definition of the numerical fluxes on boundary elements, together with a corresponding modification of the jump indicators $h_K^{-1}\|\mu^{-1/2}[\![\mathbf{u}_h]\!]_T\|_{0,\partial K}$ on $\partial K \cap \Gamma$. For general $\mathbf{g} \in L_2(\Gamma)^3$, a data oscillation term which takes into account the error in the finite element approximation of the boundary datum must be included within the above error estimator.*

Remark 6 *To bound the term $\|\mu^{-1/2}\nabla_h \times (\mathbf{u} - \mathbf{u}_h)\|_{0,\Omega}$, we closely follow the approach in [3] which was recently developed for diffusion problems. In particular, we will make use of a suitable Helmholtz decomposition of the error and of the conservation property derived in Lemma 3.1. A similar Helmholtz decomposition, combined with a duality argument, is then employed to bound the term $\|\varepsilon^{1/2}(\mathbf{u}-\mathbf{u}_h)\|_{0,\Omega}$. In contrast to diffusion problems, we note that, due to the limited regularity of $\mathbf{u}$, it is not clear how to obtain sharper* a posteriori *bounds for the L_2–norm of the error (or for functional error estimation), even when standard duality techniques are employed.*

Proof of Theorem 4.1. For simplicity, we carry out the proof for $\mu = \varepsilon = 1$; the extension to general smooth coefficients follows analogously. From the definition of the norm $|\!|\!| \cdot |\!|\!|_{\mathrm{DG}}$, the continuity of the tangential trace of $\mathbf{u}$ at the inter-element boundaries, and the fact that $p = 0$, we have

$$\|\!|\,(\mathbf{u}-\mathbf{u}_h, p-p_h)\,|\!\|^2_{\mathrm{DG}} = \|\mathbf{u}-\mathbf{u}_h\|^2_{0,\Omega} + \sum_{K\in\mathcal{T}_h} \|\nabla\times(\mathbf{u}-\mathbf{u}_h)\|^2_{0,K}$$
$$+\|\mathrm{h}^{-1/2}[\![\mathbf{u}_h]\!]_T\|^2_{0,\mathcal{F}_h} + \sum_{K\in\mathcal{T}_h} \|\nabla p_h\|^2_{0,K} + \|\mathrm{h}^{-1/2}[\![p_h]\!]_N\|^2_{0,\mathcal{F}_h}.$$

Thereby, we only need to estimate $\|\mathbf{e}\|^2_{0,\Omega}$ and $\sum_{K\in\mathcal{T}_h}\|\nabla\times\mathbf{e}\|^2_{0,K}$, where we have set, for convenience, $\mathbf{e}=\mathbf{u}-\mathbf{u}_h$; to this end, we proceed as follows.

Step 1. Estimate of $\sum_{K\in\mathcal{T}_h}\|\nabla\times\mathbf{e}\|^2_{0,K}$. We follow the approach of [3] and write the L_2–orthogonal decomposition of $\nabla_h\times\mathbf{e}\in L_2(\Omega)^3$ as

$$\nabla_h\times\mathbf{e} = \nabla\varphi + \nabla\times\mathbf{w}, \tag{9}$$

with $\varphi\in H^1(\Omega)/\mathbb{R}$ and $\mathbf{w}\in H_0(\mathrm{curl};\Omega)\cap H(\mathrm{div}^0;\Omega)$, with $\|\mathbf{w}\|_{\mathrm{curl},\Omega}\le C\,\|\nabla\times\mathbf{w}\|_{0,\Omega}$. This follows from the decomposition $L_2(\Omega)^3=\nabla H^1(\Omega)\oplus H_0(\mathrm{div}^0;\Omega)$, and from [1, Theorem 3.17 and Corollary 3.19]. From the embeddings in (5), we have that $\mathbf{w}$ belongs to $H^s(\Omega)^3$ and satisfies $\|\mathbf{w}\|_{s,\Omega}\le C\,\|\nabla\times\mathbf{w}\|_{0,\Omega}$. Therefore, the following stability estimate of the decomposition (9) holds:

$$\|\nabla\varphi\|^2_0 + \|\mathbf{w}\|^2_{s,\Omega} \le C\sum_{K\in\mathcal{T}_h}\|\nabla\times\mathbf{e}\|^2_{0,K}. \tag{10}$$

Employing this decomposition, we have that

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}\|\nabla\times\mathbf{e}\|^2_{0,K} &= \sum_{K\in\mathcal{T}_h}\int_K \nabla\times\mathbf{e}\cdot\nabla\varphi\,d\mathbf{x} + \sum_{K\in\mathcal{T}_h}\int_K \nabla\times\mathbf{e}\cdot\nabla\times\mathbf{w}\,d\mathbf{x}\\ &\equiv \mathrm{I}+\mathrm{II}.\end{aligned} \tag{11}$$

We first deal with term I: using the definition of $\mathbf{e}$, the smoothness of $\mathbf{u}$ and φ, integration by parts and the fact that $\mathbf{n}\times\mathbf{u}=\mathbf{0}$ on ∂K, we have

$$\sum_{K\in\mathcal{T}_h}\int_K \nabla\times\mathbf{e}\cdot\nabla\varphi\,d\mathbf{x} = -\sum_{K\in\mathcal{T}_h}\langle\mathbf{n}_K\times(\nabla\varphi\times\mathbf{n}_K),\mathbf{n}_K\times\mathbf{u}_h\rangle_{\partial K}.$$

Next, fixing $\mathbf{v}$ in $\mathbf{V}_h\cap H_0(\mathrm{curl};\Omega)$, we observe that

$$\sum_{K\in\mathcal{T}_h}\langle\mathbf{n}_K\times(\nabla\varphi\times\mathbf{n}_K),\mathbf{n}_K\times\mathbf{v}\rangle_{\partial K}=0.$$

Thereby,

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}\int_K \nabla\times\mathbf{e}\cdot\nabla\varphi\,d\mathbf{x} &= \sum_{K\in\mathcal{T}_h}\langle\mathbf{n}_K\times(\nabla\varphi\times\mathbf{n}_K),\mathbf{n}_K\times(\mathbf{v}-\mathbf{u}_h)\rangle_{\partial K}\\ &\le \sum_{K\in\mathcal{T}_h}\|\mathbf{n}_K\times\nabla\varphi\|_{-1/2,\partial K}\|\mathbf{n}_K\times(\mathbf{v}-\mathbf{u}_h)\|_{1/2,\partial K}.\end{aligned}$$

Using the trace estimate

$$\|\mathbf{n}_K \times \mathbf{r}\|_{-1/2,\partial K} \leq C[\|\mathbf{r}\|_{0,K} + h_K \|\nabla \times \mathbf{r}\|_{0,K}], \qquad \mathbf{r} \in H(\mathrm{curl}; K),$$

together with (10), we obtain

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} \int_K \nabla \times \mathbf{e} \cdot \nabla\varphi \, d\mathbf{x} &\leq C \, \|\nabla\varphi\|_{0,\Omega} \Big(\sum_{K\in\mathcal{T}_h} \|\mathbf{n}_K \times (\mathbf{v} - \mathbf{u}_h)\|^2_{1/2,\partial K} \Big)^{\frac{1}{2}} \\ &\leq C \, \|\nabla_h \times \mathbf{e}\|_{0,\Omega} \Big(\sum_{K\in\mathcal{T}_h} \|\mathbf{n}_K \times (\mathbf{v} - \mathbf{u}_h)\|^2_{1/2,\partial K} \Big)^{\frac{1}{2}}.\end{aligned}$$

Inverse estimates and the discrete trace inequality yield

$$\begin{aligned}&\sum_{K\in\mathcal{T}_h} \|\mathbf{n}_K \times (\mathbf{v} - \mathbf{u}_h)\|^2_{1/2,\partial K} \leq C \sum_{K\in\mathcal{T}_h} h_K^{-1} \|\mathbf{n}_K \times (\mathbf{v} - \mathbf{u}_h)\|^2_{0,\partial K} \\ &\leq C \sum_{K\in\mathcal{T}_h} h_K^{-2} \|\mathbf{v} - \mathbf{u}_h\|^2_{0,K} \leq C \sum_{K\in\mathcal{T}_h} \Big(h_K^{-2} \|\mathbf{v} - \mathbf{u}_h\|^2_{0,K} + \|\nabla \times (\mathbf{v} - \mathbf{u}_h)\|^2_{0,K} \Big).\end{aligned}$$

A careful inspection of the proof of Theorem 4.1 in [6] shows that

$$\inf_{\mathbf{v}\in\mathbf{V}_h\cap H_0(\mathrm{curl};\Omega)} \sum_{K\in\mathcal{T}_h} \Big(h_K^{-2} \|\mathbf{v}-\mathbf{u}_h\|^2_{0,K} + \|\nabla\times(\mathbf{v}-\mathbf{u}_h)\|^2_{0,K} \Big) \leq C \, \|\mathtt{h}^{-1/2} [\![\mathbf{u}_h]\!]_T\|^2_{0,\mathcal{F}_h}.$$

As a consequence, we obtain

$$\sum_{K\in\mathcal{T}_h} \int_K \nabla \times \mathbf{e} \cdot \nabla\varphi \, d\mathbf{x} \leq C \, \|\nabla_h \times \mathbf{e}\|_{0,\Omega} \Big(\sum_{K\in\mathcal{T}_h} h_K^{-1} \|[\![\mathbf{u}_h]\!]_T\|^2_{0,\partial K} \Big)^{1/2}. \tag{12}$$

In order to deal with term II, we first write $\mathbf{w}_h$ to denote the L_2–projection of $\mathbf{w}$ onto $P^0(\mathcal{T}_h)$, so that

$$\|\mathbf{w} - \mathbf{w}_h\|_{0,K} + h_K^{1/2} \|\mathbf{w} - \mathbf{w}_h\|_{0,\partial K} \leq C \, h_K^s \|\mathbf{w}\|_{s,K} \quad \forall K \in \mathcal{T}_h. \tag{13}$$

Thereby, replacing $\mathbf{w}$ by $\mathbf{w} - \mathbf{w}_h$ in term II, integrating by parts and using the fact that $\mathbf{u}$ satisfies (1) with $p \equiv 0$, we obtain

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} \int_K \nabla \times \mathbf{e} \cdot \nabla \times \mathbf{w} \, d\mathbf{x} &= \sum_{K\in\mathcal{T}_h} \int_K (\mathbf{f} - \nabla \times \nabla \times \mathbf{u}_h) \cdot (\mathbf{w} - \mathbf{w}_h) \, d\mathbf{x} \\ &\quad - \sum_{K\in\mathcal{T}_h} \langle \tau_K(\mathbf{e}), \mathbf{w} \rangle_{\partial K} + \sum_{K\in\mathcal{T}_h} \langle \tau_K(\mathbf{e}), \mathbf{w}_h \rangle_{\partial K}.\end{aligned}$$

From the continuity properties of $\mathbf{u}$ and $\mathbf{w}$, and the fact that the numerical fluxes $\widehat{\tau}_K$ are single-valued at inter-element boundaries, we have

$$\sum_{K\in\mathcal{T}_h} \langle \tau_K(\mathbf{e}), \mathbf{w}\rangle_{\partial K} = \sum_{K\in\mathcal{T}_h} \langle \widehat{\tau}_K(\mathbf{u}_h) - \tau_K(\mathbf{u}_h), \mathbf{w}\rangle_{\partial K}.$$

Employing Lemma 3.1, we can write

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} \langle \tau_K(\mathbf{e}), \mathbf{w}_h\rangle_{\partial K} = &\sum_{K\in\mathcal{T}_h} \langle \widehat{\tau}_K(\mathbf{u}_h) - \tau_K(\mathbf{u}_h), \mathbf{w}_h\rangle_{\partial K} \\ &+ \sum_{K\in\mathcal{T}_h} \frac{1}{2}\langle [\![p_h]\!]_N, \mathbf{w}_h\rangle_{\partial K} - \sum_{K\in\mathcal{T}_h} \int_K \nabla p_h \cdot \mathbf{w}_h \, d\mathbf{x}.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} \int_K \nabla\times\mathbf{e}\cdot\nabla\times\mathbf{w}\, d\mathbf{x} = &\sum_{K\in\mathcal{T}_h} \int_K (\mathbf{f} - \nabla\times\nabla\times\mathbf{u}_h + \nabla p_h)\cdot(\mathbf{w}-\mathbf{w}_h)\, d\mathbf{x} \\ &- \sum_{K\in\mathcal{T}_h} \langle \widehat{\tau}_K(\mathbf{u}_h) - \tau_K(\mathbf{u}_h), \mathbf{w}-\mathbf{w}_h\rangle_{\partial K} - \sum_{K\in\mathcal{T}_h} \frac{1}{2}\langle [\![p_h]\!]_N, \mathbf{w}-\mathbf{w}_h\rangle_{\partial K},\end{aligned}$$

where we also have added and subtracted $\sum_{K\in\mathcal{T}_h}\int_K \nabla p_h \cdot \mathbf{w}\, d\mathbf{x}$, used integration by parts and the fact that $\nabla\cdot\mathbf{w} = 0$. Finally, using estimates (13) and (10), we obtain the following bound:

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} \int_K \nabla\times\mathbf{e}\cdot\nabla\times\mathbf{w}\, d\mathbf{x} \le C\Big(&\sum_{K\in\mathcal{T}_h} h_K^{2s}\|\mathbf{f} - \nabla\times\nabla\times\mathbf{u}_h + \nabla p_h\|_{0,K}^2 \\ &+ \sum_{K\in\mathcal{T}_h} h_K^{2s-1}\|\widehat{\tau}_K(\mathbf{u}_h) - \tau_K(\mathbf{u}_h)\|_{0,\partial K}^2 + \sum_{K\in\mathcal{T}_h} h_K^{2s-1}\|[\![p_h]\!]_N\|_{0,\partial K}^2\Big)^{1/2} \\ &\times\|\nabla_h\times\mathbf{e}\|_{0,\Omega}. \qquad (14)\end{aligned}$$

Substituting (12) and (14) into (11) and dividing through by $\|\nabla_h\times\mathbf{e}\|_{0,\Omega}$, we conclude that

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} \|\nabla\times\mathbf{e}\|_{0,K} \le C\Big(&\sum_{K\in\mathcal{T}_h} h_K^{2s}\|\mathbf{f} - \nabla\times\nabla\times\mathbf{u}_h + \nabla p_h\|_{0,K}^2 \\ &+ \sum_{K\in\mathcal{T}_h} h_K^{2s-1}\|\widehat{\tau}_K(\mathbf{u}_h) - \tau_K(\mathbf{u}_h)\|_{0,\partial K}^2 + \sum_{K\in\mathcal{T}_h} h_K^{-1}\|[\![\mathbf{u}_h]\!]_T\|_{0,\partial K}^2 \\ &+ \sum_{K\in\mathcal{T}_h} h_K^{2s-1}\|[\![p_h]\!]_N\|_{0,\partial K}^2\Big)^{1/2}.\end{aligned}$$

Step 2. Estimate of $\|\mathbf{e}\|_{0,\Omega}$. We decompose $\mathbf{e}\in L_2(\Omega)^3$ into $\mathbf{e} = \nabla\psi + \mathbf{\Psi}$, with $\psi\in H_0^1(\Omega)$ and $\mathbf{\Psi}\in H(\mathrm{div}^0;\Omega)$. This decomposition is L_2–orthogonal; hence

$$\|\mathbf{e}\|_{0,\Omega}^2 = \|\nabla\psi\|_{0,\Omega}^2 + \|\mathbf{\Psi}\|_{0,\Omega}^2.$$

Since $\nabla\psi$ is curl-free and has zero tangential component along the boundary, we have that $\nabla_h \times \mathbf{\Psi} = \nabla_h \times \mathbf{e}$ and $\nabla \cdot \mathbf{\Psi} = 0$.

In order to bound $\|\mathbf{\Psi}\|_{0,\Omega}$, we consider the problem of finding $\mathbf{z}$ such that $\nabla \times \nabla \times \mathbf{z} = \mathbf{\Psi}$ and $\nabla \cdot \mathbf{z} = 0$ in Ω, with $\mathbf{n} \times \mathbf{z} = \mathbf{0}$ on Γ. We notice that, from (5), $\mathbf{z} \in H^s(\Omega)^3$, $\nabla \times \mathbf{z} \in H^s(\Omega)^3$, and $\|\mathbf{z}\|_{s,\Omega} + \|\nabla \times \mathbf{z}\|_{s,\Omega} \leq C\, \|\mathbf{\Psi}\|_{0,\Omega}$. By integration by parts and elementary manipulations, taking into account that $\nabla \times \mathbf{z} \in H(\mathrm{curl};\Omega)$, we have

$$
\begin{aligned}
\|\mathbf{\Psi}\|_{0,\Omega}^2 &= \int_\Omega \mathbf{\Psi} \cdot \mathbf{e}\, d\mathbf{x} = \sum_{K\in\mathcal{T}_h} \int_K \nabla \times \nabla \times \mathbf{z} \cdot \mathbf{e}\, d\mathbf{x} \\
&= \sum_{K\in\mathcal{T}_h} \int_K \nabla \times \mathbf{z} \cdot \nabla \times \mathbf{e}\, d\mathbf{x} + \sum_{K\in\mathcal{T}_h} \langle \tau_K(\mathbf{z}), \mathbf{e}\rangle_{\partial K} \\
&= \sum_{K\in\mathcal{T}_h} \int_K \nabla \times \mathbf{z} \cdot \nabla \times \mathbf{e}\, d\mathbf{x} + \sum_{K\in\mathcal{T}_h} \frac{1}{2} \langle \nabla \times \mathbf{z}, [\![\mathbf{u}_h]\!]_T\rangle_{\partial K\setminus\Gamma} \\
&\quad + \sum_{K\in\mathcal{T}_h} \langle \nabla \times \mathbf{z}, \mathbf{n}_K \times \mathbf{u}_h\rangle_{\partial K\cap\Gamma}.
\end{aligned}
$$

From the stability estimate $\|\nabla \times \mathbf{z}\|_{s,\Omega} \leq C\, \|\mathbf{\Psi}\|_{0,\Omega}$, the weighted Cauchy-Schwarz inequality, and the trace inequality $\|\nabla \times \mathbf{z}\|_{0,\partial K}^2 \leq C\, h_K^{-1} \|\nabla \times \mathbf{z}\|_{s,K}^2$ (see [9]), we obtain, after division by $\|\mathbf{\Psi}\|_{0,\Omega}$, the following

$$
\|\mathbf{\Psi}\|_{0,\Omega} \leq C \Big(\sum_{K\in\mathcal{T}_h} \|\nabla \times \mathbf{e}\|_{0,K}^2 + \sum_{K\in\mathcal{T}_h} h_K^{-1} \|[\![\mathbf{u}_h]\!]_T\|_{0,\partial K}^2 \Big)^{1/2}.
$$

The first term on the right–hand of the above inequality can then be bounded in terms of computable quantities using the estimate obtained in Step 1. To deal with the term $\|\nabla\psi\|_{0,\Omega}$, we first note that

$$
\|\nabla\psi\|_{0,\Omega}^2 = \int_\Omega \nabla\psi \cdot (\nabla\psi + \mathbf{\Psi})\, d\mathbf{x} = \int_\Omega \nabla\psi \cdot \mathbf{e}\, d\mathbf{x} = -b_h(\mathbf{e},\psi).
$$

Further, we let ψ_h be the standard Clément interpolation of ψ that satisfies

$$
\|\psi - \psi_h\|_{0,K} + h_K \|\psi - \psi_h\|_{1,K} + h_K^{1/2} \|\psi - \psi_h\|_{0,\partial K} \leq C\, h_K \|\psi\|_{1,\delta K}, \qquad (15)
$$

where δK is the patch of elements adjacent to K (see [8, Section 5.6.1]). Since $[\![\psi - \psi_h]\!]_N = \mathbf{0}$, using Galerkin orthogonality, integration by parts and the fact that $\nabla \cdot \mathbf{u} = 0$, we can write

$$
\begin{aligned}
b_h(\mathbf{e},\psi) = b_h(\mathbf{e},\psi - \psi_h) = &- \sum_{K\in\mathcal{T}_h} \int_K \nabla \cdot \mathbf{u}_h (\psi - \psi_h)\, d\mathbf{x} \\
&+ \sum_{K\in\mathcal{T}_h} \frac{1}{2} \langle \psi - \psi_h, [\![\mathbf{u}_h]\!]_N\rangle_{0,\partial K\setminus\Gamma}.
\end{aligned}
$$

Therefore, using the weighted Cauchy-Schwarz inequality and (15), after dividing by $\|\nabla\psi\|_{0,\Omega}$, we obtain

$$\|\nabla\psi\|_{0,\Omega} \leq C\left(\sum_{K\in\mathcal{T}_h} h_K^2\|\nabla\cdot\mathbf{u}_h\|_{0,K}^2 + \sum_{K\in\mathcal{T}_h} h_K\|[\![\mathbf{u}_h]\!]_N\|_{0,\partial K\setminus\Gamma}^2\right)^{1/2}.$$

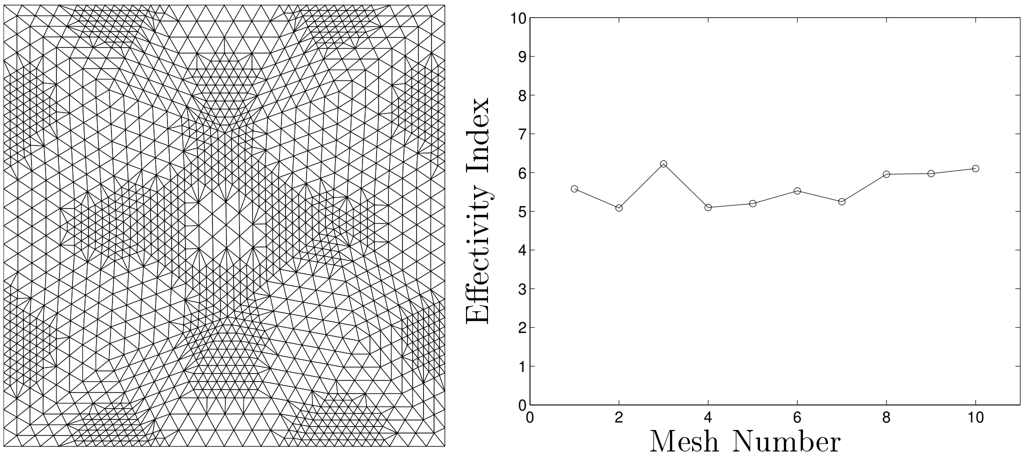
Conclusion of the proof. Combining the results obtained in Step 1 and Step 2 gives the estimate stated in the theorem. □

5 Numerical experiments

In this section we present a series of numerical examples to illustrate the practical performance of the proposed *a posteriori* error estimator within an automatic adaptive refinement procedure. Here, we restrict ourselves to two–dimensional model problems with constant coefficients $\mu \equiv \varepsilon \equiv 1$. Furthermore, in each of the examples shown below, we set the polynomial degree ℓ equal to 1 and the constants α and γ arising in the interior penalty parameters, cf. (8), equal to 10 and 1, respectively. The adaptive meshes are constructed by employing the fixed fraction strategy, with refinement and derefinement fractions set to 25% and 10%, respectively. Here, the emphasis will be to demonstrate the asymptotic exactness of the proposed *a posteriori* error indicator on non-uniform adaptively refined meshes. Thereby, as in [3], we set the constant C arising in Theorem 4.1 equal to one and ensure that the corresponding effectivity indices are roughly constant on all of the meshes employed; in general, to ensure the reliability of the error estimator, C must be numerically determined for the underlying problem at hand, cf. [4], for example.

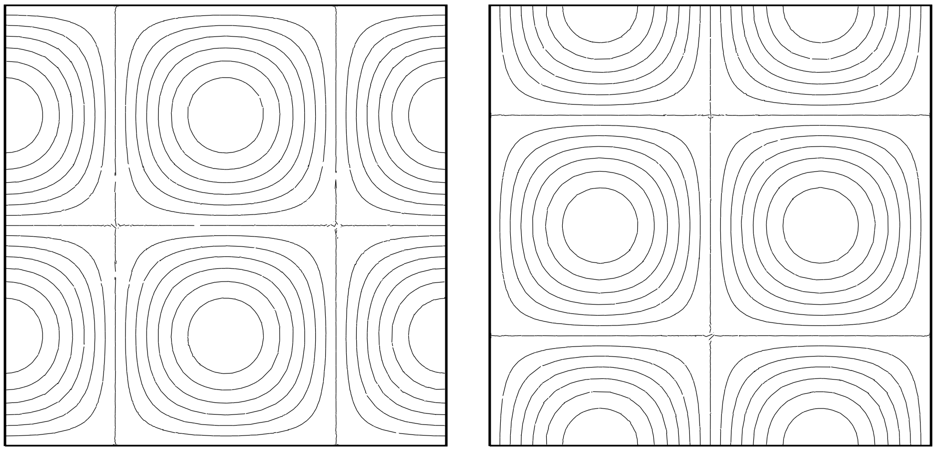
5.1 Example 1

In this first example, we let Ω be the convex domain $(-1,1)^2$ (i.e. so that $s = 1$ in (5)) and $\mathbf{f} = (2\cos(\pi x)\sin(\pi y)\pi^2, -2\cos(\pi y)\sin(\pi x)\pi^2)$; thereby, the analytical solution to (1) is given by $\mathbf{u} = (\cos(\pi x)\sin(\pi y), -\cos(\pi y)\sin(\pi x))$ and $p = 0$. In Fig. 1(a) we show the mesh generated using the proposed *a posteriori* error indicator after 8 adaptive refinement steps. Here, we see that while the mesh has been largely uniformly refined throughout the entire computational domain, additional refinement has been performed where the solution has local maxima and minima; cf. Fig. 2 where we plot the isolines of the numerical approximation $\mathbf{u}_h$ computed on this mesh. Finally, in Fig. 1(b) we show the history of the effectivity indices on each of the meshes generated by our adaptive algorithm. Here, we observe that the error estimator



(a) (b)

Fig. 1. *Example 1.* (a) Computational mesh after 8 adaptive refinements with 3210 elements; (b) Effectivity Indices.



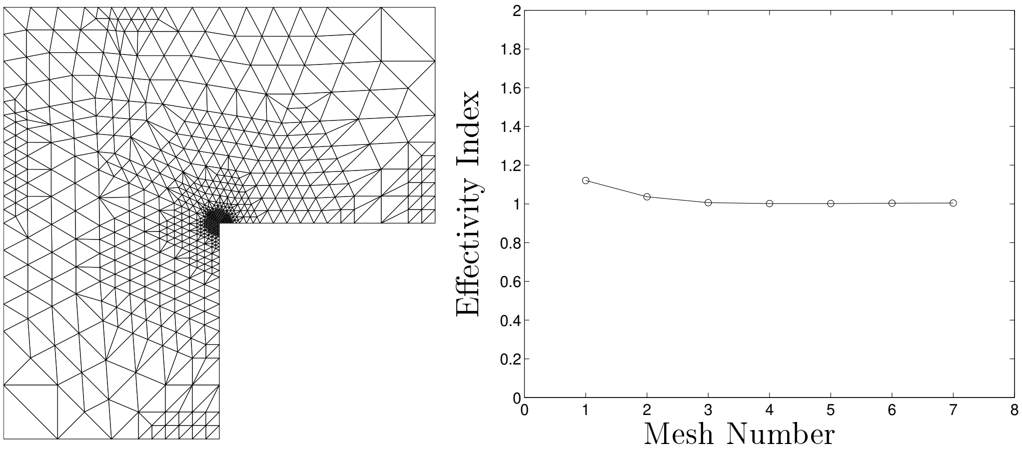
(a) (b)

Fig. 2. *Example 1.* Numerical approximation to: (a) u_1; (b) u_2.

over-estimates the true error by a consistent factor of between 5–6, thereby confirming the asymptotic exactness of the proposed error indicator for this smooth problem.

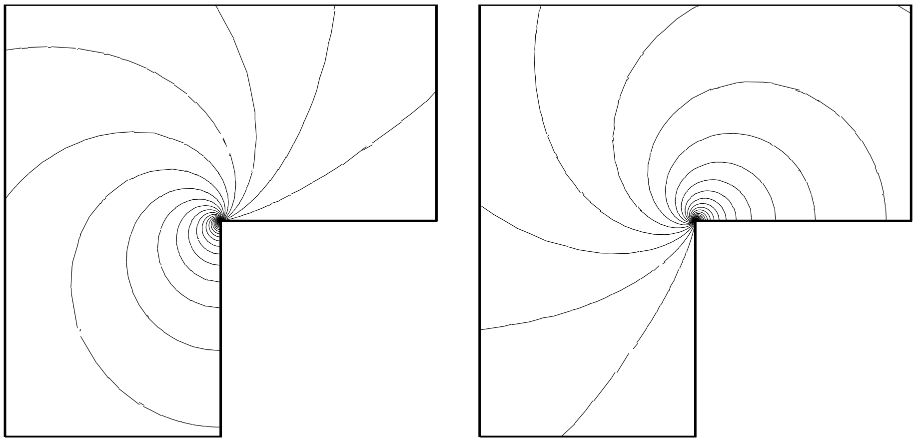
5.2 Example 2

In this final example, we select Ω to be the (non-convex) L–shaped domain with vertices $(1,0)$, $(1,1)$, $(-1,1)$, $(-1,-1)$, $(0,-1)$ and $(0,0)$. Furthermore, we set $\mathbf{f}=\mathbf{0}$ and impose $\mathbf{n}\times\mathbf{u}=\mathbf{g}$ on Γ, where $\mathbf{g}$ is chosen so that $\mathbf{u}(x,y)=\nabla(r^{2/3}\sin(2\vartheta/3))$, in terms of the polar coordinates (r,ϑ); thereby, on the basis of Remark 4, we set $s=2/3$. In Fig. 3(a) we show the mesh generated



(a) (b)

Fig. 3. *Example 2.* (a) Computational mesh after 6 adaptive refinements with 1308 elements; (b) Effectivity Indices.



(a) (b)

Fig. 4. *Example 2.* Numerical approximation to: (a) u_1; (b) u_2.

using the local error indicators η_K after 6 adaptive refinement steps. Here, we see that the mesh has been largely refined in the vicinity of the re-entrant corner located at the origin. In Fig. 3(b) we show the history of the effectivity indices on each of the meshes generated by our adaptive algorithm. As in the previous example, we observe that the error estimator over-estimates the true error by a consistent factor, thereby confirming the asymptotic exactness of the proposed error indicator; however, here we see that for this non-smooth example, the effectivity indices are smaller than the corresponding quantities computed for the smooth example considered in Section 5.1. Finally, in Fig. 2 we plot the isolines of the numerical approximation $\mathbf{u}_h$ computed on the final mesh.

6 Concluding remarks

In this article we have derived a residual–based energy norm *a posteriori* error bound for the mixed DG approximation of the time–harmonic Maxwell operator. The analysis is based on employing Helmholtz decompositions of the error, together with the conservation properties of the underlying method. Numerical experiments presented in this article clearly highlight the asymptotic exactness of the proposed estimator on adaptively refined meshes.

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