Discontinuous Galerkin Methods for Maxwell's Equations

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We present discontinuous Galerkin methods for the numerical discretization of time-harmonic eddy current problems in conducting and insulating materials. The involved operators are discretized by using suitable variants of the classical interior penalty technique; here, the divergence-free constraint on the electric field in insulating materials is imposed either by using a regularization approach or by a Lagrange multiplier technique.

1 Introduction

In recent years, there has been a considerable interest in nonconforming finite element methods that are based on discontinuous piecewise polynomial approximation spaces. Such approaches are referred to as discontinuous Galerkin (DG) methods (for a recent overview, see [1]). The main advantages of these methods lie in their conservation properties, their ability to treat a wide range of problems within the same unified framework, and their great flexibility in the mesh-design. Indeed, DG methods can easily handle non-matching grids and non-uniform, even anisotropic, polynomial approximation degrees. Moreover, orthogonal bases can easily be constructed which lead to diagonal mass matrices; this is particularly advantageous in unsteady problems. Finally, DG methods can easily be parallelized.

Here, following [2] and [3], we describe DG methods for time-harmonic eddy-current problems. We start by considering the case of conducting materials, where the involved forms are coercive in natural norms; these methods are essentially based on discretizing the curl-curl operator in a DG fashion. For simplicity, we only consider the classical Interior Penalty (IP) technique. Next, we consider the case of insulating materials; the main difficulty there is the treatment of the divergence-free constraint on the electric field. We present two methods: the first one is based on a regularization approach [2], and the second one uses recently devised mixed DG discretizations [3]. We point out that the case of heterogeneous media can be easily dealt with by combining the methods for conducting and insulating materials. In the regularization approach no additional unknowns are introduced, but, as for its conforming counterpart, the method needs appropriate corrections in the neighborhood of solution singularities. On the other hand, the mixed formulation also requires the discretization of a (scalar) Lagrange multiplier variable, but leads to optimal a priori bounds also for strongly singular analytical solutions. Before describing the methods, we first introduce some notation.

Domain and Functional Spaces. The computational domain Ω is assumed to be a simply connected bounded Lipschitz polyhedron in $\mathbb{R}^3$, with connected boundary $\Gamma = \partial\Omega$, and outward normal unit vector $\mathbf{n}$ to Γ. For a domain $D \subset \mathbb{R}^d$, we denote by $H^s(D)$ the Sobolev space of order s. We write $H_0^1(D)$ for the subspace of $H^1(D)$ of functions with zero trace on ∂D, and set $L^2(D) = H^0(D)$. We also make use of the spaces $H(\mathrm{div}; D) = \{\mathbf{v} \in L^2(D)^3 : \nabla \cdot \mathbf{v} \in L^2(D)\}$ and $H(\mathbf{curl}; D) = \{\mathbf{v} \in L^2(D)^3 : \nabla \times \mathbf{v} \in L^2(D)^3\}$.

Meshes, Finite Element Spaces and Traces. We consider shape regular meshes $\mathcal{T}_h$ that partition Ω into tetrahedra or parallelepipeds, with possible hanging nodes. We denote by h_K the diameter of $K \in \mathcal{T}_h$ and set $h = \max_K h_K$. An *interior face* of $\mathcal{T}_h$ is the (non-empty) two-dimensional interior of $\partial K^+ \cap \partial K^-$, where K^+ and K^- are two adjacent elements, not necessarily matching. A *boundary face* of $\mathcal{T}_h$ is the (non-empty) two-dimensional interior of $\partial K \cap \partial\Omega$, where K is a boundary element. We denote by $\mathcal{E}_h^{\mathcal{I}}$ and $\mathcal{E}_h^{\mathcal{D}}$ the union of all interior and boundary faces, respectively, and set $\mathcal{E}_h = \mathcal{E}_h^{\mathcal{I}} \cup \mathcal{E}_h^{\mathcal{D}}$. We define the *local meshsize* function $\mathtt{h}$ on $\mathcal{E}_h$ by setting $\mathtt{h}(\mathbf{x}) = \min\{h_{K^+}, h_{K^-}\}$, if $\mathbf{x}$ is in the interior of $\partial K^+ \cap \partial K^-$, and $\mathtt{h}(\mathbf{x}) = h_K$, if $\mathbf{x}$ is in the interior of $\partial K \cap \partial\Omega$.

The generic discontinuous finite element space is given by

$$\mathcal{S}^\ell(\mathcal{T}_h) := \{u \in L^2(\Omega) : u|_K \in S^\ell(K) \ \ \forall K \in \mathcal{T}_h\},$$

where $S^\ell(K)$ is the space of polynomials of degree $\leq \ell$ or of degree $\leq \ell$ in each variable on K, if K is a tetrahedron or a parallelepiped, respectively.

For $e \subset \mathcal{E}_h^{\mathcal{I}}$ shared by two elements K^+ and K^- with outward unit normals $\mathbf{n}^\pm$, respectively, we define, with obvious notation, the *jumps* across e by $[\![\mathbf{v}]\!]_T = \mathbf{n}^+ \times \mathbf{v}^+ + \mathbf{n}^- \times \mathbf{v}^-$, $[\![\mathbf{v}]\!]_N = \mathbf{v}^+ \cdot \mathbf{n}^+ + \mathbf{v}^- \cdot \mathbf{n}^-$, $[\![q]\!]_N = q^+\mathbf{n}^+ + q^-\mathbf{n}^-$, and the *averages* by $\{\!\{\mathbf{v}\}\!\} = (\mathbf{v}^+ + \mathbf{v}^-)/2$ and $\{\!\{q\}\!\} = (q^+ + q^-)/2$. On $e \subset \mathcal{E}_h^{\mathcal{D}}$, we set $[\![\mathbf{v}]\!]_T = \mathbf{n} \times \mathbf{v}$, $[\![q]\!]_N = q\,\mathbf{n}$ and $\{\!\{\mathbf{v}\}\!\} = \mathbf{v}$.

2 Conducting Materials

We consider the time-harmonic eddy current problem in conducting materials: find the (complex-valued) electric field phasor $\mathbf{E} \in H(\mathbf{curl}; \Omega)$ such that

$$\nabla \times \nabla \times \mathbf{E} + i\omega\sigma\,\mathbf{E} = \mathbf{J} \text{ in } \Omega, \qquad \mathbf{n} \times \mathbf{E} = \mathbf{0} \text{ on } \Gamma. \tag{1}$$

Here, $\omega \neq 0$ is a given frequency, $\sigma = \sigma(\mathbf{x})$ the (uniformly positive definite) electric conductivity tensor, and $\mathbf{J} := -i\omega\,\mathbf{J}_s$, where $\mathbf{J}_s$ is a given divergence-free source field. For simplicity, the magnetic permeability and electric permittivity are set equal to 1.

Let $\mathbf{V}_h = S^\ell(\mathcal{T}_h)^3$, for a partition $\mathcal{T}_h$ of Ω and $\ell \geq 1$. The Interior Penalty DG-FEM (IPDG-FEM) for (1) consists in finding $\mathbf{E}_h \in \mathbf{V}_h$ such that

$$\mathcal{A}_h(\mathbf{E}_h, \mathbf{v}) + i\omega \int_\Omega \sigma\,\mathbf{E}_h \cdot \mathbf{v}\,d\mathbf{x} = \int_\Omega \mathbf{J} \cdot \overline{\mathbf{v}}\,d\mathbf{x} \tag{2}$$

for all $\mathbf{v} \in \mathbf{V}_h$, where

$$\begin{aligned}\mathcal{A}_h(\mathbf{E}, \mathbf{v}) = &\int_\Omega \nabla_h \times \mathbf{E} \cdot \nabla_h \times \overline{\mathbf{v}}\,d\mathbf{x} - \int_\mathcal{E} [\![\mathbf{E}]\!]_T \cdot \{\!\{\nabla_h \times \overline{\mathbf{v}}\}\!\}\,ds \\ &- \int_\mathcal{E} [\![\overline{\mathbf{v}}]\!]_T \cdot \{\!\{\nabla_h \times \mathbf{E}\}\!\}\,ds + \int_\mathcal{E} \mathtt{a}\,[\![\mathbf{E}]\!]_T \cdot [\![\overline{\mathbf{v}}]\!]_T\,ds.\end{aligned} \tag{3}$$

The form $\mathcal{A}_h$ corresponds to the IP discretization of the curl-curl operator. The parameter $\mathtt{a} \in L^\infty(\mathcal{E})$ is the discontinuity stabilization function chosen as $\mathtt{a} = \alpha\,\mathtt{h}^{-1}$, where α is a positive constant independent of meshsize.

The method in (2) is consistent and stable, provided that $\alpha \geq \alpha_{\min}$, for a constant $\alpha_{\min} > 0$ independent of the meshsize. The error analysis of this method can be carried out in the framework of [5] and, for smooth analytical solutions, leads to optimal error estimates in the DG-norm $|\!|\!|\,\mathbf{v}\,|\!|\!|_h^2 = \|\sigma^{\frac{1}{2}}\mathbf{v}\|_{L^2(\Omega)^3}^2 + \|\nabla_h \times \mathbf{v}\|_{L^2(\Omega)^3}^2 + \|\mathtt{h}^{-\frac{1}{2}}[\![\mathbf{v}]\!]_T\|_{L^2(\mathcal{E})^3}^2$; cf. [2, 4].

3 Insulating Materials

In insulating materials ($\sigma \equiv 0$), problem (1) has to be completed with a divergence-free constraint: find the (real-valued) field $\mathbf{E} \in H(\mathbf{curl}; \Omega) \cap H(\mathrm{div}; \Omega)$ such that

$$\nabla \times \nabla \times \mathbf{E} = \mathbf{J} \text{ in } \Omega, \qquad \nabla \cdot \mathbf{E} = 0 \text{ in } \Omega, \qquad \mathbf{n} \times \mathbf{E} = \mathbf{0} \text{ on } \Gamma. \tag{4}$$

In the following subsections we consider two approaches for imposing $\nabla\cdot\mathbf{E} = 0$.

3.1 IPDG-FEM Based on a Regularization Approach

The regularized IPDG-FEM reads as follows: find $\mathbf{E}_h \in \mathbf{V}_h$ such that

$$\mathcal{A}_h(\mathbf{E}_h, \mathbf{v}) + \mathcal{A}_h^*(\mathbf{E}_h, \mathbf{v}) = \int_\Omega \mathbf{J} \cdot \mathbf{v}\,d\mathbf{x}$$

for all $\mathbf{v} \in \mathbf{V}_h$, where $\mathcal{A}_h(\mathbf{E}, \mathbf{v})$ is the curl-curl form in (3), and $\mathcal{A}_h^*$ is given by the following analogous IP discretization of the grad-div operator:

$$\mathcal{A}_h^*(\mathbf{E},\mathbf{v}) = \int_\Omega \nabla_h \cdot \mathbf{E}\, \nabla_h \cdot \mathbf{v}\, d\mathbf{x} - \int_{\mathcal{E}_\mathcal{I}} [\![\mathbf{E}]\!]_N \,\{\!\{\, \nabla_h \cdot \mathbf{v}\}\!\}\, ds$$
$$- \int_{\mathcal{E}_\mathcal{I}} [\![\mathbf{v}]\!]_N \,\{\!\{\, \nabla_h \cdot \mathbf{E}\}\!\}\, ds + \int_{\mathcal{E}_\mathcal{I}} \mathtt{b}\, [\![\mathbf{E}]\!]_N\, [\![\mathbf{v}]\!]_N\, ds.$$

The parameters $\mathtt{a}$ and $\mathtt{b}$ are chosen as $\mathtt{a} = \alpha\, \mathtt{h}^{-1}$ and $\mathtt{b} = \beta\, \mathtt{h}^{-1}$, with positive α and β, large enough to guarantee stability. Optimal error estimates in the DG-norm, in the case of smooth analytical solutions, have been proven in [2, 4]. Numerical experiments confirming the sharpness of these theoretical results can be found in [4].

3.2 Mixed DG Discretization

Alternatively, we can rewrite problem (4) in mixed form: find $\mathbf{E} \in H(\mathbf{curl};\Omega)$ and a multiplier $p \in H_0^1(\Omega)$ such that

$$\nabla \times \nabla \times \mathbf{E} - \nabla p = \mathbf{J} \text{ in } \Omega, \qquad \nabla \cdot \mathbf{E} = 0 \text{ in } \Omega, \qquad \mathbf{n} \times \mathbf{E} = \mathbf{0} \text{ on } \Gamma. \quad (5)$$

Again, we define $\mathbf{V}_h = S^\ell(\mathcal{T}_h)^3$ and take $Q_h = S^\ell(\mathcal{T}_h)$. We discretize problem (5) by the following DG method: find $(\mathbf{E}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$\mathcal{A}_h(\mathbf{E}_h,\mathbf{v}) + \mathcal{S}_h(\mathbf{E}_h, v) + \mathcal{B}_h(\mathbf{v}, p_h) = \int_\Omega \mathbf{J} \cdot \mathbf{v}\, d\mathbf{x}, \quad (6)$$
$$\mathcal{B}_h(\mathbf{E}_h, q) - \mathcal{C}_h(p_h, q) = 0, \quad (7)$$

for any $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$, where $\mathcal{A}_h$ is the IP discretization for the curl-curl operator in (3), and the forms $\mathcal{B}_h$, $\mathcal{S}_h$ and $\mathcal{C}_h$ are given by

$$\mathcal{B}_h(\mathbf{v}, p) = -\int_\Omega \mathbf{v} \cdot \nabla_h p\, d\mathbf{x} + \int_\mathcal{E} \{\!\{\mathbf{v}\}\!\} \cdot [\![p]\!]_N\, ds, \quad (8)$$
$$\mathcal{S}_h(\mathbf{E},\mathbf{v}) = \int_{\mathcal{E}_\mathcal{I}} \mathtt{b}\, [\![\mathbf{E}]\!]_N [\![\mathbf{v}]\!]_N\, ds, \qquad \mathcal{C}_h(p, q) = \int_\mathcal{E} \mathtt{c} [\![p]\!]_N \cdot [\![q]\!]_N\, ds. \quad (9)$$

The form $\mathcal{B}_h$ discretizes the divergence operator in a DG fashion; $\mathcal{S}_h$ and $\mathcal{C}_h$ are stabilization forms that penalize the normal jumps of $\mathbf{E}$ and the jumps of p, respectively. The parameters $\mathtt{a}$, $\mathtt{b}$ and $\mathtt{c}$ are positive stabilization functions that are chosen as $\mathtt{a} = \alpha \mathtt{h}^{-1}$, $\mathtt{b} = \beta \mathtt{h}$ and $\mathtt{c} = \gamma \mathtt{h}^{-1}$, with positive α, β and γ. Again, α has to be taken large enough. Contrary to the regularized formulation in Section 3.1, the normal jumps are penalized with a stabilization function of the order $\mathcal{O}(h)$. With this choice, the method (6)–(7) is stable and delivers optimal convergence rates in a DG-norm, even for non-smooth solutions [3].

We test the method in the two-dimensional case for $\ell = 1$ on triangular grids. First, we solve the problem corresponding to the smooth exact solution $\mathbf{u} = [e^x(y\cos y + \sin y), e^x y \sin y]$ in $\Omega = (-1, 1)^2$. In Table 1 we report convergence rates in the DG-norm and the L^2–norm. In the DG-norm of $(\mathbf{u}, p)$, first order convergence is observed, while we obtain second order convergence in the L^2–norm for p. The L^2–rate for $\mathbf{u}$ is slightly worse than 2.

Table 1. Smooth solution: errors and convergence rates

meshsize reduct.	DG-norm of $(\mathbf{u}, p)$ error	rate	L^2-norm of $\mathbf{u}$ error	rate	L^2-norm of p error	rate
–	1.6594e00	–	1.3423e-01	–	4.8247e-02	–
2.0	6.9193e-01	1.26	2.9279e-02	2.20	9.1419e-03	2.40
2.0	2.8731e-01	1.27	7.0537e-03	2.06	1.9341e-03	2.24
2.0	1.2265e-01	1.23	2.2581e-03	1.64	4.5793e-04	2.08

The second test consists in solving the problem corresponding to the analytical solutions, given in polar coordinates, by $\mathbf{u} = \nabla S(r, \theta)$, with $S(r, \theta) = r^{2n/3} \sin(2n\theta/3)$, $n = 2, 1$, in the L–shaped domain $(-1, 1) \times (-1, 0) \cup (0, 1) \times (0, 1)$. Notice that $\mathbf{u} \in H^{2n/3}(\Omega)^2$. Numerical results for these cases are reported in Table 2; convergence rates of $\min\{\ell, 2n/3\}$ in the DG-norm are observed.

Table 2. Smooth solution: errors and convergence rates

sol.	meshsize reduc.	DG-norm of $(\mathbf{u}, p)$ error	rate	DG-norm of $\mathbf{u}$ error	rate
	–	5.8778e-01	–	1.3989e-01	–
$H^{4/3}$	2.0	3.1082e-01	0.92	6.0366e-02	1.21
$(n = 2)$	2.0	1.4316e-01	1.12	2.7084e-02	1.16
	2.0	6.0843e-02	1.23	1.1799e-02	1.20
	–	1.9982e+00	–	5.7246e-01	–
$H^{2/3}$	2.0	1.7238e+00	0.21	3.6870e-01	0.63
$(n = 1)$	2.0	1.2944e+00	0.41	2.2061e-01	0.74
	2.0	8.9294e-01	0.54	1.3147e-01	0.75

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