Injective Hopf bimodules, cohomologies of infinite dimensional Hopf algebras and graded-commutativity of the Yoneda product.

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Abstract

We prove that the category of Hopf bimodules over any Hopf algebra has enough injectives, which enables us to extend some results on the unification of Hopf bimodule cohomologies of [T1, T2] to the infinite dimensional case. We also prove that the cup-product defined on these cohomologies is graded-commutative.

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1 Introduction

The object of this paper is to extend the results in [T1, T2], on cohomologies for Hopf algebras, to a more general and more useful context, and to prove some properties of the objects introduced there.

For a finite dimensional Hopf algebra H, we identified various cohomologies with the Ext^* functor over an 'enveloping' associative algebra of H introduced by C. Cibils and M. Rosso ([CR]). In this way, the category of Hopf bimodules over H and the category of modules over

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this enveloping algebra can be identified, and we are then able to use all the usual properties of the Ext^* functor for a module category. When H is infinite dimensional however, such an 'enveloping' algebra does not exist. Nevertheless, many interesting examples of Hopf algebras are in fact infinite dimensional (*e.g.* enveloping algebras of Lie algebras), and it is important that such cases be considered. In fact, the identification of the cohomologies with an Ext^* functor (still for the category of Hopf bimodules over H, but this category is not equivalent to a module category in this case) does still hold: this generalisation is the object of Sections 2 and 3. The obstacle to this generalisation was the existence of enough projectives in the category of Hopf bimodules over H: it is not known whether any Hopf bimodule is a quotient of a projective Hopf bimodule. However, this problem can be circumvented: the proofs of the identification (which use universal properties of the Ext^* functor) can be adapted so that the obstacle becomes the existence of enough *injectives* in the category of Hopf bimodules (*see* Theorem 2.13), and this obstacle can be removed: we prove in Section 3 that there are indeed enough injectives in the category of Hopf bimodules, that is that any Hopf bimodule can be embedded in an injective Hopf bimodule. As a result, the identification of the cohomologies will hold for any Hopf algebra, that is each of the cohomologies we are considering is isomorphic to the Ext^* functor for the category of Hopf bimodules.

We also defined in [T1, T2] a cup-product on some of these cohomologies, and proved that it corresponds to the Yoneda product of extensions *via* this identification. (This correspondence was proved when the Hopf algebra is finite dimensional, since we needed the identification, but the proofs are valid for infinite dimensional Hopf algebras). We are in fact interested in the algebraic structure of one of these cohomologies, $\mathrm{H}_b^*(H,H)$, which is without coefficients: we would like to know whether this cohomology is a Gerstenhaber algebra. Gerstenhaber algebras are graded algebras, which are graded-commutative, and which are endowed with a graded Lie product (for a different grading) compatible with the cup-product. This graded Lie bracket appears in many cohomologies adapted to the study of deformations of various structures; it describes the obstructions to deforming the structures for which these cohomologies are defined. It also enabled M. Gerstenhaber and S.D. Schack in [GS2] to give a short proof of the Hochschild-Kostant-Rosenberg theorem.

In this paper, we prove that the cup-product is graded-commutative, using techniques of S. Schwede (*cf.* [S]), which tends to encourage the possibility of the cohomology being a Gerstenhaber algebra. We also give a candidate for a graded Lie bracket, which is graded-anticommutative.

This paper is organised as follows: in the first section we recall the definitions of the cohomologies and give some of the proofs for the unification (using injective Hopf bimodules) in the finite dimensional case. In the next section we prove that every Hopf bimodule over any Hopf algebra is a sub-Hopf bimodule of an injective Hopf bimodule, and that as a consequence the unification is true for an infinite dimensional Hopf algebra. We then consider the algebraic structure of $\mathrm{H}_b^*(H,H)$, first proving that the cup-product defined in [T1, T2] is graded-commutative in Section 4, and finally in the last section we give a candidate for a graded Lie bracket.

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2 Unification of cohomologies associated to a finite dimensional Hopf algebra

Let us define first of all the cohomologies we are interested in, and briefly give the proofs of the identification. In all the following, k is a commutative field.

2.1 Preliminaries

Let $(H, \mu, \eta, \Delta, \varepsilon, S)$ be a Hopf algebra. We shall define some Hopf bimodule structures on the tensor product of Hopf bimodules; $\otimes$ denotes the tensor product over k.

Definition-Proposition 2.1 *Let M and N be Hopf bimodules over H, and consider the vector space $M \otimes N$. This can be endowed with two different Hopf bimodule structures.*

The first one we shall denote by $M \underline{\otimes} N$; its actions are the regular ones and its coactions are codiagonal:

$$\begin{array}{rrcl}
\mu_L: & H \otimes M \underline{\otimes} N & \longrightarrow & M \underline{\otimes} N; \\
& h \otimes m \otimes n & \mapsto & hm \otimes n \\[1em]
\mu_R: & M \underline{\otimes} N \otimes H & \longrightarrow & M \underline{\otimes} N; \\
& m \otimes n \otimes h & \mapsto & m \otimes nh \\[1em]
\delta_L: & M \underline{\otimes} N & \longrightarrow & H \otimes M \underline{\otimes} N; \\
& m \otimes n & \mapsto & m_{(-1)} n_{(-1)} \otimes m_{(0)} \otimes n_{(0)} \\[1em]
\delta_R: & M \underline{\otimes} N & \longrightarrow & M \underline{\otimes} N \otimes H; \\
& m \otimes n & \mapsto & m_{(0)} \otimes n_{(0)} \otimes m_{(1)} n_{(1)}.
\end{array}$$

Dually, we denote by $M\bar{\otimes}N$ the Hopf bimodule whose actions are diagonal and whose coactions are regular:

$$\begin{array}{rrcl}
\mu_L: & H\otimes M\bar{\otimes}N & \longrightarrow & M\bar{\otimes}N; \\
 & h\otimes m\otimes n & \mapsto & h^{(1)}m\otimes h^{(2)}n \\
 & & & \\
\mu_R: & M\bar{\otimes}N\otimes H & \longrightarrow & M\bar{\otimes}N; \\
 & m\otimes n\otimes h & \mapsto & mh^{(1)}\otimes nh^{(2)} \\
 & & & \\
\delta_L: & M\bar{\otimes}N & \longrightarrow & H\otimes M\bar{\otimes}N; \\
 & m\otimes n & \mapsto & m_{(-1)}\otimes m_{(0)}\otimes n \\
 & & & \\
\delta_R: & M\bar{\otimes}N & \longrightarrow & M\bar{\otimes}N\otimes H; \\
 & m\otimes n & \mapsto & m\otimes n_{(0)}\otimes n_{(1)}.
\end{array}$$

Remark 2.2 Suppose M is an H-bimodule. Then, with the structures described above, the space $H\bar{\otimes}M\bar{\otimes}H$ is a well-defined Hopf bimodule.

We shall now recall the properties of the bar and cobar resolutions of Hopf bimodules (*see* for instance [Sh-St] or [T1]):

Proposition 2.3 Let M and N be Hopf bimodules. We view the spaces in the right module (*resp.* bimodule) bar resolution of M and the spaces in the left comodule (resp. bicomodule) cobar resolution of N as Hopf bimodules, with $Bar_q(M)=M\underline{\otimes}H^{\underline{\otimes}q+1}$ (*resp.* $B_q(M)=H^{\underline{\otimes}q+1}\underline{\otimes}M\underline{\otimes}H^{\underline{\otimes}q+1}$) and $Cob^p(N)=H^{\bar{\otimes}p+1}\bar{\otimes}N$ (*resp.* $C^p(N)=H^{\bar{\otimes}p+1}\bar{\otimes}N\bar{\otimes}H^{\bar{\otimes}p+1}$).

Then these resolutions are Hopf bimodule resolutions. Furthermore, $Bar_\bullet(H)$ and $B_\bullet(M)$ split as sequences of bicomodules, and dually $Cob^\bullet(H)$ and $C^\bullet(N)$ split as sequences of bimodules.

These resolutions have more properties which we shall need. To describe them, we need some definitions:

Definition 2.4 ([Sh-St]) *Let* $\mathrm{Hom}_{\mathcal{H}}(M,N)$ *denote the space of Hopf bimodule morphisms from M to N. A Hopf bimodule N is called a* relative injective *if the functor* $\mathrm{Hom}_{\mathcal{H}}(-,N)$ *takes exact sequences of Hopf bimodules that split as sequences of bimodules to exact sequences of $k-$vector spaces.*

Example 2.5 An injective Hopf bimodule is a relative injective.

Example 2.6 ([Sh-St]) If V is a bimodule, then the Hopf bimodule $H\bar{\otimes}V\bar{\otimes}H$ is a relative injective.

Definition 2.7 *A resolution of a Hopf bimodule is a* relative injective resolution *if all its terms are relative injectives, and if it splits as a sequence of bimodules.*

Relative injective resolutions have properties which are similar to those of injective resolutions, and we shall use:

Proposition 2.8 (*cf.* [Sh-St] Proposition 10.5.3) Two relative injective resolutions are homotopy equivalent as Hopf bimodule complexes.

We can define relative projectives and relative projective resolutions in a dual way. Then:

Proposition 2.9 The bar resolutions $Bar_\bullet(H)$ and $B_\bullet(M)$ are relative projective resolutions, the cobar resolutions $Cob^\bullet(H)$ and $C^\bullet(N)$ are relative injective resolutions.

We now have the background necessary to define and unify the cohomologies.

2.2 The cohomologies

We are interested in three cohomologies: two were defined by M. Gerstenhaber and S.D. Schack in [GS1] and [GS2], the third by C. Ospel in his thesis [Os]. They are denoted by H^*_{GS}, H^*_b and H^*_{H4} respectively, and are defined as follows:

Definition 2.10 *The three cohomologies are all defined using the bar and cobar resolutions. Let M and N be Hopf bimodules:*

1. $\mathrm{H}^*_{GS}(M,N)$ *is the cohomology of the double complex* $\mathrm{Hom}_{\mathcal{H}}(B_\bullet(M), C^\bullet(N))$; *at point* (p,q) *the entry of the double complex is* $\mathrm{Hom}_{\mathcal{H}}(B_q(M), C^p(N))$, *the space of Hopf bimodule maps from* $B_q(M)$ *to* $C^p(N)$, *the vertical differential is composition with the bar differential, and the horizontal differential is composition with the cobar differential.*

2. $\mathrm{H}^*_b(H,H)$ *is the cohomology of the double complex* $\mathrm{Hom}_{\mathcal{H}}(Bar_\bullet(H), Cob^\bullet(H))$. *This cohomology does not have coefficients.*

3. H^*_{H4} *is the cohomology of the double complex* $\mathrm{Hom}_{\mathcal{H}}(Bar_\bullet(M), Cob^\bullet(N))$.

Remark 2.11 It is obvious from the definitions that the third cohomology H^*_{H4} generalises the second H^*_b. In fact, H^*_{GS} also generalises H^*_b, and is furthermore isomorphic to H^*_{H4}. To prove this, we have identified each one with the Ext^* functor over an 'enveloping' algebra X of H, defined by C. Cibils and M. Rosso:

Theorem 2.12 ([CR] Theorem 3.10) Let H be a finite dimensional Hopf algebra. Then there exists an associative algebra X such that there is a vector space-preserving equivalence of categories between the category of left modules over X and the category $\mathcal{H}$ of Hopf bimodules over H.

We can now state:

Theorem 2.13 ([T1], [T2]) Let H be a finite dimensional Hopf algebra. Then the following isomorphisms hold for any Hopf bimodules M and N:

(a) $\mathrm{H}^*_{GS}(M,N) \cong \mathrm{Ext}^*_X(M,N)$

(b) $\mathrm{H}^*_{H4}(M,N) \cong \mathrm{Ext}^*_X(M,N)$

Proof: (This proof is similar to that in [T1] or [T2], but here we use injectives instead of projectives). Each isomorphism is proved using the universal property of Ext^*_X; let us remark that the category of Hopf bimodules over H has enough injectives, since it is equivalent to the category of modules over X. The functor Ext^*_X is then characterized by the following (*cf.* [McL]):

(1) $\mathrm{Ext}^0_X(M,N) \cong \mathrm{Hom}_X(M,N) = \mathrm{Hom}_{\mathcal{H}}(M,N)$,

(2) $\mathrm{Ext}^n_X(M,I) = 0$ for every $n \geqslant 1$ and every injective Hopf bimodule I,

(3) $\mathrm{Ext}^*_X(M,-)$ is a cohomological $\delta-$functor (*see* [W] p30).

Therefore we need to prove that these three properties are satisfied for H^*_{GS} and H^*_{H4}.

The proof of **(1)** is straightforward in both cases, and the proofs of **(3)** are similar to the proofs given in [T1] and [T2], they rely on alternative definitions of the double complexes and on the structure of Hopf bimodules. We shall now go through the proof of **(2)** in each case.

Lemma 2.14 For every injective Hopf bimodule I and every integer $n \geqslant 1$, the $k-$vector space $\mathrm{H}^n_{GS}(M,I)$ vanishes.

Proof of Lemma: Let I be an injective Hopf bimodule. Then $0 \to I \to I \to 0 \to \cdots \to 0 \to \cdots$ is a relative injective resolution of I. It is therefore homotopy equivalent to the cobar resolution $C^\bullet(I)$ (Proposition 2.8). Gerstenhaber and Schack's cohomology is therefore the cohomology of the double complex in which all the terms are zero except those on the first line, which are equal to $\mathrm{Hom}_{\mathcal{H}}(B_q(M),I)$, for $q \geqslant 0$. This line is acyclic, since I is injective and $B_\bullet(M)$ is exact. Its cohomology is therefore zero in positive degree. □

Lemma 2.15 For every injective Hopf bimodule I and every integer $n \geqslant 1$, the $k-$vector space $\mathrm{H}^n_{H4}(M,I)$ vanishes.

Proof of Lemma: Let I be an injective Hopf bimodule. Let us consider its cobar resolution

$$Cob^\bullet(I) : 0 \to I \xrightarrow{\lambda^{-1}} H\bar{\otimes} I \xrightarrow{\lambda^0} H^{\bar{\otimes}2}\bar{\otimes} I \to \cdots \to H^{\bar{\otimes}p+1}\bar{\otimes} I \xrightarrow{\lambda^p} H^{\bar{\otimes}p+2}\bar{\otimes} I \to \cdots,$$

with $\lambda^{-1}(u) = \rho_L(u) = u_{(-1)}\bar{\otimes} u_{(0)}$.

Since I is injective and λ^{-1} is one-to-one, λ^{-1} has a retraction: there exists a morphism of Hopf bimodules $r : H\bar{\otimes}I \to I$ satisfying $r(\lambda^{-1}(u)) = u$ for every $u \in I$.

Set

$$\begin{array}{rcl} \chi^p \ : H^{\bar{\otimes}p+2}\bar{\otimes}I & \longrightarrow & H^{\bar{\otimes}p+1}\bar{\otimes}I \\ h_0\bar{\otimes}\dots\bar{\otimes}h_{p+1}\bar{\otimes}u & \mapsto & h_0\bar{\otimes}\dots\bar{\otimes}h_p\bar{\otimes}r(h_{p+1}\bar{\otimes}u). \end{array}$$

It is a Hopf bimodule morphism, and $(\lambda^{p-1}\chi^{p-1}+\chi^p\lambda^p) = \mathrm{id}$. Therefore $\chi^\bullet$ is a Hopf bimodule homotopy from id to 0.

Now fix $q \in \mathbb{N}$, and consider the complex $\mathrm{Hom}_{\mathcal{H}}(Bar_q(M), Cob^\bullet(I))$; the homotopy $\chi^\bullet$ on $Cob^\bullet(I)$ yields a homotopy $\chi^\bullet \circ -$ from id to 0 on this complex. Therefore, the double complex:

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ \mathrm{Hom}_{\mathcal{H}}(Bar_1(M), I) & \longrightarrow & \mathrm{Hom}_{\mathcal{H}}(Bar_1(M), Cob^0(I)) & \longrightarrow & \mathrm{Hom}_{\mathcal{H}}(Bar_1(M), Cob^1(I)) \quad \dots \\ \uparrow & & \uparrow & & \uparrow \\ \mathrm{Hom}_{\mathcal{H}}(Bar_0(M), I) & \longrightarrow & \mathrm{Hom}_{\mathcal{H}}(Bar_0(M), Cob^0(I)) & \longrightarrow & \mathrm{Hom}_{\mathcal{H}}(Bar_0(M), Cob^1(I)) \quad \dots \end{array}$$

which is the double complex defining $\mathrm{H}^*_{H4}(M, I)$ to which we have added one column, has exact rows. Its homology is therefore that of the first column $\mathrm{Hom}_{\mathcal{H}}(Bar_\bullet(M), I)$ (*cf.* [W] p59-60). Since I is injective and $Bar_\bullet(M)$ is exact, the homology of this complex is zero in positive degree. $\square$

These two lemmas prove the second part of the characterization of Ext^*_X for both cohomologies. $\blacksquare$

Remark 2.16 We have only used the existence of X to say that the category of Hopf bimodules has enough injectives.

3 On the category of Hopf bimodules over an infinite dimensional Hopf algebra; unification of cohomologies associated to an infinite dimensional Hopf algebra

In this section, we shall prove that the category of Hopf bimodules over any Hopf algebra has enough injectives, and then extend the unification of Section 2.

Theorem 3.1 Let H be any Hopf algebra. Then every Hopf bimodule over H can be embedded in an injective Hopf bimodule.

Proof: Let M be a Hopf bimodule over H. Then M can be viewed as an H-bimodule, and as such can be embedded in an injective H-bimodule I. Let $\varphi : M \hookrightarrow I$ denote this embedding.

Now consider the Hopf bimodule $H\bar{\otimes}I\bar{\otimes}H$ (*see* Remark 2.2). We shall now prove that this is an injective Hopf bimodule and that there is an embedding of Hopf bimodules from M to $H\bar{\otimes}I\bar{\otimes}H$.

Let us consider the functor $\mathrm{Hom}_{\mathcal{H}}(-, H\bar{\otimes}I\bar{\otimes}H)$. Since $\mathrm{Hom}_{\mathcal{H}}(-, H\bar{\otimes}I\bar{\otimes}H)$ is isomorphic to $\mathrm{Hom}_{H-H}(-, I)$ (*see* [T2] Remark 1.22) and I is an injective bimodule, this functor is exact, thus proving that $H\bar{\otimes}I\bar{\otimes}H$ is an injective Hopf bimodule.

Now define a map ψ from M to $H\bar{\otimes}I\bar{\otimes}H$ by setting $\psi(m) := m_{(-1)}\bar{\otimes}\varphi(m_{(0)})\bar{\otimes}m_{(1)}$. This is a Hopf bimodule map, and is clearly injective, since $(\varepsilon\bar{\otimes}\mathrm{id}\bar{\otimes}\varepsilon) \circ \psi$ is equal to φ which is injective. ■

Corollary 3.2 Let H be any Hopf algebra. Then the following isomorphisms hold for any Hopf bimodules M and N:

(a) $\mathrm{H}^*_{GS}(M, N) \cong \mathrm{Ext}^*_{\mathcal{H}}(M, N)$

(b) $\mathrm{H}^*_{H4}(M, N) \cong \mathrm{Ext}^*_{\mathcal{H}}(M, N)$

Proof: This is proved in the same way as Theorem 2.13, now that we know that there are enough injective Hopf bimodules in the category $\mathcal{H}$. Note that we can consider the extensions $\mathrm{Ext}^*_{\mathcal{H}}(M, N)$, since the category $\mathcal{H}$ is abelian. ■

Remark 3.3 Recall that when H is finite dimensional, the category of Yetter-Drinfel'd modules is equivalent to the category of modules over the Drinfel'd double $\mathcal{D}(H)$ (as braided categories), and therefore all these cohomologies are isomorphic to $\mathrm{Ext}^*_{\mathcal{D}(H)}(M^R, N^R)$.

In particular, $\mathrm{H}^*_b(H, H) \cong \mathrm{Ext}^*_{\mathcal{D}(H)}(k, k)$, which answers a question of F. Panaite and D. Ştefan (*see* [PS]).

Remark 3.4 We proved in [T1] (Corollary 2.18) that the cohomologies we have considered here are Morita invariant when the Hopf algebra is finite dimensional. It is clear that this result extends to the infinite dimensional case, that is that if H and H' are Hopf algebras which are Morita equivalent as algebras and such that the functors $\mathcal{F}$ and $\mathcal{G}$ giving the equivalence are monoidal, then there is an isomorphism

$$\mathrm{Ext}^*_{\mathcal{H}}(M, N) \cong \mathrm{Ext}^*_{\mathcal{H}'}(\mathcal{F}(M), \mathcal{F}(N))$$

for all H-Hopf bimodules M and N. Here, $\mathcal{H}'$ denotes the category of Hopf bimodules over H'. In particular,

$$\mathrm{H}^*_b(H, H) \cong \mathrm{H}^*_b(H', H').$$

4 Graded-commutativity of the Yoneda product of Hopf bimodule extensions

In [T1] and [T2], we have defined a cup-product on H^*_{H4} which corresponds to the Yoneda product of extensions up to sign *via* the identification of Corollary 3.2 (the proof of the correspondence of the products remains valid for an infinite dimensional Hopf algebra). This part is devoted to the study of this product, in particular proving that it is graded-commutative, using techniques of S. Schwede [S].

4.1 The products on the cohomologies H^*_{H4} and H^*_b

Let us first define the cup-product on the cohomology H^*_{H4} :

Proposition 4.1 ([T1], [T2]) Let $f \in \mathrm{Hom}_{H-}^{-H}(M \otimes H^{\otimes p-s}, H^{\otimes s} \otimes L)$ be a $p-$cochain (a map of left H-modules and right H-comodules) and $g \in \mathrm{Hom}_{H-}^{-H}(L \otimes H^{\otimes q-r}, H^{\otimes r} \otimes N)$ be a $q-$cochain. Set $n = p+q$ and $t = s+r$. Define the $n-$cochain $f \smile g \in \mathrm{Hom}_{H-}^{-H}(M \otimes H^{\otimes n-t}, H^{\otimes t} \otimes N)$ by:

$$\begin{aligned} &f \smile g(m \otimes \mathbf{a_{1,n-t}}) = \\ &\qquad (-1)^{s(q-r)}(1^{\otimes s} \otimes g)\,[f(m \otimes \mathbf{a_{1,p-s}}).(\Delta^{(s-1)}(a^{(1)}_{p-s+1} \dots a^{(1)}_{n-t}) \otimes 1) \otimes \mathbf{a^{(2)}_{p-s+1,n-t}}]. \end{aligned}$$

The differential D of the total complex associated to the Hopf bimodule double complex is a right derivation for the cup-product $\smile$, that is

$$D(f \smile g) = Df \smile g + (-1)^p f \smile Dg,$$

so that the formula for $\smile$ yields a product $\mathrm{H}^*_{H4}(M,L) \otimes \mathrm{H}^*_{H4}(L,N) \to \mathrm{H}^*_{H4}(M,N)$.

Remark 4.2 The cup-product in $\mathrm{H}^*_b(H,H)$ is as follows: let f be in $\mathrm{Hom}_k(H^{\otimes p-s}, H^{\otimes s})$ and g in $\mathrm{Hom}_k(H^{\otimes q-r}, H^{\otimes r})$; then

$$\begin{aligned} (f \smile g)(a_1 \otimes \dots \otimes a_{n-t}) \;\; = \;\; & (-1)^{s(q-r)} f(a^{(1)}_1 \otimes \dots \otimes a^{(1)}_{p-s})\ \Delta^{(s-1)}(a^{(1)}_{p-s+1} \dots a^{(1)}_{n-t}) \\ & \otimes \Delta^{(r-1)}(a^{(2)}_1 \dots a^{(2)}_{p-s})\ g(a^{(2)}_{p-s+1} \otimes \dots \otimes a^{(2)}_{n-t}). \end{aligned}$$

We shall now relate the cup-product $\smile$ with the Yoneda product of extensions.

Theorem 4.3 ([T1], [T2]) Let H be a Hopf algebra. Let M, N and L be Hopf bimodules over H. Let $\varphi_{MN} : \mathrm{H}^*_{H4}(M,N) \to \mathrm{Ext}^*_X(M,N)$ be the isomorphism extending $\mathrm{id}_{\mathrm{Hom}_{\mathcal{H}}(M,N)}$ and let $\sharp$ denote the Yoneda product of extensions (*see* below).

Then, if $f \in \mathrm{H}^p_{H4}(M,L)$ and $g \in \mathrm{H}^q_{H4}(L,N)$, the relationship between the products is given by

$$\varphi^{p+q}_{MN}(f \smile g) = (-1)^{pq}\, \varphi^q_{LN}(g) \sharp \varphi^p_{ML}(f) :$$

the cup product and the Yoneda product are equal up to sign.

Proof: (Sketch) After proving universal properties of the cup-product and the Yoneda product, we reduce to the case when $q = 0$; this case is proved by induction on p, thanks to a partial associativity property, and the knowledge of φ^1. ■

Therefore, to study the algebraic structure of $\mathrm{H}_b^*(H, H)$, we may consider the algebra $\mathrm{Ext}_{\mathcal{H}}^*(H, H)$ endowed with the Yoneda product.

4.2 Graded-commutativity of the Yoneda product

We will follow S. Schwede's method ([S]) to prove that the Yoneda product of Hopf bimodule extensions of H by H is graded-commutative. We refer to [S] for the definitions and results on the homotopy of (the nerve of) a category which we shall need. Let us fix notations.

We shall denote by $\mathcal{E}xt_{\mathcal{H}}^n(H, H)$ the category of extensions of Hopf bimodules from H to H. Then $\pi_0\mathcal{E}xt_{\mathcal{H}}^n(H, H) \cong \mathrm{Ext}_{\mathcal{H}}^n(H, H)$ and $\pi_1\mathcal{E}xt_{\mathcal{H}}^n(H, H) \cong \mathrm{Ext}_{\mathcal{H}}^{n-1}(H, H)$, where paths and loops are defined as in [S]. Now given two extensions $\mathbf{E} : 0 \stackrel{i_{\mathbf{E}}}{\to} H \to E_{m-1} \to E_{m-2} \to \cdots \to E_0 \stackrel{p_{\mathbf{E}}}{\to} H \to 0$ in $\mathcal{E}xt_{\mathcal{H}}^m(H, H)$ and $\mathbf{F} : 0 \to H \to F_{n-1} \to F_{n-2} \to \cdots \to F_0 \to H \stackrel{p_{\mathbf{F}}}{\to} 0$ in $\mathcal{E}xt_{\mathcal{H}}^n(H, H)$, we define their Yoneda product to be the class in $\pi_0\mathcal{E}xt_{\mathcal{H}}^{m+n}(H, H)$ of the extension obtained by 'splicing' $\mathbf{E}$ and $\mathbf{F}$:

$$\mathbf{E}\sharp\mathbf{F} : 0 \to H \to F_{n-1} \to \cdots \to F_0 \stackrel{i_{\mathbf{E}}p_{\mathbf{F}}}{\longrightarrow} E_{m-1} \to \cdots \to E_0 \to H \to 0.$$

We denote by $(-1)\mathbf{E}$ the sequence obtained from $\mathbf{E}$ by replacing $p_{\mathbf{E}}$ by $-p_{\mathbf{E}}$; this represents the inverse of $\mathbf{E}$ in $\pi_0\mathcal{E}xt_{\mathcal{H}}^m(H, H)$ with respect to Baer sum.

To prove that this product is graded-commutative, we need to prove that the classes in $\pi_0\mathcal{E}xt_{\mathcal{H}}^{m+n}(H, H)$ of $\mathbf{F}\sharp\mathbf{E}$ and of $(-1)^{mn}\mathbf{E}\sharp\mathbf{F}$ are equal, so we need to find a path from $\mathbf{F}\sharp\mathbf{E}$ to $(-1)^{mn}\mathbf{E}\sharp\mathbf{F}$ in $\mathcal{E}xt_{\mathcal{H}}^{m+n}(H, H)$. This will involve a tensor product of extensions of Hopf bimodules; the first step is to define this tensor product over H of Hopf bimodules and hence of extensions of Hopf bimodules, and to check that it makes sense in the category of Hopf bimodules $\mathcal{H}$.

Definition 4.4 *Let E and F be Hopf bimodules over H. Define $E\underline{\otimes}_H F$ as the quotient*

$$\frac{E\underline{\otimes}F}{< \{eh\underline{\otimes}f - e\underline{\otimes}hf \mid \ e \in E,\ h \in H,\ f \in F\} >}.$$

Lemma 4.5 The space $E\underline{\otimes}_H F$ is a Hopf bimodule.

Proof: Let I be the vector space generated by $\{eh\underline{\otimes}f - e\underline{\otimes}hf \mid \ e \in E,\ h \in H,\ f \in F\}$. It is well-known that I is a sub-bimodule of $E\underline{\otimes}F$. It is straightforward to check that it is also a sub-bicomodule of $E\underline{\otimes}F$ (recall that the coactions on $E\underline{\otimes}F$ are codiagonal). Therefore I is a sub-Hopf bimodule of $E\underline{\otimes}F$, and the quotient is then a Hopf bimodule. ■

Now we need to see that the tensor product of extensions of Hopf bimodules is again an extension; let us consider a special case first:

Lemma 4.6 If E is a Hopf bimodule and $F' \to F \to F''$ is any exact sequence of Hopf bimodules, then

$$(S) \quad E \underline{\otimes}_H F' \to E \underline{\otimes}_H F \to E \underline{\otimes}_H F''$$

is an exact sequence of Hopf bimodules.

Proof: The sequence (S) is a sequence of Hopf bimodules, since the spaces are Hopf bimodules by the previous lemma and the maps are tensor products of Hopf bimodule morphisms and are therefore Hopf bimodule morphisms.

It remains to be checked that the sequence is exact. Since E is a Hopf bimodule, we know that it is free as a right H-module (*see* for instance [Mo] Theorem 1.9.4), therefore E is flat as a right H-module. So (S) is exact as a sequence of vector spaces, hence also as a sequence of Hopf bimodules. ■

We may now consider tensor products of extensions:

Proposition 4.7 Let $\mathbf{E} : 0 \to H \to E_{m-1} \to E_{m-2} \to \cdots \to E_0 \overset{p_{\mathbf{E}}}{\to} H \to 0$ and $\mathbf{F} : 0 \to H \to F_{n-1} \to F_{n-2} \to \cdots \to F_0 \to H \overset{p_{\mathbf{F}}}{\to} 0$ be extensions of Hopf bimodules. Then their tensor product over H, defined by

$$(\mathbf{E} \underline{\otimes}_H \mathbf{F})_r := \bigoplus_{\substack{s+t=r \\ s,t \geqslant 0}} E_s \underline{\otimes}_H F_t$$

for $0 \leqslant r \leqslant m+n$ and $(\mathbf{E} \underline{\otimes}_H \mathbf{F})_{-1} := H$, is also a Hopf bimodule extension of H by itself.

Proof: Fix $-1 \leqslant s \leqslant m$: from Lemma 4.6, the sequence $E_s \underline{\otimes}_H \mathbf{F}$ is an extension of Hopf bimodules. We then take the direct sum over $-1 \leqslant s \leqslant m$ of these exact sequences of Hopf bimodules to obtain an extension of Hopf bimodules $\mathbf{E} \underline{\otimes}_H \mathbf{F}$. ■

Given two extensions $\mathbf{E}$ and $\mathbf{F}$ as above, we may now construct a path from $\mathbf{F} \sharp \mathbf{E}$ to $(-1)^{mn} \mathbf{E} \sharp \mathbf{F}$. In fact, as in [S], we are going to construct a loop in $\mathcal{E}xt^{m+n}_{\mathcal{H}}(H,H)$ going through these two extensions.

Consider the following two maps, $\lambda_{\mathbf{E},\mathbf{F}} : \mathbf{E} \underline{\otimes}_H \mathbf{F} \to \mathbf{F} \sharp \mathbf{E}$ and $\rho_{\mathbf{E},\mathbf{F}} : \mathbf{E} \underline{\otimes}_H \mathbf{F} \to (-1)^{mn} \mathbf{E} \sharp \mathbf{F}$, defined on each degree by:

$$
\begin{aligned}
\text{if } 0 \leqslant i < m, \quad (\lambda_{\mathbf{E},\mathbf{F}})_i &: \quad (\mathbf{E}\underline{\otimes}_H\mathbf{F})_i \overset{projection}{\longrightarrow} E_0\underline{\otimes}_H F_i \overset{p_{\mathbf{E}}\underline{\otimes}_H \mathrm{id}_{F_i}}{\longrightarrow} F_i \\
\text{if } m \leqslant i \leqslant m+n, \quad (\lambda_{\mathbf{E},\mathbf{F}})_i &: \quad (\mathbf{E}\underline{\otimes}_H\mathbf{F})_i \overset{projection}{\longrightarrow} E_{i-m}\underline{\otimes}_H F_m \overset{\sim}{\longrightarrow} E_{i-m} \\
\text{if } 0 \leqslant j < n, \quad (\rho_{\mathbf{E},\mathbf{F}})_j &: \quad (-1)^{mn}\left[(\mathbf{E}\underline{\otimes}_H\mathbf{F})_j \overset{projection}{\longrightarrow} E_j\underline{\otimes}_H F_0 \overset{\mathrm{id}_{\mathbf{E}_j}\underline{\otimes}_H p_{\mathbf{F}}}{\longrightarrow} E_j\right] \\
\text{if } n \leqslant j \leqslant m+n, \quad (\rho_{\mathbf{E},\mathbf{F}})_j &: \quad (-1)^{m+n-j}\left[(\mathbf{E}\underline{\otimes}_H\mathbf{F})_j \overset{projection}{\longrightarrow} E_n\underline{\otimes}_H F_{j-n} \overset{\sim}{\longrightarrow} F_{j-n}\right].
\end{aligned}
$$

These are morphims of complexes (this is a straightforward computation) of Hopf bimodules, since each of the maps is given by composition of projections, natural identifications $E_i\underline{\otimes}_H H \cong E_i$ or $H\underline{\otimes}_H F_j \cong F_j$, and the tensor product of an identity map with a morphism of Hopf bimodules, so they are all morphisms of Hopf bimodules.

We may now consider the following loop of extensions in $\mathcal{E}xt^{m+n}_{\mathcal{H}}(H,H)$:

$$
\begin{tikzcd}
 & \mathbf{E}\underline{\otimes}_H\mathbf{F} \arrow[dl, "\lambda_{\mathbf{E},\mathbf{F}}"'] \arrow[dr, "\rho_{\mathbf{E},\mathbf{F}}"] & \\
\mathbf{F}\sharp\mathbf{E} & & (-1)^{mn}\mathbf{E}\sharp\mathbf{F} \\
 & (-1)^{mn}\mathbf{F}\underline{\otimes}_H\mathbf{E} \arrow[ul, "\rho_{\mathbf{F},\mathbf{E}}"] \arrow[ur, "\lambda_{\mathbf{F},\mathbf{E}}"'] &
\end{tikzcd}
$$

oriented counter-clockwise. We call it $\Omega(\mathbf{F},\mathbf{E})$.

There is therefore a path in $\mathcal{E}xt^{m+n}_{\mathcal{H}}(H,H)$ from $\mathbf{F}\sharp\mathbf{E}$ to $(-1)^{mn}\mathbf{E}\sharp\mathbf{F}$ (either the upper part or the lower part of the diagram above), so $\mathbf{F}\sharp\mathbf{E}$ and $(-1)^{mn}\mathbf{E}\sharp\mathbf{F}$ define the same element in $\pi_0\mathcal{E}xt^{m+n}_{\mathcal{H}}(H,H) \cong \mathrm{Ext}^{m+n}_{\mathcal{H}}(H,H)$.

Furthermore, as in [S], the construction of the loop $\Omega(\mathbf{F},\mathbf{E})$ is functorial in $\mathbf{F}$ and $\mathbf{E}$, so this defines a map:

$$\Omega : \pi_0\mathcal{E}xt^{m}_{\mathcal{H}}(H,H) \times \pi_0\mathcal{E}xt^{n}_{\mathcal{H}}(H,H) \longrightarrow \pi_1\mathcal{E}xt^{m+n}_{\mathcal{H}}(H,H),$$

called the loop bracket.

Remark 4.8 The graded-commutativity can be deduced from the above construction without involving homotopy groups: indeed, we have constructed maps of extensions between $\mathbf{E}\underline{\otimes}_H\mathbf{F}$ and $\mathbf{F}\sharp\mathbf{E}$, and between $\mathbf{E}\underline{\otimes}_H\mathbf{F}$ and $(-1)^{mn}\mathbf{E}\sharp\mathbf{F}$. Therefore, by definition of the equivalence relation between extensions (*see* [McL]), these three extensions are equivalent, and represent the same element in $\mathrm{Ext}^{m+n}_{\mathcal{H}}(H,H)$.

The homotopy groups become useful when defining the bracket, which we shall now discuss further.

5 The loop bracket

The cohomologies we have described above (H^*_{A4} and H^*_b) have many points in common with Hochschild cohomology of algebras. They involve bar resolutions, are isomorphic to an Ext^* functor, and are endowed with a cup-product which corresponds to the Yoneda product of extensions and which is graded-commutative. It is therefore natural to wonder if $\mathrm{H}^*_b(H,H)$ might be a G-algebra, that is if it might be endowed with a graded Lie bracket compatible with the cup-product, as M. Gerstenhaber proved is the case for Hochschild cohomology. We have been unable to answer this question, but we describe below some points in favour of this being the case.

Let us recall the definition of a G-algebra:

Definition 5.1 *A* G-algebra *is a graded k-module $\Lambda = \oplus_n \Lambda_n$ equipped with two multiplications, $(\lambda,\nu) \mapsto \lambda \smile \nu$ and $(\lambda,\nu) \mapsto [\lambda,\nu]$, satisfying the following properties:*

(1) *$\smile$ is an associative graded (by degree) commutative product;*

(2) *$[-,-]$ is a graded Lie bracket for which the grading is reduced degree, this being one less than the degree;*

(3) *$[-,\eta^p]$ is a degree $p-1$ graded derivation of the associative algebra structure for all $\eta^p \in \Lambda_p$:*

$$[\lambda^m \smile \nu^n, \eta^p] = [\lambda,\eta] \smile \nu + (-1)^{m(p-1)} \lambda \smile [\nu,\eta] \quad \forall \lambda,\nu.$$

We know that the first condition is satisfied (Section 4). Regarding the bracket, in the previous section we described a map

$$\Omega : \pi_0 \mathcal{E}xt^m_{\mathcal{H}}(H,H) \times \pi_0 \mathcal{E}xt^n_{\mathcal{H}}(H,H) \longrightarrow \pi_1 \mathcal{E}xt^{m+n}_{\mathcal{H}}(H,H).$$

Transporting this *via* the isomorphisms φ between $\mathrm{H}^*_b(H,H)$ and $\pi_0\mathcal{E}xt^*_{\mathcal{H}}(H,H)$ and ψ between $\mathrm{H}^{*-1}_b(H,H)$ and $\pi_1\mathcal{E}xt^*_{\mathcal{H}}(H,H)$ gives maps $\mathrm{H}^m_b(H,H) \times \mathrm{H}^n_b(H,H) \to \mathrm{H}^{m+n-1}_b(H,H)$. We therefore have a candidate for the product $[-,-]$. However, we do not know any of the isomorphisms explicitly, so we cannot prove directly conditions **(2)** and **(3)**. More precisely, since we want to keep as close as possible to the Hochschild situation (*see* [S]), we consider the map $[-,-] : \mathrm{H}^m_b(H,H) \times \mathrm{H}^n_b(H,H) \to \mathrm{H}^{m+n-1}_b(H,H)$ such that the diagram

$$\begin{tikzcd}
\mathrm{H}^m_b(H,H) \times \mathrm{H}^n_b(H,H) \arrow[r, "{[-,-]}"] \arrow[d, "\varphi\times\varphi"'] & \mathrm{H}^{m+n-1}_b(H,H) \arrow[d, "\psi"] \\
\pi_0\mathcal{E}xt^m_{\mathcal{H}}(H,H) \times \pi_0\mathcal{E}xt^n_{\mathcal{H}}(H,H) \arrow[r, "\Omega"'] & \pi_1\mathcal{E}xt^{m+n}_{\mathcal{H}}(H,H)
\end{tikzcd}$$

commutes *up to the sign* $(-1)^n$, that is, if $\mathbf{E} = \varphi(e)$ is of degree m and $\mathbf{F} = \varphi(f)$ is of degree n, then $\Omega(\mathbf{E},\mathbf{F}) = \psi((-1)^n[e,f])$. However, we do not know any of the isomorphisms explicitly, so we do not have an expression for $[-,-]$, and we cannot prove directly conditions **(2)** and **(3)**.

Therefore, we now need to find out how the conditions **(2)** and **(3)** translate for Ω. These conditions involve in particular the sum of elements of $\mathrm{H}_b^*(H,H)$. This corresponds to concatenation of loops:

Proposition 5.2 The isomorphism ψ is an isomorphism of groups, where the group law on $\mathrm{H}_b^*(H,H)$ is induced by Baer sum and that on $\pi_1\mathcal{E}xt_{\mathcal{H}}^{*+1}(H,H)$ is induced by concatenation of loops.

Proof: We have an isomorphism $\mathrm{H}_b^*(H,H) \xrightarrow{\sim} \pi_0\mathcal{E}xt_{\mathcal{H}}^*(H,H) = \mathrm{Ext}_{\mathcal{H}}^*(H,H)$: this is the isomorphism of Corollary 3.2, which is a morphism of (abelian) groups (*cf.* [McL]), where the group law on extensions is Baer sum. We are interested in the isomorphism $\mathrm{H}_b^*(H,H) \xrightarrow{\sim} \pi_1\mathcal{E}xt_{\mathcal{H}}^{*+1}(H,H) = \mathrm{Ext}_{\mathcal{H}}^*(H,H)$ which we know exists from [R]. We know ([R] Theorem 2) that the isomorphism $\pi_0\mathcal{E}xt_{\mathcal{H}}^n(H,H) \cong \pi_1\mathcal{E}xt_{\mathcal{H}}^{n+1}(H,H)$ comes from a functor F_{n+1} which is a *group* isomorphism (with Baer sum on the left and concatenation of loops on the right). Therefore by composition, the isomorphism ψ is an isomorphism of groups. ∎

We now wish to prove properties **(2)** and **(3)** adapted to Ω. They are satisfied in the case of an associative algebra when considering bimodule extensions, since S. Schwede proved in [S] that Ω corresponds to Gerstenhaber's Lie bracket in that case. However, he doesn't give a direct proof of these properties for Ω, and describing the Jacobi identity for Ω requires the construction of a loop (or of an element in a higher homotopy group) associated not to two extensions but to one extension and one loop, and property **(3)** requires the definition of the Yoneda product of a loop and an extension. We do not yet know how to do this; however, the graded anti-commutativity of the bracket can easily be translated as follows:

Proposition 5.3 The graded anti-commutativity of the Lie bracket translates as

$$\Omega(\mathbf{E},\mathbf{F}) = \Omega(\mathbf{F},\mathbf{E})^{(-1)^{mn}}$$

where $\mathbf{E}$ is of degree m and $\mathbf{F}$ is of degree n.

Proof: Indeed, if $\mathbf{E} = \varphi(e)$ and $\mathbf{F} = \varphi(f)$, we have

$$\begin{aligned}\Omega(\mathbf{E},\mathbf{F}) &= \psi((-1)^n[e,f]) \\ &= \psi((-1)^n(-1)^{(m-1)(n-1)+1}[f,e]) \\ &= \psi((-1)^m[f,e])^{(-1)^{mn}} \\ &= \Omega(\mathbf{F},\mathbf{E})^{(-1)^{mn}}.\end{aligned}$$

∎

Let us now prove that Ω does indeed satisfy this property:

Proposition 5.4 The loop bracket Ω satisfies:

$$\Omega(\mathbf{E},\mathbf{F}) = \Omega(\mathbf{F},\mathbf{E})^{(-1)^{mn}}$$

where $\mathbf{E}$ is of degree m and $\mathbf{F}$ is of degree n.

Proof: The first case we consider is when either m or n is even. Then we wish to prove that $\Omega(\mathbf{E},\mathbf{F}) = \Omega(\mathbf{F},\mathbf{E})$, and this is obvious when looking at the loops: one is obtained from the other by rotating by 180 degrees.

In fact, we notice that in general $\Omega(\mathbf{E},\mathbf{F}) = \Omega(\mathbf{F},(-1)^{mn}\mathbf{E})$. Therefore in the case when both m and n are odd, we wish to prove that $\Omega(\mathbf{F},-\mathbf{E}) = \Omega(\mathbf{F},\mathbf{E})^{-1}$, or that $\Omega(\mathbf{F},\mathbf{E}).\Omega(\mathbf{F},-\mathbf{E})$ is the trivial loop. To do this, we shall consider the behaviour of Ω towards Baer sums.

Note that the components of the category of extensions have canonically isomorphic groups, the isomorphisms are given by Baer sum with a fixed extension (and converses are given by Baer sum with the opposite extension). Therefore, even if the loops we are going to consider are not always in the same component originally, they may be considered so.

Now given three extensions $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$, we wish to compare the loops $\Omega(\mathbf{G},\mathbf{E}+\mathbf{F})$ and $\Omega(\mathbf{G},\mathbf{E}).\Omega(\mathbf{G},\mathbf{F})$. Since both the Yoneda product $\sharp$ and the tensor product $\otimes_H$ commute with Baer sum, the left hand term is obtained by adding the extensions in the loops $\Omega(\mathbf{G},\mathbf{E})$ and $\Omega(\mathbf{G},\mathbf{F})$ componentwise.

Therefore the question may be set more generally as follows: given two loops of extensions $L : Y \overset{\alpha_0}{\longleftrightarrow} C_1 \overset{\alpha_1}{\longleftrightarrow} \cdots \overset{\alpha_{r-1}}{\longleftrightarrow} C_r \overset{\alpha_r}{\longleftrightarrow} Y$ and $L' : Y' \overset{\alpha'_0}{\longleftrightarrow} C'_1 \overset{\alpha'_1}{\longleftrightarrow} \cdots \overset{\alpha'_{r-1}}{\longleftrightarrow} C'_r \overset{\alpha'_r}{\longleftrightarrow} Y'$, we wish to compare the loops of extensions $L'' : Y+Y' \overset{(\alpha_0,\alpha'_0)}{\longleftrightarrow} C_1+C'_1 \overset{(\alpha_1,\alpha'_1)}{\longleftrightarrow} \cdots \overset{(\alpha_{r-1},\alpha'_{r-1})}{\longleftrightarrow} C_r+C'_r \overset{(\alpha_r,\alpha'_r)}{\longleftrightarrow} Y+Y'$ and $L''' : Y+Y' \overset{(\alpha_0,\mathrm{id})}{\longleftrightarrow} C_1+Y' \overset{(\alpha_1,\mathrm{id})}{\longleftrightarrow} \cdots \overset{(\alpha_{r-1},\mathrm{id})}{\longleftrightarrow} C_r+Y' \overset{(\alpha_r,\mathrm{id})}{\longleftrightarrow} Y+Y' \overset{(\mathrm{id},\alpha'_0)}{\longleftrightarrow} Y+C'_1 \overset{(\mathrm{id},\alpha'_1)}{\longleftrightarrow} \cdots \overset{(\mathrm{id},\alpha'_{r-1})}{\longleftrightarrow} Y+C'_r \overset{(\mathrm{id},\alpha'_r)}{\longleftrightarrow} Y+Y'$. Note that in the last loop, the original loops L and L' have been shifted so that they are in the same component, based at $Y+Y'$, and can therefore be composed.

Lemma 5.5 The loops of extensions L'' and L''' are homotopic.

Proof of Lemma: To simplify notation, we shall look at the case when r=1; the general case is similar.

In this case, L'' takes the form $Y+Y' \overset{(\alpha_0,\alpha'_0)}{\longleftrightarrow} C_1+C'_1 \overset{(\alpha_1,\alpha'_1)}{\longleftrightarrow} Y+Y'$, which is homotopic to $Y+Y' \overset{(\alpha_0,\mathrm{id})}{\longleftrightarrow} C_1+Y' \overset{(\mathrm{id},\alpha'_0)}{\longleftrightarrow} C_1+C'_1 \overset{(\alpha_1,\mathrm{id})}{\longleftrightarrow} Y+C'_1 \overset{(\mathrm{id},\alpha'_1)}{\longleftrightarrow} Y+Y'$ by definition of the homotopy of paths. This loop is homotopic to $Y+Y' \overset{(\alpha_0,\mathrm{id})}{\longleftrightarrow} C_1+Y' \overset{(\alpha_1,\alpha'_0)}{\longleftrightarrow} Y+C'_1 \overset{(\mathrm{id},\alpha'_1)}{\longleftrightarrow} Y+Y'$ which is finally homotopic to $Y+Y' \overset{(\alpha_0,\mathrm{id})}{\longleftrightarrow} C_1+Y' \overset{(\alpha_1,\mathrm{id})}{\longleftrightarrow} Y+Y' \overset{(\mathrm{id},\alpha'_0)}{\longleftrightarrow} Y+C'_1 \overset{(\mathrm{id},\alpha'_1)}{\longleftrightarrow} Y+Y' = L'''$. $\square$

Finally we obtain:

Lemma 5.6 Given three extensions $\mathbf{E}$, $\mathbf{F}$, and $\mathbf{G}$, and changing base points so that the identities make sense, we have:

$$\begin{aligned}
\Omega(\mathbf{G}, \mathbf{E}+\mathbf{F}) &= \Omega(\mathbf{G}, \mathbf{E}).\Omega(\mathbf{G}, \mathbf{F}) \text{ and} \\
\Omega(\mathbf{F}+\mathbf{G}, \mathbf{E}) &= \Omega(\mathbf{F}, \mathbf{E}).\Omega(\mathbf{G}, \mathbf{E}).
\end{aligned}$$

Now using Lemma 5.6, we know that $\Omega(\mathbf{F}, \mathbf{E}).\Omega(\mathbf{F}, -\mathbf{E}) = \Omega(\mathbf{F}, \mathbf{0})$, where $\mathbf{0}$ is the trivial extension of degree m. So if $\Omega(\mathbf{F}, \mathbf{0})$ is the trivial loop, we have proved Proposition 5.4.

Let us consider the loop $\Omega(\mathbf{F}, \mathbf{0})$: computing the maps in this loop gives $\lambda_{\mathbf{0},\mathbf{F}} = \rho_{\mathbf{F},\mathbf{0}}$ and $\lambda_{\mathbf{F},\mathbf{0}} = \rho_{\mathbf{0},\mathbf{F}}$. We also have $\mathbf{F}\underline{\otimes}_H\mathbf{0} = \mathbf{0}\underline{\otimes}_H\mathbf{F}$ and $\mathbf{F}\sharp\mathbf{0} = \mathbf{0}\sharp\mathbf{F}$. Therefore, using the homotopy relations, the loops

$$\begin{tikzcd}
& \mathbf{0}\underline{\otimes}_H\mathbf{F} \arrow[dl, "\lambda_{\mathbf{0},\mathbf{F}}"'] \arrow[dr, "\lambda_{\mathbf{F},\mathbf{0}}"] & \\
\mathbf{F}\sharp\mathbf{0} & \Omega(\mathbf{F}, \mathbf{0}) & \mathbf{0}\sharp\mathbf{F} \\
& \mathbf{F}\underline{\otimes}_H\mathbf{0} \arrow[ul, "\lambda_{\mathbf{0},\mathbf{F}}"] \arrow[ur, "\lambda_{\mathbf{F},\mathbf{0}}"'] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbf{0}\underline{\otimes}_H\mathbf{F} \arrow[d, "\lambda_{\mathbf{F},\mathbf{0}}"] \\
\mathbf{0}\sharp\mathbf{F} \\
\mathbf{0}\underline{\otimes}_H\mathbf{F} \arrow[u, "\lambda_{\mathbf{F},\mathbf{0}}"']
\end{tikzcd}$$

are homotopic, and the last one is homotopic to the trivial loop based at $\mathbf{0}\underline{\otimes}_H\mathbf{F}$. This concludes the proof. ■

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