GENERALIZED ASSOCIAHEDRA VIA QUIVER REPRESENTATIONS

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ABSTRACT. We provide a quiver-theoretic interpretation of certain smooth complete simplicial fans associated to arbitrary finite root systems in recent work of S. Fomin and A. Zelevinsky. The main properties of these fans then become easy consequences of the known facts about tilting modules due to K. Bongartz, D. Happel and C. M. Ringel.

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1. INTRODUCTION

The goal of this paper is to provide a conceptual interpretation for *generalized associahedra* introduced in [7]. Recall that in [7, Section 3], to every finite (crystallographic) root system Φ is associated a smooth complete simplicial fan $\Delta(\Phi)$ in the ambient real vector space. (Here smoothness means that every full-dimensional cone of $\Delta(\Phi)$ is generated by a $\mathbb{Z}$-basis of the root lattice Q, and completeness means that the union of all cones is the whole ambient space.) As proved in [5], $\Delta(\Phi)$ is the normal fan of a simple convex polytope. With some abuse of terminology, both $\Delta(\Phi)$ and the corresponding simple polytope are referred to as the generalized associahedron associated to Φ. This construction includes as special cases the well-known associahedron (or Stasheff polytope) [12] for Φ of type A, and the Bott-Taubes cyclohedron [4] for Φ of type B or C.

The construction in [7, Section 3] appears somewhat ad hoc (as indicated in [7], it is motivated by the developing theory of cluster algebras [6] but this connection is still unpublished). In this paper, we present a natural interpretation of generalized

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associahedra in terms of quiver representations. With this interpretation, the main results in [7, Section 3] become easy consequences of the known facts about tilting representations [8].

To explain our results in more detail, let us recall the main ingredients of the construction in [7]:

1. The set of *almost positive roots* $\Phi_{\geq -1} = \Phi_{>0} \cup (-\Pi)$, where $\Pi = \{\alpha_i : i \in I\}$ is a fixed set of simple roots in the root system Φ, and $\Phi_{>0}$ is the corresponding set of positive roots. Almost positive roots are precisely the generators of one-dimensional cones in the fan $\Delta(\Phi)$.
2. The function $\Phi_{\geq -1} \times \Phi_{\geq -1} \to \mathbb{Z}_{\geq 0}$ assigning to each $\alpha, \beta \in \Phi_{\geq -1}$ their *compatibility degree* $(\alpha \| \beta)$. A subset $C \subset \Phi_{\geq -1}$ generates a cone in $\Delta(\Phi)$ if and only if it is *compatible*, i.e., $(\alpha \| \beta) = 0$ for any $\alpha, \beta \in C$.
3. The group of piecewise-linear transformations of the root lattice Q generated by "truncated simple reflections" σ_i for $i \in I$. Every σ_i preserves $\Phi_{\geq -1}$.
4. In the case when Φ is irreducible, the group generated by the σ_i contains two distinguished involutions τ_+ and τ_- that preserve $\Delta(\Phi)$.

Our aim is to provide natural interpretations of all these notions in terms of quiver representations. For simplicity, we only work with simply-laced root systems, i.e., those corresponding to Dynkin diagrams of the ADE-type. (The general case reduces to this with the help of a familiar "folding" procedure, see e. g. [10, Chapter 14], or by modifying our constructions in the spirit of [9, 13].) Let Γ be a quiver (= oriented graph) whose underlying graph is a disjoint union of Dynkin diagrams of type A_n, D_n, E_6, E_7, E_8. By Gabriel's theorem, the dimension vector map identifies isomorphism classes of indecomposable representations of Γ with positive roots of the corresponding root system Φ. As a first step in our program, we include the category of representations of Γ into a bigger category of *decorated representations* (see Section 2 below). A suitable modification of the dimension vector map establishes a bijection $\alpha \mapsto U_\alpha^\Gamma$ between almost positive roots in Φ and isomorphism classes of indecomposable decorated representations of Γ.

As a next step, we find a natural interpretation for the compatibility degree function. To do this, in Section 3 we construct a bifunctor E_Γ assigning a finite-dimensional vector space to every two decorated representations of Γ. We then define the Γ-compatibility degree function $\Phi_{\geq -1} \times \Phi_{\geq -1} \to \mathbb{Z}_{\geq 0}$ by setting $(\alpha \| \beta)_\Gamma = \dim E_\Gamma(U_\alpha^\Gamma, U_\beta^\Gamma)$. As in (2) above, we call a subset $C \subset \Phi_{\geq -1}$ Γ-*compatible* if $(\alpha \| \beta)_\Gamma = 0$ for any $\alpha, \beta \in C$. We prove that the cones generated by Γ-compatible subsets form a smooth complete simplicial fan in the ambient space $Q_\mathbb{R}$ (Theorem 3.7). This result follows without much effort from known results about tilting representations due to D. Happel and C. M. Ringel [8] and K. Bongartz [2].

The original compatibility degree and the corresponding fan $\Delta(\Phi)$ appear as a special case of this construction corresponding to an *alternating* quiver Γ_0 for which every vertex is a source or a sink. Our verification of this fact involves the final step in our program: finding a quiver-theoretic interpretation of the piecewise-linear involutions σ_i acting on the root lattice Q and preserving $\Phi_{\geq -1}$. We do this by extending the familiar reflection functors S_i (see e.g., [1]) to the functors Σ_i on the category of decorated representations; more precisely, each Σ_i sends decorated representations of a quiver Γ to those of the quiver $s_i\Gamma$ obtained from Γ by reversing all arrows at a vertex i which is a source or a sink in Γ. The main advantage of the extended reflection functors is that each Σ_i induces a bijection between isomorphism classes of decorated representations of Γ and those of $s_i\Gamma$ (recall that the classical reflection functors S_i send some non-zero quiver representations to zero). Furthermore, $\Sigma_i^2(M)$ is isomorphic to M for every decorated representation M. Note however that Σ_i is *not* an equivalence of categories; in fact, it sends some non-zero morphisms to zero.

The key property of the functors Σ_i for our purposes is that they are compatible with the bifunctors E_Γ in the following sense: the vector space $E_\Gamma(M, N)$ is isomorphic to $E_{s_i\Gamma}(\Sigma_i M, \Sigma_i N)$ for every two decorated representations M and N of Γ (Theorem 4.7 below). Returning to piecewise-linear involutions σ_i, we show that each σ_i is obtained by combining the reflection functor Σ_i with the signed dimension vector map for decorated quiver representations (see Lemma 4.5 for a precise statement). This implies the following important property of the Γ-compatibility degree: $(\sigma_i\alpha||\sigma_i\beta)_{s_i\Gamma} = (\alpha||\beta)_\Gamma$ (Proposition 4.10).

The role of two distinguished piecewise-linear involutions τ_+ and τ_- also becomes clear in this context. The functors Σ_i give rise to a *groupoid* whose objects correspond to different orientations Γ of a fixed Dynkin graph, and each Σ_i is viewed as a morphism from Γ to $s_i\Gamma$. It is natural to enlarge this groupoid by the *duality functor* D that sends every quiver Γ to the opposite quiver obtained from Γ by reversing all arrows. We study these two groupoids in Section 5 (in a more general setting where the Dynkin graph is replaced by an arbitrary finite tree). We prove in particular that for the groupoid generated by the Σ_i and D, the automorphism group of an alternating quiver Γ_0 is generated by two involutions T_+ and T_-, whose induced action on the root lattice Q is given by the involutions τ_+ and τ_- from [7].

We conclude the paper by an amusing application of our results to the theory of ordinary quiver representations. We say that a set C of isomorphism classes of indecomposable representations of a quiver Γ is *Ext-free* if $\mathrm{Ext}^1(M, N) = 0$ for every $M, N \in C$. The Ext-free sets form a simplicial complex, and we show that the f-vector of this complex (i.e., the number of Ext-free sets of any given size) is an invariant of the underlying Dynkin graph, i.e., is independent of the choice of the orientation Γ (Proposition 6.1). Note that the complexes of Ext-free sets corresponding to different orientations of the same Dynkin graph are in general not isomorphic. Passing from ordinary quiver representations to decorated ones allows us to view the complex of Ext-free sets for Γ as a subcomplex of the complex of Γ-compatible sets discussed above; these bigger complexes are all isomorphic to each other, by the action of extended reflection functors Σ_i.

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2. Decorated quiver representations

Let Γ be a Dynkin quiver (that is, the underlying unoriented graph is a disjoint union of Dynkin diagrams of type A_n, D_n, E_6, E_7, E_8), and let I be the set of vertices. Fix once and for all an algebraically closed field k. Let $\operatorname{rep}\Gamma$ be the category of finite dimensional k-representations of Γ; the objects M of $\operatorname{rep}\Gamma$ thus consist of tuples of k-vector spaces $(M_i)_{i\in I}$ and tuples of linear maps $(M_a : M_i \to M_j)_{a:i\to j}$ parametrized by the arrows a in Γ. For each vertex $i \in I$, denote by E_i the associated simple representation. The space of homomorphisms (resp. extensions) between representations $M, N \in \operatorname{rep}\Gamma$ is denoted by $\operatorname{Hom}_\Gamma(M,N)$ (resp. $\operatorname{Ext}^1_\Gamma(M,N)$). The k-linear dual of a k-vector space X is denoted by X^*.

Let Φ be the root system associated with the Dynkin graph underlying Γ, let $\Phi_{>0}$ be the positive roots, and let $\Phi_{\geq -1}$ be the union of $\Phi_{>0}$ and the negative simple roots $-\Pi = \{-\alpha_i : i \in I\}$. For a subset $J \subset I$, let $\Phi(J)$ be the root subsystem of roots whose support is contained in J.

Let Q be the root lattice, and let Q_+ be the positive span of $\Phi_{>0}$ in Q. For $\gamma \in Q$ and $i \in I$, denote by $[\gamma : \alpha_i]$ the multiplicity of α_i in γ.

The Grothendieck group of the category $\operatorname{rep}\Gamma$ can be identified canonically with Q via the dimension vector map $\underline{\dim} : K_0(\operatorname{rep}\Gamma) \to Q$ given on objects $M \in \operatorname{rep}\Gamma$ by

$$\underline{\dim}(M) = \sum_{i\in I} (\dim M_i)\alpha_i.$$

Associate to Γ a new quiver $\widetilde{\Gamma} = \Gamma \cup \{i^- : i \in I\}$, with no arrows pointing from or to the added vertices i^-. The category $\operatorname{rep}\widetilde{\Gamma}$ will be called the decorated representation category for Γ. The simple representation E_{i^-} will be denoted by E_i^-; the objects of $\operatorname{rep}\widetilde{\Gamma}$ are thus of the form $M = M^+ \oplus V$, where $M^+ \in \operatorname{rep}\Gamma$, and $V = \oplus_{i\in I}(E_i^- \otimes V_i)$ is viewed as an I-graded vector space. More explicitly, an object of $\operatorname{rep}\widetilde{\Gamma}$ consists of tuples of vector spaces M_i and V_i for $i \in I$, together with linear maps $M_a : M_i \to M_j$ for all arrows $a : i \to j$ in Γ; morphisms between two such pairs of tuples (M_i, V_i) and (N_i, W_i) consist of linear maps $f_i : M_i \to N_i$, together with linear maps $\varphi_i : V_i \to W_i$, such that $f_j \circ M_a = N_a \circ f_i$ for all arrows $a : i \to j$.

The space of I-graded homomorphisms of I-graded vector spaces V, W will be denoted by $\operatorname{Hom}^I(V,W)$.

The category $\operatorname{rep}\Gamma$ is viewed as a subcategory of $\operatorname{rep}\widetilde{\Gamma}$ in the obvious way. Define the signed dimension vector $\underline{\mathrm{sdim}}(M)$ of $M = M^+ \oplus V$ by

$$\underline{\mathrm{sdim}}(M) = \underline{\dim} M^+ - \sum_{i\in I}(\dim V_i)\alpha_i = \sum_{i\in I}(\dim M_i - \dim V_i)\alpha_i.$$

By Gabriel's theorem, the dimension vector induces a bijection between the isoclasses of indecomposable representations in $\operatorname{rep}\Gamma$ and the positive roots $\Phi_{>0}$.

We have the following extension of Gabriel's theorem, which is immediate from the definition of the decorated representation category:

Proposition 2.1. *The map $\underline{sdim}$ induces a bijection $\alpha \mapsto [U_\alpha^\Gamma]$ between the roots $\alpha \in \Phi_{\geq -1}$ and isoclasses $[U_\alpha^\Gamma]$ of indecomposable objects in $\operatorname{rep}\widetilde{\Gamma}$.*

3. Clusters from decorated representations

For decorated representations $M = M^+ \oplus V$ and $N = N^+ \oplus W$ in $\operatorname{rep}\widetilde{\Gamma}$, we define

$$E_\Gamma(M,N) = \operatorname{Ext}^1_\Gamma(M^+,N^+) \oplus \operatorname{Ext}^1_\Gamma(N^+,M^+)^* \oplus \operatorname{Hom}^I(M^+,W) \oplus \operatorname{Hom}^I(V,N^+). \tag{3.1}$$

Obviously, this defines a bifunctor

$$E_\Gamma(\_,\_) : (\operatorname{rep}\widetilde{\Gamma})^{\mathrm{op}} \times \operatorname{rep}\widetilde{\Gamma} \to \mathrm{vect},$$

contravariant in the first argument, and covariant in the second one. Here vect denotes the category of finite-dimensional vector spaces.

We define the Γ-compatibility degree of two roots $\alpha,\beta \in \Phi_{\geq -1}$ by

$$(\alpha||\beta)_\Gamma = \dim E_\Gamma(U^\Gamma_\alpha, U^\Gamma_\beta). \tag{3.2}$$

This definition makes it clear that $(||)_\Gamma$ is symmetric and compatible with root subsystems in the following sense: if $\Gamma(J)$ denotes the full subquiver of Γ supported on a subset of vertices $J \subset I$, and if $\alpha,\beta \in \Phi_{\geq -1}(J)$, then $(\alpha||\beta)_{\Gamma(J)} = (\alpha||\beta)_\Gamma$ (compare [7, Proposition 3. 3.]). There is an additional symmetry property involving the natural duality functor $D : \operatorname{rep}\widetilde{\Gamma} \to \operatorname{rep}\widetilde{\Gamma}^{\mathrm{op}}$ that acts by replacing all vector spaces and maps by their duals, thus replacing Γ with the opposite orientation Γ^{op}:

Proposition 3.1. *For all roots $\alpha,\beta \in \Phi_{\geq -1}$, we have $(\alpha||\beta)_\Gamma = (\alpha||\beta)_{\Gamma^{\mathrm{op}}}$.*

This follows at once from the natural identification

$$E_{\Gamma^{\mathrm{op}}}(DN, DM) \simeq E_\Gamma(M,N).$$

We can now define Γ-compatible subsets and Γ-clusters as in [7, Definition 3. 4.].

Definition 3.2. A subset C of $\Phi_{\geq -1}$ is called Γ-*compatible* if $(\alpha||\beta)_\Gamma = 0$ for all $\alpha,\beta \in C$. The subset C is called a Γ-*cluster* if it is a maximal (by inclusion) Γ-compatible subset.

Definition 3.3. The *negative support* $S(C)$ of a subset C of $\Phi_{\leq -1}$ is defined by $S(C) = \{i \in I : -\alpha_i \in C\}$. The subset C is called *positive* if $C \subset \Phi_{>0}$, i.e., $S(C) = \emptyset$.

Moreover, [7, Proposition 3. 6.] holds with the same proof.

Proposition 3.4. *For every subset $J \subset I$, the correspondence $C \mapsto C - \{-\alpha_i : i \in J\}$ defines a bijection between Γ-compatible subsets (resp. clusters) for Φ with negative support J and positive Γ-compatible subsets (resp. clusters) for $\Phi(I-J)$.*

We can now prove the main properties of Γ-clusters.

Proposition 3.5 (Purity). *All Γ-clusters have the same size n, the rank of the root system Φ. Moreover, each Γ-cluster is a $\mathbb{Z}$-basis for the root lattice Q.*

Proof. In view of Proposition 3.4, it suffices to prove our statements for positive Γ-clusters. To any subset $C \subset \Phi_{>0}$ let us associate a Γ-representation $X_C = \oplus_{\alpha\in C} U^\Gamma_\alpha$. By our definitions, C is Γ-compatible if and only if $\operatorname{Ext}^1(X_C, X_C) = 0$. Now we use the theory of tilting modules introduced in [8]. In our situation, they can be defined as follows (see [8, Theorem 4.5]): a Γ-representation T is *tilting* if $\mathrm{Ext}^1(T,T)=0$, and T has precisely n different isoclasses of indecomposable direct summands.

Now let C be a positive Γ-cluster. By [2, Lemma 2.1], X_C is a direct summand of a tilting module T. By the maximality property of C, the representation X_C contains all isoclasses of indecomposable direct summands of T and so $|C|=n$. The fact that C is a $\mathbb{Z}$-basis for Q is contained in the proof of [8, Lemma 4.3]. $\square$

Proposition 3.6 (Cluster expansion). *Each element $\gamma \in Q$ has a unique Γ-cluster expansion, that is, $\gamma = \sum_{\alpha\in\Phi_{\geq -1}} m_\alpha \alpha$ such that $m_\alpha \in \mathbb{Z}_{\geq 0}$ for all α, and $m_\alpha m_\beta = 0$ whenever $\alpha, \beta \in \Phi_{\geq -1}$ are not Γ-compatible.*

Proof. We prove that for each $\gamma \in Q$, there exists a unique decorated representation $M = M^+ \oplus V \in \mathrm{rep}\,\widetilde{\Gamma}$ such that $\underline{\mathrm{sdim}}M = \gamma$ and $E_\Gamma(M,M)=0$ (the proposition follows from this by definition of the Γ-compatibility degree). From the definition of E_Γ it is clear that $E_\Gamma(M,M)=0$ if and only if $\mathrm{Ext}^1_\Gamma(M^+,M^+)=0$, and M^+ and V have disjoint supports in I. Denote by $d=\sum_{i\in I} d_i\alpha_i$ the dimension vector of M^+. Note that it is uniquely determined by γ, namely $d = \sum_{i\in I}\max([\gamma:\alpha_i],0)\alpha_i$. Define the variety R_d of k-representations of Γ of dimension vector d by $R_d = \oplus_{a:i\to j}\mathrm{Hom}(k^{d_i},k^{d_j})$. The group $G_d = \prod_{i\in I}\mathrm{GL}(k^{d_i})$ acts on R_d via $(g_i)(M_a) = (g_j M_a g_i^{-1})_{a:i\to j}$. By definition, the orbits of G_d in R_d correspond bijectively to the isomorphism classes of representations of Γ of dimension vector d. Denote by $\mathcal{O}_A$ the G_d-orbit of a representation A. Since R_d is irreducible and there are only finitely many orbits by Gabriel's theorem, R_d contains a unique dense orbit. It is well known that (see, e.g., [3])

$$\mathrm{codim}\,_{R_d}\mathcal{O}_A = \dim \mathrm{Ext}^1_\Gamma(A,A).$$

It follows that there exists a unique (up to isomorphism) representation M^+ of dimension vector d such that $\mathrm{Ext}^1_\Gamma(M^+,M^+)=0$. Since the I-graded k-space V is uniquely determined by its signed dimension vector $\gamma - d$, uniqueness and existence follow. $\square$

As in [7, Proof of Theorem 1. 10.], Propositions 3.5 and 3.6 imply the following:

Theorem 3.7. *The simplicial cones $\mathbf{R}_{\geq 0}C$ generated by all Γ-clusters C form a smooth complete simplicial fan Δ_Γ in the ambient real vector space $Q_{\mathbf{R}}$ of the root system.*

Remark 3.8. The "positive part" of Δ_Γ was studied by L. Unger [14] following a suggestion of C. M. Ringel.

4. Extended reflection functors

We recall the well-known reflection functors on quiver representations [1]. Let $i \in I$ be a source (resp. sink) in Γ, and let $s_i\Gamma$ be the quiver obtained by reversing all arrows pointing from (resp. to) i. Given a representation $M \in \mathrm{rep}\,\Gamma$, define a representation $S_i(M) \in \mathrm{rep}\,s_i\Gamma$ by

(4.1)

$$S_i(M)_i = \left\{ \begin{array}{lcl} \mathrm{Coker}(\oplus_{a:i\to j}M_a) & , & i \text{ a source} \\ \mathrm{Ker}(\oplus_{a:j\to i}M_a) & , & i \text{ a sink} \end{array} \right\}, \quad S_i(M)_j = M_j \text{ for all } j \neq i,$$

and the maps $S_i(M)_a$ are either unchanged, or derived in an obvious way from the canonical injection or projection maps. The functor S_i is defined on morphisms using the canonical induced maps on cokernels (resp. kernels).

Let $\mathrm{rep}\,{}^i\Gamma$ denote the full subcategory of $\mathrm{rep}\,\Gamma$ consisting of representations without E_i as a direct summand. Then S_i induces an equivalence of categories

$$S_i : \mathrm{rep}\,{}^i\Gamma \xrightarrow{\sim} \mathrm{rep}\,{}^i s_i\Gamma.$$

4.1. **Definition and basic properties.** Assume that $i \in I$ is a source or a sink in Γ. We will now define a reflection functor

$$\Sigma_i : \mathrm{rep}\,\widetilde{\Gamma} \to \mathrm{rep}\,\widetilde{s_i\Gamma},$$

which has the advantage – over the usual reflection functor S_i – that it induces a bijection on isomorphism classes of decorated representations.

Let $M = M^+ \oplus V$ be an object of $\mathrm{rep}\,\widetilde{\Gamma}$. Suppose first that i is a source in Γ, and consider the canonical exact sequence of Γ-representations

$$0 \to E_i \otimes \mathrm{Hom}_\Gamma(E_i, M^+) \to M^+ \to M' \to 0. \tag{4.2}$$

Note for future reference the following well-known exact sequences:

$$0 \to \mathrm{Hom}_\Gamma(E_i, M^+) \to M_i \to \bigoplus_{i\to j} M_j \to \mathrm{Ext}^1_\Gamma(E_i, M^+) \to 0. \tag{4.3}$$

The analogue for i a sink is

$$0 \to \mathrm{Hom}_\Gamma(M^+, E_i) \to M_i^* \to \bigoplus_{j\to i} M_j^* \to \mathrm{Ext}^1_\Gamma(M^+, E_i) \to 0. \tag{4.4}$$

Both these sequences are special cases of [11, Lemma 2.1].

In particular, (4.3) implies the natural isomorphism

$$\mathrm{Hom}_\Gamma(E_i, M^+) \simeq \mathrm{Ker}(\oplus_{a:i\to j} M_a).$$

Returning to (4.2), it is clear that $\mathrm{Hom}_\Gamma(E_i, M') = 0$, thus $M' \in \mathrm{rep}\,{}^i\Gamma$. We define $\Sigma_i(M) = \overline{M}^+ \oplus \overline{V}$ by setting

$$\overline{M}^+ = S_i(M') \oplus (E_i \otimes V_i), \quad \overline{V} = (E_i^- \otimes \mathrm{Hom}_\Gamma(E_i, M^+)) \oplus \bigoplus_{j\neq i} (E_j^- \otimes V_j). \tag{4.5}$$

More explicitly, $\Sigma_i(M)$ replaces M_i by $\mathrm{Coker}(\oplus_{a:i\to j} M_a) \oplus V_i$, replaces V_i by $\mathrm{Ker}(\oplus_{a:i\to j} M_a)$, and leaves the rest of the spaces unchanged.

To define Σ_i on morphisms, let $N = N^+ \oplus W$ be another object of $\mathrm{rep}\,\widetilde{\Gamma}$, and define N', $\overline{N}$, $\overline{W}$ in the same way as above. Let $f = f^+ \oplus (\bigoplus_j \varphi_j)$ be a morphism from M to N in $\mathrm{rep}\,\widetilde{\Gamma}$. Since $\mathrm{Hom}_\Gamma(E_i, N') = 0$, the map f^+ induces maps

$$\psi : \mathrm{Hom}_\Gamma(E_i, M^+) \to \mathrm{Hom}_\Gamma(E_i, N^+), \quad f' : M' \to N'.$$

We then define the morphism $\Sigma_i(f)$ as the direct sum of the morphisms

$$S_i(f') \oplus \varphi_i : \overline{M}^+ \to \overline{N}^+$$

and

$$\psi \oplus \bigoplus_{j\neq i} \varphi_j : \overline{V} \to \overline{W}.$$

More explicitly, $\Sigma_i(f)$ replaces the map f_i by $g \oplus \varphi_i$ (with respect to the above sum decomposition), where g is the map induced by f_i on the cokernels, replaces φ_i by the map h induced on the kernels, and leaves the rest of the maps unchanged.

To treat the case when i is a sink, we use the duality functor D (see Section 3). This allows us to reduce the case of a sink to that of a source by defining

$$\Sigma_i = D \circ \Sigma_i \circ D.$$

Explicitly, take the canonical exact sequence

$$0 \to M' \to M^+ \to E_i \otimes \mathrm{Hom}_\Gamma(M^+, E_i)^* \to 0.$$

Then we have $\Sigma_i(M) = \overline{M}^+ \oplus \overline{V}$, where

$$\overline{M}^+ = S_i(M') \oplus (E_i \otimes V_i)$$

and

$$\overline{V} = (E_i^- \otimes \mathrm{Hom}_\Gamma(M^+, E_i)^*) \oplus \bigoplus_{j \neq i} (E_j^- \otimes V_j).$$

The description of the action of Σ_i on morphisms is similar to the case of a source and is left to the reader.

The following proposition summarizes some simple properties of the extended reflection functors. All of them are immediate from the definitions.

Proposition 4.1. 1. *If $M \in \mathrm{rep}\,{}^i\Gamma \subset \mathrm{rep}\,\widetilde{\Gamma}$, then $\Sigma_i(M) = S_i(M)$.*
2. *We have $\Sigma_i(E_i) = E_i^-$ and $\Sigma_i(E_i^-) = E_i$.*
3. *We have $\Sigma_i(E_j^-) = E_j^-$ whenever $i \neq j$.*
4. *Suppose i is a source (resp. a sink) in Γ, and $M \in \mathrm{rep}\,{}^i\Gamma \subset \mathrm{rep}\,\widetilde{\Gamma}$. Let f be any morphism from M to E_i (resp. from E_i to M). Then $\Sigma_i(f) = 0$.*

Proposition 4.2. *For any decorated representation $M \in \mathrm{rep}\,\widetilde{\Gamma}$ and any vertex $i \in I$ we have $\Sigma_i^2(M) \simeq M$.*

Proof. Assume $i \in I$ to be a source in Γ (the case of a sink follows by duality). For a decorated representation $M = M^+ \oplus V \in \mathrm{rep}\,\widetilde{\Gamma}$, denote by $N = N^+ \oplus W$ the decorated representation $\Sigma_i(M)$. (In the notation of 4.5, we have $N = \overline{M}$ and $W = \overline{V}$.) By definition, we have:

$$\Sigma_i^2(M) = \overline{N}^+ \oplus \overline{W}.$$

Here

$$\overline{N}^+ = S_i(N') \oplus (E_i \otimes W_i) \simeq M' \oplus (E_i \otimes \mathrm{Hom}_\Gamma(E_i, M^+),$$

and

$$\begin{aligned} \overline{W} &= (E_i^- \otimes \mathrm{Hom}_{s_i\Gamma}(S_i(N') \oplus (E_i \otimes V_i), E_i))^* \oplus \bigoplus_{j \neq i} (E_j^- \otimes W_j) \\ &\simeq (E_i^- \otimes V_i) \oplus \bigoplus_{j \neq i} (E_j^- \otimes V_j) \simeq V. \end{aligned}$$

Note that $M' \oplus (E_i \otimes \mathrm{Hom}_\Gamma(E_i, M^+)$ is (non-canonically) isomorphic to M^+ since the sequence (4.2) splits. Therefore $\Sigma_i^2(M) \simeq M^+ \oplus V = M$. □

Corollary 4.3. *The functor Σ_i induces an isomorphism of Grothendieck groups $\Sigma_i : K_0(\mathrm{rep}\,\widetilde{\Gamma}) \xrightarrow{\sim} K_0(\mathrm{rep}\,\widetilde{s_i\Gamma})$.*

We define piecewise linear reflections σ_i on Q as in [7, 2.2]. For the simply-laced case, this can be stated as follows:

Definition 4.4. *For $i \in I$, define $\sigma_i : Q \to Q$ by*

$$[\sigma_i \alpha : \alpha_j] = \begin{cases} [\alpha : \alpha_j] & , \quad j \neq i \\ -[\alpha : \alpha_i] + \sum\limits_{k — i} \max([\alpha : \alpha_k], 0) & , \quad j = i \end{cases},$$

where k —— i means that k and i are joined by an edge in I.

Lemma 4.5. *For any source or sink $i \in I$ of Γ, and any object $M = M^+ \oplus V \in \operatorname{rep} \widetilde{\Gamma}$ such that $\dim M_i \cdot \dim V_i = 0$, we have $\underline{sdim}(\Sigma_i(M)) = \sigma_i(\underline{sdim}(M))$.*

Proof. Let i be a source in Γ (the case of a sink is dual). The reflection functor Σ_i does not change any component M_j or V_j for $j \neq i$, which corresponds to the first part of the above definition of σ_i. So we concentrate on the i-components. By definition, we have $[\underline{\mathrm{sdim}} M : \alpha_i] = \dim M_i - \dim V_i$. Using the definition of Σ_i, we compute:

$$[\underline{\mathrm{sdim}} \Sigma_i(M) : \alpha_i] = \dim S_i(M')_i + \dim V_i - \dim \operatorname{Hom}_\Gamma(E_i, M).$$

In view of the exact sequence (4.3),

$$\dim S_i(M')_i - \dim \operatorname{Hom}_\Gamma(E_i, M) = -\dim M_i + \sum_{i \to j} \dim M_j.$$

Therefore,

$$\begin{aligned} [\underline{\mathrm{sdim}} \Sigma_i(M) : \alpha_i] &= -\dim M_i + \sum_{i \to j} \dim M_j + \dim V_i = \\ &= -[\underline{\mathrm{sdim}} M : \alpha_i] + \sum_{i \to j} \dim M_j. \end{aligned}$$

Under the assumption $\dim M_i \cdot \dim V_i = 0$, this last expression is the same as in Definition 4.4. $\square$

Corollary 4.6. *For all roots $\alpha \in \Phi_{\geq -1}$, and any $i \in I$ which is a source or a sink in Γ, we have*

$$\underline{sdim}(\Sigma_i(U_\alpha^\Gamma)) = \sigma_i(\underline{sdim}(U_\alpha^\Gamma)).$$

Proof. Notice that for any indecomposable object $M \in \operatorname{rep} \widetilde{\Gamma}$, we have $\dim M_i \cdot \dim V_i = 0$ for all $i \in I$. $\square$

4.2. **Extended reflection functors and compatibility degree.** Recall the bifunctor E_Γ on decorated representations introduced in (3.1).

Theorem 4.7. *If $i \in I$ is a source or a sink in Γ then $E_{s_i \Gamma}(\Sigma_i(M), \Sigma_i(N)) \simeq E_\Gamma(M, N)$ for any decorated representations M and N.*

Proof. We start with a lemma which is an immediate consequence of (4.3) and (4.4).

Lemma 4.8. *Let A be a representation of Γ. If i is a source in Γ, then*

$$S_i(A)_i \simeq \operatorname{Ext}^1_\Gamma(E_i, A).$$

If i is a sink in Γ, then

$$S_i(A)_i \simeq \operatorname{Ext}^1_\Gamma(A, E_i)^*.$$

Returning to the proof of Theorem 4.7, we will only treat the case of a source $i \in I$; the case of a sink is dual. As usual, we write $M = M^+ \oplus V$ and $N = N^+ \oplus W$. Recall the (non-canonical) decompositions

$$M^+ = M' \oplus (E_i \otimes X), \quad N^+ = N' \oplus (E_i \otimes Y),$$

where $X = \mathrm{Hom}_\Gamma(E_i, M^+)$ and $Y = \mathrm{Hom}_\Gamma(E_i, N^+)$. Using the definition of Σ_i, we decompose

$$\begin{aligned} &E_{s_i\Gamma}(\Sigma_i(M), \Sigma_i(N)) \\ &\quad = \mathrm{Ext}^1_{s_i\Gamma}(S_i(M'), S_i(N')) \oplus (\mathrm{Ext}^1_{s_i\Gamma}(S_i(M'), E_i) \otimes W_i) \oplus \\ &\qquad \mathrm{Ext}^1_{s_i\Gamma}(S_i(N'), S_i(M'))^* \oplus (\mathrm{Ext}^1_{s_i\Gamma}(S_i(N'), E_i) \otimes V_i)^* \oplus \\ &\qquad \mathrm{Hom}(S_i(M')_i, Y) \oplus \mathrm{Hom}(V_i, Y) \oplus \bigoplus_{j \neq i} \mathrm{Hom}(M_j, W_j) \oplus \\ &\qquad \mathrm{Hom}(X, S_i(N')_i) \oplus \mathrm{Hom}(X, W_i) \oplus \bigoplus_{j \neq i} \mathrm{Hom}(V_j, N_j). \end{aligned}$$

Since the classical reflection functor S_i induces an equivalence of categories $\mathrm{rep}\, {}^i\Gamma \tilde{\to} \mathrm{rep}\, {}^i s_i\Gamma$, we have

$$\mathrm{Ext}^1_{s_i\Gamma}(S_i(M'), S_i(N')) \simeq \mathrm{Ext}^1_\Gamma(M', N'), \tag{4.6}$$

$$\mathrm{Ext}^1_{s_i\Gamma}(S_i(N'), S_i(M'))^* \simeq \mathrm{Ext}^1_\Gamma(N', M')^*. \tag{4.7}$$

Furthermore, applying the second part of Lemma 4.8 to $A = S_i(M')$ and using Proposition 4.2, we obtain

$$\mathrm{Ext}^1_{s_i\Gamma}(S_i(M'), E_i) \otimes W_i \simeq \mathrm{Hom}(M'_i, W_i). \tag{4.8}$$

Similarly

$$(\mathrm{Ext}^1_{s_i\Gamma}(S_i(N'), E_i) \otimes V_i)^* \simeq \mathrm{Hom}(V_i, N'_i). \tag{4.9}$$

Substituting the expressions provided by (4.6) - (4.9) into the above decomposition of $E_{s_i\Gamma}(\Sigma_i(M), \Sigma_i(N))$ and regrouping the terms, we can write it as the direct sum of the following four terms:

$$(\mathrm{Ext}^1_\Gamma(M', N') \oplus \mathrm{Hom}(X, S_i(N')_i)) \oplus \tag{4.10}$$

$$(\mathrm{Ext}^1_\Gamma(N', M')^* \oplus \mathrm{Hom}(S_i(M')_i, Y)) \oplus \tag{4.11}$$

$$(\mathrm{Hom}(M'_i, W_i) \oplus \bigoplus_{j \neq i} \mathrm{Hom}(M_j, W_j) \oplus \mathrm{Hom}(X, W_i)) \oplus \tag{4.12}$$

$$(\mathrm{Hom}(V_i, N'_i) \oplus \mathrm{Hom}(V_i, Y) \oplus \bigoplus_{j \neq i} \mathrm{Hom}(V_j, N_j)). \tag{4.13}$$

To complete the proof, we show that these four terms can be identified with the four terms in (3.1) in the same order.

First we have

$$\begin{aligned} \mathrm{Ext}^1_\Gamma(M^+, N^+) &= \mathrm{Ext}^1_\Gamma(M' \oplus (E_i \otimes X), N' \oplus (E_i \otimes Y)) \\ &= \mathrm{Ext}^1_\Gamma(M', N') \oplus (\mathrm{Ext}^1_\Gamma(E_i, N') \otimes X^*) \end{aligned}$$

This agrees with (4.10) by Lemma 4.8. The identification of $\mathrm{Ext}^1_\Gamma(N^+, M^+)^*$ with (4.11) is proved dually. Finally, the terms in (4.12) and (4.13) obviously equal the last two terms in (3.1). □

Remark 4.9. The isomorphism given in the theorem is not natural, since the Σ_i do not provide an equivalence of categories. But it can be shown (with the same proof as above) that there exists an isomorphism of functors $E_{s_i\Gamma}(\Sigma_i, \Sigma_i) \simeq E_\Gamma(\Sigma_i^2, \Sigma_i^2)$.

As an immediate consequence of the definition (3.2), Corollary 4.6 and Theorem 4.7, we get:

Proposition 4.10. *If $i \in I$ is a source or a sink in Γ, then $(\sigma_i\alpha||\sigma_i\beta)_{s_i\Gamma} = (\alpha||\beta)_\Gamma$ for all $\alpha, \beta \in \Phi_{\geq -1}$.*

This result has the following implications.

Corollary 4.11. *If Γ and Γ' are two quivers with the same underlying Dynkin graph, then the simplicial complexes Δ_Γ and $\Delta_{\Gamma'}$ are isomorphic to each other.*

Proof. If $\Gamma' = s_i\Gamma$ for some source or sink i, then by Proposition 4.10, the permutation σ_i of $\Phi_{\geq -1}$ transforms Δ_Γ to $\Delta_{\Gamma'}$. Any two quivers can be obtained from each other by a sequence of such transformations (see, e. g. [1, Theorem 1.2]). □

Corollary 4.12. *For an alternating quiver Γ_0 (i. e. such that each vertex is a source or a sink) the Γ_0-compatibility degree coincides with the compatibility degree in* [7, Section 3.1]. *Therefore, Γ_0-clusters coincide with the clusters in the sense of* [7].

Proof. Denote the set of sources of Γ_0 by I^+, the set of sinks by I^-. The compatibility degree is characterized in [7] by the following two properties:

1. $(-\alpha_i||\beta) = \max([\beta : \alpha_i], 0)$ for all $i \in I$ and $\beta \in \Phi_{\geq -1}$,
2. $(\tau_\epsilon\alpha||\tau_\epsilon\beta) = (\alpha||\beta)$, where τ_ϵ denotes the element $\prod_{i\in I^\epsilon} \sigma_i$ for $\epsilon \in \{+, -\}$.

So it suffices to show that the Γ_0-compatibility degree satisfies these two properties. The first one follows from the definition of the compatibility degree (3.2). For the second, notice that τ_+ and τ_- send Γ_0 to Γ_0^{op}, and apply Propositions 4.10 and 3.1. □

5. The reflection groupoid

In this section, we show that the group generated by the two involutions τ_+, τ_- has a natural interpretation in the context of decorated quiver representations. To do this, we consider the following more general setup. Suppose that I is an arbitrary finite tree. We associate to I two groupoids defined as follows.

Let $R_0(I)$ denote the groupoid whose objects are given by the set $\mathrm{Quiv}(I)$ of all quivers obtained by specifying an orientation of I, and whose morphisms are defined as follows. First of all, for any $\Gamma \in \mathrm{Quiv}(I)$ and any source or sink i in Γ, there is an "elementary" isomorphism $\Sigma_i : \Gamma \xrightarrow{\sim} s_i\Gamma$. The morphisms in $R_0(I)$ are arbitrary compositions of these isomorphisms, subject to the following relations (where 1 denotes the appropriate identity automorphism):

(R1) $\Sigma_i^2 = 1$,
(R2) $\Sigma_i\Sigma_j = \Sigma_j\Sigma_i$ whenever i and j are not linked in I.

In both cases, the relation holds whenever the maps and their compositions are defined.

We also consider another groupoid $R(I)$ with the same set of objects $\mathrm{Quiv}(I)$, obtained from $R_0(I)$ by adjoining one more "elementary isomorphism" $D : \Gamma \xrightarrow{\sim} \Gamma^{\mathrm{op}}$,

for any $\Gamma \in \mathrm{Quiv}(I)$, satisfying the following relations (as before, they must hold whenever the morphisms and compositions on both sides are defined):
(R3) $D^2 = 1$,
(R4) $D\Sigma_i = \Sigma_i D$ for all $i \in I$.
Let $I = I^+ \cup I^-$ be a disjoint union of I into two completely disconnected subsets, i.e. no two vertices in I^+ (respectively, I^-) have a common edge. Such a decomposition exists because I is a tree, and is unique up to interchanging I^+ and I^-.

Denote by Γ_0 the alternating orientation of I in which every element of I^+ is a source. Let $\Sigma_+ = \prod_{i \in I^+} \Sigma_i$ and $\Sigma_- = \prod_{i \in I^-} \Sigma_i$. Note that, by the relation (R2), these morphisms do not depend on the order in which the compositions are taken. We can regard Σ_+, Σ_- as morphisms (in either of the groupoids $R_0(I)$ or $R(I)$) from Γ_0 to the opposite orientation Γ_0^{op}, or from Γ_0^{op} to Γ_0. The main result of this section can now be stated as follows:

Theorem 5.1. *(a) The automorphism group* $\mathrm{Aut}_{R_0(I)}(\Gamma_0)$ *is generated by* $\Sigma_+\Sigma_-$ *and* $\Sigma_-\Sigma_+$.
(b) The automorphism group $\mathrm{Aut}_{R(I)}(\Gamma_0)$ *is generated by* $D\Sigma_+$ *and* $D\Sigma_-$.

Before embarking on the proof of this theorem, we note the following:

Remark 5.2. Let $T_+ = D\Sigma_+ : \Gamma_0 \tilde{\to} \Gamma_0$ and let $T_- = D\Sigma_- : \Gamma_0 \tilde{\to} \Gamma_0$. Theorem 5.1 states that T_+ and T_- generate the automorphism group $\mathrm{Aut}_{R(I)}(\Gamma_0)$. In the case where I is a simply-laced Dynkin diagram, we can interpret the isomorphisms Σ_i in $R(I)$ as the extended reflection functors of Section 4 and the isomorphism D as the usual duality $\mathrm{rep}\,\widetilde{\Gamma} \tilde{\to} \mathrm{rep}\,\widetilde{\Gamma^{\mathrm{op}}}$. Using Corollary 4.6 and the fact that $\underline{\mathrm{sdim}}(DM) = \underline{\mathrm{sdim}}(M)$ for all objects M of $\mathrm{rep}\,\widetilde{\Gamma}$ (for any Γ), we see that if T_+ and T_- are regarded as bijections of the set of objects of $\mathrm{rep}\,\widetilde{\Gamma}_0$, they induce the involutions τ_+ and τ_- of the root lattice Q introduced in [7].

Theorem 5.1 has the following corollary:

Corollary 5.3. *Let* Γ *be any element of* $\mathrm{Quiv}(I)$. *Then:*
(a) The automorphism group $\mathrm{Aut}_{R_0(I)}(\Gamma)$ *is cyclic.*
(b) The automorphism group $\mathrm{Aut}_{R(I)}(\Gamma)$ *is generated by two involutions.*

Proof. This follows immediately from Theorem 5.1, using the fact that $\mathrm{Quiv}(I)$ is connected under the operations s_i (by [1, Theorem 1.2]), and so all automorphism groups in the groupoid are conjugate. □

Proof of Theorem 5.1. We note first of all that (b) is an easy formal consequence of (a): an element of $\mathrm{Aut}_{R_0(I)}(\Gamma_0)$ is either $(\Sigma_+\Sigma_-)^k = (D\Sigma_+ D\Sigma_-)^k$, or $(\Sigma_-\Sigma_+)^k = (D\Sigma_- D\Sigma_+)^k$; and an element of $\mathrm{Aut}_{R(I)}(\Gamma_0) \setminus \mathrm{Aut}_{R_0(I)}(\Gamma_0)$ can be written as $D\Sigma_+\Sigma'$ with $\Sigma' \in \mathrm{Aut}_{R_0(I)}(\Gamma_0)$.

Turning to (a), let Σ be any morphism in $R_0(I)$. By analogy with Coxeter groups, we define the *length* of Σ, denoted $\ell(\Sigma)$, to be the minimum length of an expression $\Sigma = \Sigma_{i_1}\Sigma_{i_2}\cdots\Sigma_{i_r}$. We call an expression of minimal length for Σ a *reduced expression* for Σ. In the sequel, we will often write i instead of Σ_i to simplify the notation.

The following Lemma is an immediate consequence of the relations (R1) and (R2):

Lemma 5.4. *An expression* $\Sigma_{i_1}\Sigma_{i_2}\cdots\Sigma_{i_r}$ *is reduced if and only if between any two occurrences of the same* i *there is an occurrence of* j *for some* j *linked to* i.

We can strengthen this as follows:

Lemma 5.5. *An expression $\Sigma_{i_1}\Sigma_{i_2}\cdots\Sigma_{i_r}$ for a morphism in $R_0(I)$ is reduced if and only if between any two occurrences of i there is precisely one occurrence of j for each j linked to i.*

Proof. The "if" part follows from Lemma 5.4 so we only have to show the "only if" statement. So suppose that $\Sigma_{i_1}\Sigma_{i_2}\cdots\Sigma_{i_r}$ is a reduced expression. We argue by induction on the distance between two consecutive occurrences of i. If the distance is 1 then the corresponding interval is $\Sigma_i\Sigma_j\Sigma_i$. Since this expression is reduced, j is linked to i, and in order for the second Σ_i to be applicable, j must be the only element of I that is linked to i, so we are done.

For the inductive step, we must first show that every j linked to i appears at most once between two consecutive occurrences of i. This is true since, by induction, if we had two occurrences of some j linked to i, between them i should occur, a contradiction. Then notice that all j linked to i must appear between the two consecutive occurrences of i, because we have to reverse all arrows at i. □

Lemma 5.6. *Suppose that $\Sigma_{i_1}\Sigma_{i_2}\cdots\Sigma_{i_p}$ is a reduced expression for a morphism in $R_0(I)$ that starts at an arbitrary orientation of I and ends up at Γ_0 or Γ_0^{op}. Then, for every i that appears in the reduced expression, we have precisely one of the following:*

(i) *no j linked to i appears before the first occurrence of i, or*
(ii) *every j linked to i appears exactly once before the first occurrence of i.*

Proof. This follows easily from Lemma 5.5. Let $\Sigma' = \Sigma_i\Sigma$; note that this composition is well-defined since Σ ends up at Γ_0 or Γ_0^{op}, from which every Σ_i is applicable. If $\Sigma_i\Sigma_{i_1}\Sigma_{i_2}\cdots\Sigma_{i_p}$ is reduced then (ii) holds by Lemma 5.5, otherwise (i) holds by Lemma 5.4. □

Lemma 5.7. *Suppose that Σ is a morphism of $R_0(I)$ which begins and ends at the same orientation Γ, and that Σ is not the identity morphism. Then every reduced expression for Σ contains all $i \in I$.*

Proof. It is enough to show that if i occurs in a reduced expression for Σ and j is linked to i, then j also occurs. Suppose this is not the case. Then i must occur at least twice, because the arrow linking i and j must be reversed at least twice by Σ. So j must occur by Lemma 5.5. □

Now everything is ready for the proof of Theorem 5.1(a). We need to show that every morphism Σ in $R_0(I)$ which begins and ends at the same quiver Γ, where Γ is Γ_0 or Γ_0^{op}, has a reduced expression of form $(\Sigma_+\Sigma_-)^k$ or $(\Sigma_-\Sigma_+)^k$ for some $k \in \mathbb{Z}_{\geq 0}$.

We proceed by induction on $\ell(\Sigma)$. There is nothing to prove when $\ell(\Sigma) = 0$, so let us assume that Σ is non-trivial, and that the statement is known for morphisms of smaller length. Fix a reduced expression for Σ. By Lemma 5.7, all $i \in I$ occur. By Lemma 5.6, I is the disjoint union of two subsets: those satisfying 5.6(i) and those satisfying 5.6(ii). By definition, if i and j are linked then they belong to different subsets, since whichever of i and j appears first in Σ satisfies (i), while the other satisfies (ii). It follows that one of these parts is I^+, and the other is I^-. Suppose that I^+ is the set of indices satisfying (i) (the argument if I^- is this set is similar). Commuting the first occurrences of Σ_i for each $i \in I^+$ to the beginning of our reduced expression, we obtain a reduced expression for Σ of the form $\Sigma_+\Sigma'$ with $\ell(\Sigma') = \ell(\Sigma) - |I^+|$, such that every reduced expression for Σ' starts with Σ_j for some $j \in I^-$. Note that Σ' is a morphism from Γ to Γ^{op}.

Next consider the morphism $\Sigma'' = \Sigma'\Sigma_+$ from Γ^{op} to Γ^{op}. It is clear that $\ell(\Sigma'') \leq \ell(\Sigma') + |I^+| = \ell(\Sigma)$. If $\ell(\Sigma'') < \ell(\Sigma)$ then by induction Σ'' has an expression $(\Sigma_+\Sigma_-)^k$ or $(\Sigma_-\Sigma_+)^k$, which implies the same claim for $\Sigma = \Sigma_+\Sigma' = \Sigma_+\Sigma''\Sigma_+$.

So let us assume that $\ell(\Sigma'') = \ell(\Sigma)$ and take a reduced expression for Σ'' obtained by taking the original reduced expression for Σ' (recall that it starts with some $j \in I^-$) followed by any reduced expression for Σ_+. Applying to this expression the same argument as before, we conclude that by moving some Σ_j's for $j \in I^-$ to the left we obtain a reduced expression for Σ' that starts with Σ_-. This leads to a reduced expression for Σ that starts with $\Sigma_+\Sigma_-$; by induction the rest of the reduced expression can be written in the desired form, and Theorem 5.1(a) follows. □

6. An application to quiver representations

As before, let Γ be a quiver whose underlying graph I is a Dynkin diagram of the ADE-type. We say that a set C of isomorphism classes of indecomposable representations of Γ is *Ext-free* if $\mathrm{Ext}^1(M,N) = 0$ for every $M, N \in C$. For any nonnegative integer k, let $f_\Gamma^+(k)$ denote the number of these Ext-free sets of cardinality k.

Proposition 6.1. *For every k, the number $f_\Gamma^+(k)$ depends only on the underlying Dynkin graph, not on the choice of an orientation Γ.*

Proof. Using Gabriel's theorem, we identify isoclasses of indecomposable representations of Γ with positive roots in Φ. Under this identification, the Ext-free sets become positive Γ-compatible subsets of $\Phi_{\geq -1}$, in the terminology of Definition 3.2.

For every subset $J \subset I$, let $f_\Gamma(k,J)$ (resp. $f_\Gamma^+(k,J)$) denote the number of $\Gamma|_J$-compatible subsets of cardinality k in $\Phi(J)_{\geq -1}$ (resp. $\Phi(J)_{>0}$); here $\Gamma|_J$ is the orientation induced by Γ on the full subgraph J of I. Thus, we have in particular $f_\Gamma^+(k,I) = f_\Gamma^+(k)$. In view of Proposition 3.4, these numbers satisfy the following relations:

$$f_\Gamma(k,J) = \sum_{K \subset J, |K| \leq k} f_\Gamma^+(k - |K|, J - K) \ .$$

It follows that the numbers $f_\Gamma^+(k,J)$ can be expressed as integer linear combinations of the numbers $f_\Gamma(l,K)$ by using the Möbius inversion on the partially ordered set of pairs (k,J) with the partial order given by

$$(l,K) \leq (k,J) \Leftrightarrow K \subset J, k - l = |J - K| \ .$$

Since this poset structure is independent of Γ, it remains to show that so are the numbers $f_\Gamma(k,I)$. But the latter statement is an immediate consequence of Corollary 4.11. □

For every Γ, the corresponding Ext-free sets form a simplicial complex, which was studied by L. Unger [14] (see Remark 3.8). Proposition 6.1 shows that the f-vector of this complex depends only on the underlying Dynkin graph. Despite this fact, the complexes corresponding to different orientations of the same graph may be not isomorphic to each other.

Example 6.2. Let I be of type A_3, and let Γ_0 (resp. Γ_1) be an alternating (resp. equioriented) orientation of I. The corresponding simplicial complexes are shown in Figure 1, and they are clearly non-isomorphic (in the figure, the vertices labeled by i, ij, and ijk represent roots α_i, $\alpha_i + \alpha_j$, and $\alpha_i + \alpha_j + \alpha_k$, respectively).

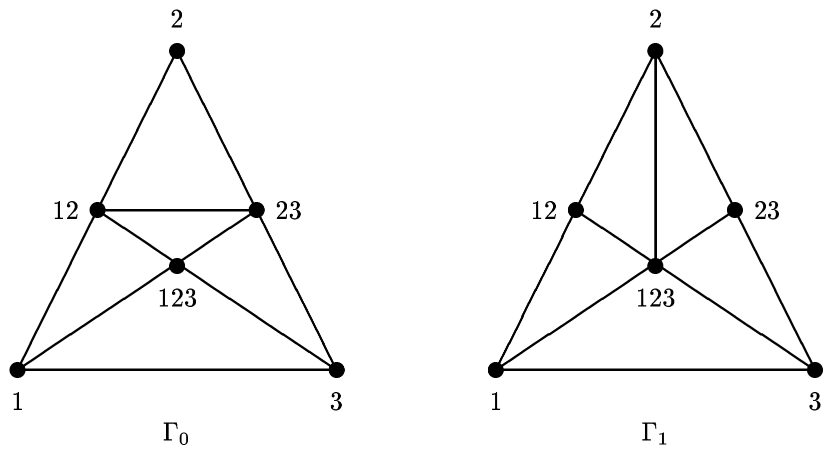

FIGURE 1. "Positive" simplicial complexes in type A_3

A recursive way to compute the numbers $f_\Gamma(k, I)$ is given by [7, Proposition 3.7]. The numbers $f_\Gamma^+(k, I)$ can then be found by the Möbius inversion described in the proof of Proposition 6.1. In particular, when Φ is irreducible, the total number $f_\Gamma^+(|I|, I)$ of positive Γ-clusters is given by [7, (3.8)]:

$$f_\Gamma^+(|I|, I) = \prod_{i \in I} \frac{e_i + h - 1}{e_i + 1} ,$$

where the e_i are the exponents of Φ, and h is the Coxeter number. It would be interesting to explain this formula using the theory of quiver representations.

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