Printed as: Technical Report 2002-01, Department of Mathematics and Computer Science, University of Leicester, January 2002

On the theory of one-step rewriting in trace monoids

Dietrich Kuske[1] and Markus Lohrey[2]

[1] Department of Mathematics and Computer Science
University of Leicester
LEICESTER, LE1 7RH, UK
[2] LIAFA, Université Paris VII, 2, place Jussieu, case 7014
75251 Paris cedex 05, France
D.Kuske@mcs.le.ac.uk, lohrey@liafa.jussieu.fr

Keywords: trace monoids, rewriting, confluence, decidability

Abstract. We introduce a general class of rewriting systems over trace monoids, called scattered rewriting systems, which generalize trace rewriting systems. We prove that the first-order theory of the one-step rewriting relation associated with a scattered rewriting system is decidable but in general not elementary. This extends known results on semi-Thue systems but our proofs use new methods; these new methods yield the decidability of local properties expressed in first-order logic augmented by modulo-counting quantifiers. Using the main decidability result, we define several subclasses of trace rewriting systems for which the confluence problem is decidable.

1 Introduction

Rewriting systems received a lot of attention in mathematics and theoretical computer science and are still an active field of research. Historically, rewriting systems were introduced to solve word problems in certain structures [47]. By the work of Markov [32] and Post [41], this hope vanished as they showed that there exist fixed semi-Thue systems with an undecidable word problem. Despite this result, there are plenty of rewriting systems with a decidable word problem, the most famous class being that of confluent and terminating systems. By Newman's Lemma, confluence can be decided for terminating semi-Thue systems as well as for terminating term rewriting systems. In general, both confluence and termination are undecidable properties of a semi-Thue system. A large deal of research tries to identify sufficient conditions for confluence/termination of rewriting systems (cf. [44]), or to describe classes of rewriting systems where confluence/termination is decidable.

These two properties which are in the heart of research in this area are typical second-order properties of the rewrite graph: its nodes are the structures that are rewritten (e.g., words in case of a semi-Thue system or terms in case of a term rewriting system), and directed edges indicate that one such structure can be rewritten into the other in one step. In order to define confluence and termination, one needs to quantify over paths in this graph. Hence the monadic second order theory of rewrite graphs is in general undecidable. The situation changes for semi-Thue systems when one considers the first-order theory: The edges of the rewrite graph of a semi-Thue system can be described by two-tape automata that move their heads synchronously on both tapes.[1]

[1] As opposed to rational graphs where the movement is asynchronous.

Using the well known closure properties of regular sets, the decidability of the first-order theory of these graphs follows [9, 23]. This result also holds for rewrite graphs of ground term rewriting systems [9], but not for term rewriting systems in general [48]. Another result in this direction is the decidability of the monadic second order theory of the rewrite graph of a prefix semi-Thue system [4] (a prefix semi-Thue system is a semi-Thue system where only prefixes can be rewritten); this follows from the decidability of the monadic second order theory of the graph of configurations of a pushdown automaton [36]. In particular confluence and termination are decidable for prefix semi-Thue systems.

This paper investigates the first-order theory of the rewrite graph of a trace rewriting system. Cartier and Foata [3] investigated the combinatorics of free partially commutative monoids that became later known as trace monoids. Mazurkiewicz [34] introduced them into computer science. They form a mathematically sound model for the concurrent behaviour of systems of high abstraction. Since trace monoids are a generalization of free monoids, it was tempting to extend the investigation of free monoids to free partially commutative monoids. This resulted, e.g., in the extensive consideration of recognizable and rational trace languages (cf. [14] for a collection of surveys on this field), trace equations [15, 33, 13], and trace rewriting systems [11, 12, 28–30].

Our main result states that for any finite trace rewriting system, the first-order theory of the associated rewrite graph is decidable. Because of the non-local effects of trace rewriting,[2] the automata-theoretic techniques from Dauchet and Tison [9] and Jacquemard [23] are not applicable here and we had to search for other ideas. The first is an application of Gaifman's locality theorem: the validity of a first-order sentence in a structure $\mathcal{S}$ depends on first-order properties of spheres around elements of $\mathcal{S}$. Since this theorem is effective, we were left with the question how to describe the set of traces that are centers of an r-sphere satisfying a given first-order formula. Our second idea was that the r-sphere around a trace can be described in the dependence graph of this trace by a sentence of monadic second order logic. Note that this logic does not speak about the infinite rewrite graph, but about a single finite dependence graph. We show that this is indeed effectively possible. Hence, by a result of Thomas [45], this implies that the set of traces that are centers of an r-sphere satisfying a given first-order formula is recognizable. Taking these two ideas together, we obtain that the first-order theory of the graph of any trace rewriting system is decidable.

The second step in our decidability proof, i.e., the interpretation of the r-sphere around a trace in the dependence graph of this trace does not only work for first-order logic, but also for its extension by modulo-counting quantifiers. Thus, we obtain that local properties of this modulo-counting logic are decidable. This result was not known for semi-Thue systems and, even more, we do not know how to prove it with the automata-theoretic methods from [9, 23]. The decidability of non-local properties is not answered in this paper, the missing tool is an analogue of Gaifman's locality

[2] With a and c the only independent letters, one can, e.g., rewrite $a^n bc^m$ into $c^m ba^n$ in just two steps using the rules $abc \to ac$ and $ca \to cba$.

theorem for this logic. Libkin [26, 27, 25] and Nurmonen [40] proved generalizations of a version of Gaifman's theorem for this and other counting extensions of first-order logic, but we could not make these generalizations serve our purposes.

We actually show a more general result since we do not only consider trace rewriting systems, but scattered rewrite rules. The idea is that of a parallel rewrite step where the intermediate factors of a trace have to satisfy some recognizable constraints and can be permuted as long as they are independent in the trace monoid.

As mentioned above, the first step in our decidability proof is an application of Gaifman's Theorem. The translation by Gaifman of a first-order sentence into a Boolean combination of local sentences is of very high complexity, i.e., our decision procedure is far from efficient. We also show that one cannot avoid this nonelementary complexity. To this aim, we construct a trace rewrite graph whose first-order theory is not elementary. Thus, our use of Gaifman's translation does not lead to an unreasonable inefficiency. We actually show a slightly stronger result, namely that the set of valid local sentences for a fixed radius is not elementary. In other words, the complexity of the decision question is already present when restricting to local sentences.

This nonelementary lower bound is shown for a nontrivial independence alphabet and the proof does not carry over to semi-Thue systems. The decision procedure for semi-Thue systems from [9, 23] is not elementary since it uses automata complementation for any quantifier alternation in a formula. This does of course not imply that there is no better decision algorithm. We show a lower bound of doubly exponential nondeterministic time for this problem. Again this lower bound holds for local sentences for a fixed radius.

In the last section, we return to the confluence problem for trace rewriting systems. For terminating rewriting systems, confluence and local confluence are equivalent. The problem with trace rewriting systems is that there can be infinitely many critical pairs which makes it impossible to check all of them in turn [12]. Even worse, by [37], it is undecidable whether a length-reducing trace rewriting system is confluent. On the other hand in [12, 30] several subclasses of terminating trace rewriting systems were defined, for which confluence is decidable. Here we will define new classes with this property. In general, even local confluence is not a first-order property of the rewriting graph. The classes of trace rewriting systems we consider in this last section ensure that local confluence is effectively expressible by a sentence of first-order logic. This then allows to apply our main result on the decidability of these first-order properties and therefore the decidability of the confluence for these classes when restricted to terminating systems.

2 General notations

Given an alphabet Σ, Σ^* denotes the set of all finite words over Σ, as well as the free monoid generated by Σ. The empty word is denoted by ϵ. Let $s \in \Sigma^*$. The length of s is denoted by $|s|$. The set of all symbols that occur in s is denoted by $\mathrm{alph}(s)$. For $\Gamma \subseteq \Sigma$ we denote by $\pi_\Gamma(s)$ the projection of the word s onto the symbols in Γ.

Let $G = (V, \to)$ be a directed graph with set of nodes V and edges $\to \subseteq V \times V$. The reflexive and transitive closure of $\to$ is denoted by $\xrightarrow{*}$. For $n \geq 0$ and $u, v \in V$ we write $u \to^{\leq n} v$ if there exists a path $w_0 \to w_1 \to \cdots \to w_m$ in $(V, \to)$ with $w_0 = u$, $w_m = v$, and $m \leq n$. The identity relation on a set A is denoted by $\mathrm{Id}_A = \{(a, a) \mid a \in A\}$.

The notion of a model is defined as usual in logic, see e.g. [22]. For a given model $\mathcal{A} = (A, F, R)$ where F is a set of operations on A and R is a set of relations over A, the *signature of* $\mathcal{A}$ contains the equality symbol $=$, and for each $f \in F$ (resp. $r \in R$) it contains a function symbol of the same arity as f (resp. a relation symbol of the same arity as r) that we denote without risk of confusion by f (resp. r) as well. Terms, atomic formulae, and (general) formulae over the signature of $\mathcal{A}$ are defined as usual. Finally the *first-order theory of* $\mathcal{A}$ denotes the set of all sentences, i.e., the set of all formulae without free variables, over the signature of $\mathcal{A}$ that are true in $\mathcal{A}$.

3 Rewriting in trace monoids

3.1 Trace monoids and recognizable trace languages

In the following we introduce some notions from trace theory, see [14] for more details. An *independence relation* on an alphabet Σ is an irreflexive and symmetric relation $I \subseteq \Sigma \times \Sigma$, the complementary relation $D = (\Sigma \times \Sigma) \backslash I$ is called a *dependence relation*. The pair (Σ, I) (resp. (Σ, D)) is called an *independence alphabet* (resp. a *dependence alphabet*). Given an independence alphabet (Σ, I), we define the *trace monoid* $\mathbb{M}(\Sigma, I)$ as the quotient monoid $\Sigma^* / \equiv_I$ where $\equiv_I$ denotes the least congruence relation on Σ^* that contains all pairs of the form (ab, ba) for $(a, b) \in I$. An element of $\mathbb{M}(\Sigma, I)$, i.e., an equivalence class of words, is called a *trace*. The trace that contains the word s is denoted by $[s]_I$. The neutral element of $\mathbb{M}(\Sigma, I)$ is the empty trace $[\epsilon]_I$ which will be denoted by 1. Concatenation of traces is defined by $[s]_I[t]_I = [st]_I$. If we want to speak explicitly of the concatenation as a binary operation over $\mathbb{M}$ then we denote it as $\circ$.

For the rest of the paper let us fix a finite independence alphabet (Σ, I) and let $\mathbb{M} = \mathbb{M}(\Sigma, I)$ and $D = (\Sigma \times \Sigma) \backslash I$. Since for all words $s, t \in \Sigma^*$, $s \equiv_I t$ implies $|s| = |t|$ and $\mathit{alph}(s) = \mathit{alph}(t)$, we can define $|[s]_I| = |s|$ and $\mathit{alph}([s]_I) = \mathit{alph}(s)$. The independence relation I can be lifted to $\mathbb{M}$ by $(u, v) \in I$ if $\mathrm{alph}(u) \times \mathrm{alph}(v) \subseteq I$. If $I = \emptyset$ then $\mathbb{M}$ is isomorphic to the *free monoid* Σ^*. On the other hand if $D = \mathrm{Id}_\Sigma$, then $\mathbb{M}$ is isomorphic to the *free commutative monoid* $\mathbb{N}^{|\Sigma|}$.

Traces can be uniquely represented as dependence graphs: the *dependence graph* $D_t = (V, E, \lambda)$ of a trace $t = [a_1 \cdots a_n]_I$ from $\mathbb{M}$ is defined as follows:

- $V = \{1, \ldots, n\}$ is the set of nodes.
- $E = \{(i, j) \mid i < j \ \wedge \ (a_i, a_j) \notin I\}$ is the edge relation.
- $\lambda(i) = a_i$ for $1 \leq i \leq n$ defines the labeling function λ.

It is easy to see that D_t is well-defined in the sense that if we choose another word that represents the trace t then we obtain an isomorphic graph. It is clear that the transitive

reflexive closure E^* of E is a partial order. A subset $U \subseteq V$ such that for all $i, j \in U$ and $k \in V$ with $(i,k),(k,j) \in E^*$ it holds $k \in U$ is called a *factor* of D_t. Factors of a dependence graph are also called *convex sets*. Note that the intersection of two convex sets is again convex. For a convex $U \subseteq V$ there exists a unique $u \in \mathbb{M}$ such that the restriction $D_t{\restriction_U}$ of D_t to U is isomorphic to D_u and t can be factorized as $t = t_1 u t_2$. Vice versa if t can be factorized as $t = t_1 u t_2$ then there exists a convex $U \subseteq V$ such that $D_t{\restriction_U}$ is isomorphic to D_u. More generally given a trace $t = t_1 t_2 \cdots t_m$, where $t_i \in \mathbb{M}$, the set of nodes V of the dependence graph $D_t = (V, E, \lambda)$ can be partitioned into convex sets $V_1, \ldots, V_m \subseteq V$ such that D_{t_i} is isomorphic to $D_t{\restriction_{V_i}}$ and for all $1 \leq i < j \leq m$ there do not exist $p \in V_i$, $q \in V_j$ with $(q,p) \in E^*$. Vice versa given a partition of V with these properties, the trace t can be factorized as $t = t_1 t_2 \cdots t_m$. Hence we call a sequence of convex sets $V_1, \ldots, V_m \subseteq V$ with the properties above a *factorization* of D_t.

Lemma 3.1. *If $U_1, \ldots, U_m$ and $V_1, \ldots, V_n$ are two factorizations of the dependence graph D_t then $\lambda[U_i \cap V_j] \times \lambda[U_k \cap V_\ell] \subseteq I$ whenever $1 \leq i < k \leq m$ and $n \geq j > \ell \geq 1$.*

This fact is well-known, it is usually used in the proof of the following lemma, which is known as Levi's lemma for traces, see e.g. [7], but we will use Lemma 3.1 also independently.

Lemma 3.2. *For $u_1, \ldots, u_m, v_1, \ldots, v_n \in \mathbb{M}$ it holds $u_1 u_2 \ldots u_m = v_1 v_2 \ldots v_n$ if and only if there exist $w_{i,j} \in \mathbb{M}$ ($1 \leq i \leq m$, $1 \leq j \leq n$) such that*

- *$u_i = w_{i,1} w_{i,2} \ldots w_{i,n}$ for every $1 \leq i \leq m$,*
- *$v_j = w_{1,j} w_{2,j} \ldots w_{m,j}$ for every $1 \leq j \leq n$, and*
- *$(w_{i,j}, w_{k,\ell}) \in I$ if $1 \leq i < k \leq m$ and $n \geq j > \ell \geq 1$.*

A *clique covering* of the dependence alphabet (Σ, D) is a sequence $(\Gamma_1, \ldots, \Gamma_n)$ of subsets of Σ such that $\Sigma = \bigcup_{i=1}^n \Gamma_i$ and $D = \bigcup_{i=1}^n \Gamma_i \times \Gamma_i$. For a fixed clique covering $(\Gamma_1, \ldots, \Gamma_n)$ of (Σ, D) we denote with $\pi_i : \mathbb{M} \to \Gamma_i^*$ the canonical lifting of the projection $\pi_{\Gamma_i} : \Sigma^* \to \Gamma_i^*$ to $\mathbb{M}$. The following lemma is well-known, see e.g. [6].

Lemma 3.3. *Let $(\Gamma_1, \ldots, \Gamma_n)$ be a clique covering of the dependence alphabet (Σ, D). Then for $s, t \in \Sigma^*$ we have $s \equiv_I t$ if and only if $\pi_{\Gamma_i}(s) = \pi_{\Gamma_i}(t)$ for all $1 \leq i \leq n$.*

A tuple $(s_1, \ldots, s_n) \in \prod_{i=1}^n \Gamma_i^*$ is *reconstructible* if there exists a trace $u \in \mathbb{M}$ such that $\pi_i(u) = s_i$ for all $i \in \{1, \ldots, n\}$ [6]. The following statement is from [15, Proposition 1.6].

Lemma 3.4. *Let $u \in \mathbb{M}$, and let $(\Gamma_1, \ldots, \Gamma_n)$ be a clique covering of the dependence alphabet (Σ, D). For all $i \in \{1, \ldots, n\}$ let $\pi_i(u) = s_i t_i$ be a factorization of $\pi_i(u)$. Then it holds $\pi_i(s_j) = \pi_j(s_i)$ for all $i, j \in \{1, \ldots, n\}$ if and only if $(s_1, \ldots, s_n)$ and $(t_1, \ldots, t_n)$ are reconstructible.*

A set $L \subseteq \mathbb{M}$ is called *recognizable* if there exists a morphism $h : \mathbb{M} \to Q$ from $\mathbb{M}$ into a finite monoid Q and a subset $F \subseteq Q$ such that $L = h^{-1}(F)$. The set of all recognizable subsets of $\mathbb{M}$ is denoted by $\mathrm{REC}(\mathbb{M})$. It is well-known that $\mathrm{REC}(\mathbb{M})$ is effectively closed under Boolean operations and concatenation of languages.[3] Furthermore emptiness and finiteness are decidable for recognizable trace languages, and if $L \in \mathrm{REC}(\mathbb{M})$ is finite then its elements can be calculated effectively.

3.2 Scattered trace rewriting

Let us fix a countable infinite set Ω of (first-order) variables ranging over $\mathbb{M}$ for the rest of this paper. In order to make notations more succinct, we associate with every first-order variable $x \in \Omega$ a recognizable trace language $L(x)$. We assume that for every $L \in \mathrm{REC}(\mathbb{M})$ there is a countably infinite supply of variables $x \in \Omega$ with $L(x) = L$. Note that since $\mathrm{REC}(\mathbb{M})$ is countable, a countable set Ω of variables suffices for this. The mapping $x \mapsto L(x)$ will be fixed for the rest of this paper. The intuition of this mapping is that in many cases (but not always) we will restrict a variable $x \in \Omega$ to its associated set $L(x)$. For $x \in \Omega$, we denote by $\mathrm{alph}(x) \subseteq \Sigma$ the smallest alphabet such that $L(x) \subseteq \mathrm{alph}(x)^*$ where $\mathrm{alph}(x)^*$ is considered as a subset of $\mathbb{M}$. On the set Ω we define an independence relation J by

$$J = \{(x, y) \mid \mathrm{alph}(x) \times \mathrm{alph}(y) \subseteq I\} \backslash \mathrm{Id}_\Omega.$$

Let $x_1, \ldots, x_n$ be pairwise different variables from Ω. A *pattern* S over $\mathbb{M}$ and the variables $x_1, \ldots, x_m$ is a sequence $y_1\, t_1\, y_2\, t_2 \cdots y_m$ where $t_j \in \mathbb{M}$ $(1 \leq i \leq m-1)$ and the function $x_i \mapsto y_i$ $(1 \leq i \leq m)$ is a permutation of $\{x_1, \ldots, x_m\}$. A pattern S over the variables $x_1, \ldots, x_m$ is also denoted by $S(x_1, \ldots, x_m)$. Note that in a pattern a variable occurs precisely once, but the variables may occur in an arbitrary order. If the variable x_i evaluates to $u_i \in \mathbb{M}$, $1 \leq i \leq m$, then the trace $S(u_1, \ldots, u_m) \in \mathbb{M}$ is defined in the obvious way. By $\mathrm{type}(S)$ we denote the projection of the pattern S onto its variables, i.e., $\mathrm{type}(S) = \pi_\Omega(S)$ if we view S as a word over the infinite alphabet $\mathbb{M} \cup \Omega$. A *scattered rewrite rule* over $\mathbb{M}$ and the variables $x_1, \ldots, x_m$ is a pair $(S(x_1, \ldots, x_m), T(x_1, \ldots, x_m))$ where S and T are patterns over $\mathbb{M}$ such that $\mathrm{type}(S) \equiv_J \mathrm{type}(T)$. The set of all scattered rewrite rules over $\mathbb{M}$ (and an arbitrary sequence of variables) is denoted by $\mathbb{S}$. A *scattered rewriting system* over $\mathbb{M}$ is a finite subset of $\mathbb{S}$. For a scattered rewrite rule $\rho = (S(x_1, \ldots, x_m), T(x_1, \ldots, x_m))$ and $s, t \in \mathbb{M}$ we write $s \to_\rho t$ if there exist $u_1 \in L(x_1), \ldots, u_m \in L(x_m)$ such that $s = S(u_1, \ldots, u_m)$ and $t = T(u_1, \ldots, u_m)$. For a scattered rewriting system $\mathcal{R}$ we write $s \to_\mathcal{R} t$ if $s \to_\rho t$ for some $\rho \in \mathcal{R}$.

An important special case of scattered rewriting systems are trace rewriting systems [11, 12]. A *trace rewriting system* over $\mathbb{M}$ may be defined as a scattered rewriting system $\mathcal{R}$ over $\mathbb{M}$ where all rules $\rho \in \mathcal{R}$ have the form $(x\ell y, xry)$ for $x, y \in \Omega$, $\ell, r \in \mathbb{M}$ such that $L(x) = L(y) = \mathbb{M}$. If $I = \emptyset$, i.e., $\mathbb{M} \simeq \Sigma^*$, then a trace rewriting

[3] In these effectiveness statements, the recognizable languages are given as a triple (Q, F, h).

system over $\mathbb{M}$ is also called a *semi-Thue system* over Σ^*, see [24, 1] for more details. On the other hand if $I = (\Sigma \times \Sigma) \backslash \mathrm{Id}_\Sigma$, i.e., $\mathbb{M} \simeq \mathbb{N}^{|\Sigma|}$, then a trace rewriting system over $\mathbb{M}$ is also called a *vector replacement system* over $\mathbb{N}^{|\Sigma|}$. A rule $(x\ell y, xry)$ of a trace rewriting system will be briefly denoted by (ℓ, r).

Next we will introduce different models, whose domain will be the set $\mathbb{M}$ of all traces. The most general model we will work with is the model

$$\mathcal{M} = (\mathbb{M}, \circ, (a)_{a \in \Sigma \cup \{1\}}, (K)_{K \in \mathrm{REC}(\mathbb{M})}, (\to_\rho)_{\rho \in \mathbb{S}}).$$

The operations of this model are the concatenation of traces $\circ$ plus the constants $a \in \Sigma \cup \{1\}$. Furthermore each recognizable language $K \in \mathrm{REC}(\mathbb{M})$ is put into $\mathcal{M}$ as a unary predicate. Instead of $K(x)$ we write $x \in K$. Finally $\mathcal{M}$ contains all binary relations $\to_\rho \subseteq \mathbb{M} \times \mathbb{M}$ for $\rho \in \mathcal{S}$. Unfortunately the model $\mathcal{M}$ has an undecidable first-order theory, already the first-order theory of $(\{a, b\}^*, \circ, a, b)$ is undecidable [16, 31]. Hence we will also deal with reducts of $\mathcal{M}$. For a scattered rewriting system $\mathcal{R}$ we define:

$$\mathcal{M}^{\mathrm{red}} = (\mathbb{M}, (K)_{K \in \mathrm{REC}(\mathbb{M})}, (\to_\rho)_{\rho \in \mathbb{S}})$$
$$\mathcal{M}^{\mathrm{red}}_{\mathcal{R}} = (\mathbb{M}, (K)_{K \in \mathrm{REC}(\mathbb{M})}, (\to_\rho)_{\rho \in \mathcal{R}})$$

3.3 The main result

The main result of this paper states that the first-order theory of any structure $\mathcal{M}^{\mathrm{red}}$ is decidable. Since our decision procedure is uniform in the underlying alphabet, we obtain

Theorem 3.5. *There is an algorithm that, on input of an independence alphabet (Σ, I) and a first-order sentence φ over the signature of the structure $\mathcal{M}^{\mathrm{red}}$, decides whether $\mathcal{M}^{\mathrm{red}} \models \varphi$.*

The proof of this theorem can be found in Section 5.4, the sections before that prepare the proof of it.

It can be easily shown that we obtain an undecidability result if we omit the restriction that for a scattered rewrite rule (S, R) it holds $\mathrm{type}(S) \equiv_J \mathrm{type}(T)$. In order to see this let us call a *permutation rewrite rule* over $\mathbb{M}$ a pair (S, T) with the same properties as a scattered rewrite rule except that $\mathrm{type}(S) \equiv_J \mathrm{type}(T)$ need not necessarily hold. A rewrite step $s \to_\rho t$, where ρ is a permutation rewrite rule, is defined exactly in the same way as it was done for scattered rewrite rules. Again a *permutation rewriting system* $\mathcal{P}$ over $\mathbb{M}$ is a finite set of permutation rewrite rules over $\mathbb{M}$, and the relation $\to_{\mathcal{P}}$ is defined as usual.

Theorem 3.6. *There exists a fixed permutation rewriting system $\mathcal{R}$ over Σ^* for some finite alphabet Σ such that the first-order theory of the model $(\Sigma^*, (\to_\rho)_{\rho \in \mathcal{R}})$ is undecidable. Furthermore we may assume that for every variable x that occurs in some $\rho \in \mathcal{R}$ we have $L(x) = \mathrm{alph}(x)^*$.*

Proof. There exists a finitely presented group with an undecidable word problem [39, 2]; let $\langle \Gamma | \{(r, \epsilon) \mid r \in R\}\rangle$ with $R \subseteq \Gamma^*$ finite be a monoid presentation for this group. Then the semi-Thue system $\mathcal{S} = \{(r, \epsilon) \mid w \in R\}$ has an undecidable word problem, too. More precisely, the question, whether for a given word $w \in \Sigma^*$ it holds $w \overset{*}{\leftrightarrow}_{\mathcal{S}} \epsilon$, is undecidable, where $\leftrightarrow_{\mathcal{S}}$ denotes the symmetric closure of $\rightarrow_{\mathcal{S}}$.

Let $\Sigma = \Gamma \cup \{\#, \$\}$ where $\#, \$ \notin \Gamma$. Define semi-Thue rules (i.e, elements of a semi-Thue system) ρ_0, ρ_1, ρ_2, and ρ_a for $a \in \Gamma$, by $\rho_0 = (\$, \$)$, $\rho_1 = (\#, \$)$, $\rho_2 = (\$\#, \$\#)$, and $\rho_a = (\$a, \$)$ for $a \in \Gamma$. For a word $s = s_1 s_2 \cdots s_m$ with $s_i \in \Gamma$, let φ_s denote the following formula with one free variable w:

$$\varphi_s(w) \quad \equiv \quad \neg(w \rightarrow_{\rho_0} w) \ \wedge \ \exists u_0, \ldots, u_m, v \left\{ \begin{array}{l} w \rightarrow_{\rho_1} u_0 \ \wedge \ w \rightarrow_{\rho_1} v \wedge \\ \bigwedge_{1 \le \ell \le m} (u_{\ell-1} \rightarrow_{\rho_{s_\ell}} u_\ell) \ \wedge \\ u_m \rightarrow_{\rho_2} u_m \ \wedge \ v \rightarrow_{\rho_2} v \end{array} \right\}$$

A word $w \in \Sigma^*$ satisfies this formula if and only if $w = w_0 \# w_1 \# \cdots w_{k-1} \# w_k$ with $w_i \in \Gamma^*$ and there exist $1 \le i, j < k$ such that $w_i = s$ and $w_j = \epsilon$. Next choose variables $x, y, x_1, y_1, x_2, y_2 \in \Omega$ with $L(x) = L(y) = \Sigma^*$ and $L(x_1) = L(x_2) = L(y_1) = L(y_2) = \Gamma^*$ and define the scattered rewrite rule $\rho_\$ = (x\#x_1\#y_1\#y, \ x\$x_1\$y_1\$y)$ and the permutation rewriting system

$$\mathcal{P} = \{(x\$x_1 \ell y_1 \$x_2 r y_2 \$y, \ x\$x_2 \ell y_2 \$x_1 r y_1 \$y) \mid (\ell, r) \in \mathcal{S} \text{ or } (r, \ell) \in \mathcal{S}\}.$$

Then we have $\exists w\{\varphi_s(w) \ \wedge \ \forall v(w \rightarrow_{\rho_\$} v \ \Rightarrow \ v \rightarrow_{\mathcal{P}} v)\}$ if and only if $s \overset{*}{\leftrightarrow}_{\mathcal{S}} \epsilon$. The important fact is that $\$u\$v\$ \rightarrow_{\mathcal{P}} \$u\$v\$$ for $u, v \in \Gamma^*$ if and only if $u \leftrightarrow_{\mathcal{S}} v$ □

Since any trace rewriting system is a scattered rewriting system, we obtain the following corollary of Theorem 3.5:

Corollary 3.7. *There is an algorithm that, on input of an independence alphabet (Σ, I), a trace rewriting system $\mathcal{R}$ over $\mathbb{M} = \mathbb{M}(\Sigma, I)$, and a first-order sentence φ over the signature of the model $(\mathbb{M}, \rightarrow_{\mathcal{R}})$, decides whether $(\mathbb{M}, \rightarrow_{\mathcal{R}}) \models \varphi$.*

This corollary will be used later in Section 7 where we consider confluence problems for trace rewriting systems. For the case that $\mathcal{R}$ is a semi-Thue system, Corollary 3.7 was shown in [9, 23]. The basic fact used in their proof is that for a semi-Thue system $\mathcal{R}$ the relation $\rightarrow_{\mathcal{R}}$ is a synchronized rational transduction [19] which means roughly that this relation can be recognized by a two-tape automaton where both heads operate synchronously. It is not hard to see that the method from [9, 23] can be generalized to the case that $\mathbb{M}$ is a direct product of free monoids (this covers in particular the case of vector replacement systems), but there seems to be no way to generalize this approach to arbitrary trace monoids. Hence in our proof of Theorem 3.5 we will follow a completely different and new strategy.

It can be easily seen that Corollary 3.7 cannot be generalized to the well-known monadic second-order logic over graphs, see e.g. [21], there exists already a fixed semi-Thue system $\mathcal{R}$ such that the graph $(\Sigma^*, \to_{\mathcal{R}})$ has an undecidable monadic second-order theory (see below for an example). But more surprisingly, also the monadic second-order theory of a *prefix rewriting system* over a trace monoid is in general undecidable. A prefix rewriting system over $\mathbb{M}$ can be defined as a scattered rewriting system $\mathcal{R}$ where all rules in $\mathcal{R}$ have the form $(x\ell y, xry)$ with $L(x) = \{1\}$ and $L(y) = \mathbb{M}$ (again we abbreviate this rule by (ℓ, r)). Based on results from [8], Caucal has shown in [4] that for a prefix rewriting system $\mathcal{R}$ over a free monoid Σ^* the monadic second-order theory of the graph $(\Sigma^*, \to_{\mathcal{R}})$ is decidable. In contrast to this, let $\mathcal{R}$ be the prefix rewriting system $\{(1, a), (1, b)\}$ over the free commutative monoid $\mathbb{M}(\{a, b\}, \{(a, b), (b, a)\})$. The graph $(\mathbb{M}, \to_{\mathcal{R}})$ is a two-dimensional grid which has an undecidable monadic second order theory [42]. Hence, in general, the monadic second order theory of the relation $\to_{\mathcal{R}}$ for a prefix rewriting system $\mathcal{R}$ is undecidable.

Before we proceed with the proof of Theorem 3.5 let us first show that Corollary 3.7 is a proper generalization of the semi-Thue case. More precisely, we can construct a trace rewriting system $\mathcal{R}$ such that the graph $(\mathbb{M}, \to_{\mathcal{R}})$ is not isomorphic to the graph of a semi-Thue system. For this we have to introduce some notions concerning confluence. Let $\mathcal{R}$ be a trace rewriting system. We say that $\mathcal{R}$ is *locally confluent* if for all $u, v_1, v_2 \in \mathbb{M}$ with $u \to_{\mathcal{R}} v_1$ and $u \to_{\mathcal{R}} v_2$ there exists a $w \in \mathbb{M}$ with $v_1 \stackrel{*}{\to}_{\mathcal{R}} w$ and $v_2 \stackrel{*}{\to}_{\mathcal{R}} w$. We say that $\mathcal{R}$ is *α-confluent*, where $\alpha \in \mathbb{N}$, if for all $u, v_1, v_2 \in \mathbb{M}$ with $u \to_{\mathcal{R}} v_1$ and $u \to_{\mathcal{R}} v_2$ there exists a $w \in \mathbb{M}$ with $v_1 \to_{\mathcal{R}}^{\leq \alpha} w$ and $v_2 \to_{\mathcal{R}}^{\leq \alpha} w$.

Lemma 3.8. *If a semi-Thue system is locally confluent then it is α-confluent for some $\alpha \in \mathbb{N}$.*

Proof. For every semi-Thue system $\mathcal{S}$ over Σ^* there exist finitely many critical pairs $(u_1, v_1), \ldots, (u_n, v_n) \in \Sigma^* \times \Sigma^*$ which result from overlapping left-hand sides. Then $\mathcal{S}$ is locally confluent if and only if for every critical pair (u_i, v_i) there exists a $w_i \in \Sigma^*$ such that $u_i \stackrel{*}{\to}_{\mathcal{S}} w_i$ and $v_i \stackrel{*}{\to}_{\mathcal{S}} w_i$, see e.g. [1]. Assume that these w_i exist. Then there exists an $\alpha \in \mathbb{N}$ such that $u_i \to_{\mathcal{S}}^{\leq \alpha} w_i$ and $v_i \to_{\mathcal{S}}^{\leq \alpha} w_i$. From the definition of critical pairs and α-confluence, it follows immediately that $\mathcal{S}$ is α-confluent. □

Theorem 3.9. *Let $\Sigma = \{a, b, c\}$, $I = \{(a, c), (c, a)\}$, and let $\mathcal{R}$ be the trace rewriting system $\mathcal{R} = \{(ba, 1), (ab, 1), (c, 1)\}$ over $\mathbb{M}(\Sigma, I)$. Then there is no semi-Thue system $\mathcal{S}$ over some alphabet Γ such that the graph $(\mathbb{M}(\Sigma, I), \to_{\mathcal{R}})$ is isomorphic to the graph $(\Gamma^*, \to_{\mathcal{S}})$.*

Proof. In [30, Example 2.2] the second author has shown that $\mathcal{R}$ is locally confluent. We claim that there is no $\alpha \in \mathbb{N}$ such that $\mathcal{R}$ is α-confluent, this proves the theorem by Lemma 3.8. Note that for all $n \geq 0$ we have $bac^nb \to_{\mathcal{R}} c^nb$ and $bac^nb = bc^nab \to_{\mathcal{R}} bc^n$, but there is no $\alpha \leq n - 1$ and no trace w such that $c^nb \to_{\mathcal{R}}^{\leq \alpha} w$ and $bc^n \to_{\mathcal{R}}^{\leq \alpha} w$. Hence there is no $\alpha \in \mathbb{N}$ such that $\mathcal{R}$ is α-confluent. □

In Section 7 we will come back to the notion of α-confluence.

4 An excursion to word and trace equations

In this section we will prove some results concerning equations in trace monoids. These results will be used in Section 5 where we prove Theorem 3.5. They state that certain equations (with recognizable constraints) are equivalent (in the model $\mathcal{M}$) to Boolean combinations of recognizable constraints. It is important to note that these equivalences are effective in the sense that starting from an equation (with recognizable constraints) that satisfies certain restrictions we can effectively construct an equivalent Boolean combination of recognizable constraints. This fact is not stated explicitly in the following lemmas, but it can be always easily checked. A similar remark applies to the results of Section 5.

Lemma 4.1. *For $s, t \in \Sigma^*$, the set $\{u \in \Sigma^* \mid su = ut\}$ is recognizable.*

Proof. Let $s, t \in \Sigma^*$. If they are of different length, then the set is empty and therefore trivially recognizable. So assume $|s| = |t| = n$. Let furthermore $u \in \Sigma^*$ with $n \leq |u|$. Then $su = ut$ if s is a prefix of u, t is a suffix of u, and, for $1 \leq k \leq |u| - n$, the k^{th} and the $(k+n)^{\text{th}}$ letter of u coincide. This can be checked by a finite automaton. Now the set in question contains, besides these "long" words u, finitely many short words u with $|u| < n$. Hence the result follows. □

The next lemma generalizes the previous lemma to the case of several variables.

Lemma 4.2. *Let $s_1, \ldots, s_{m+1}, t_1, \ldots, t_{m+1} \in \Sigma^*$. Then the formula*

$$s_1 x_1 s_2 \cdots x_m s_{m+1} = t_1 x_1 t_2 \cdots x_m t_{m+1} \tag{1}$$

is equivalent in the model $(\Sigma^*, \circ, (a)_{a\in\Sigma}, (K)_{K\in \mathrm{REC}(\Sigma^*)})$ to a Boolean combination of formulae of the form $x_i \in L$ with $L \in \mathrm{REC}(\Sigma^*)$.

Proof. We prove the lemma by an induction on m. The case $m = 0$ is trivial. So assume that $m > 0$. If neither s_1 is a prefix of t_1 nor t_1 is a prefix of s_1 then (1) is equivalent to the truth value false. Now assume w.l.o.g. that $s_1 = t_1 s$. By cancelling t_1 we obtain the equation $s x_1 s_2 \cdots x_m s_{m+1} = x_1 t_2 \cdots x_m t_{m+1}$. If $s = \epsilon$ then we can also cancel the variable x_1 which allows us to apply the induction hypothesis. Now assume that $s \neq \epsilon$. We can guess the position in $t_2 x_2 \cdots t_m x_m t_{m+1}$ where $s x_1$ ends. More precisely, the word equation $s x_1 s_2 \cdots x_m s_{m+1} = x_1 t_2 \cdots x_m t_{m+1}$ is equivalent to the finite disjunction of all formulae of one of the following two forms:

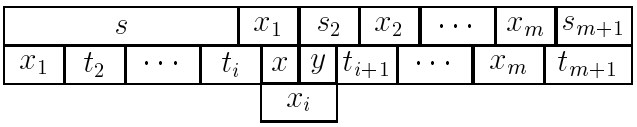
(1) $s x_1 = x_1 t_2 x_2 \cdots t_{i-1} x_{i-1} t' \wedge s_2 x_2 s_3 \cdots x_m s_{m+1} = t'' x_i t_{i+1} \cdots x_m t_{m+1}$ where $2 \leq i \leq m+1$ and $t_i = t' t''$ (i.e., $s x_1$ ends in t_i)

(2) $\exists x, y \left\{ \begin{array}{l} x_i = xy \wedge s x_1 = x_1 t_2 \cdots x_{i-1} t_i x \wedge \\ s_2 x_2 s_3 \cdots x_m s_{m+1} = y t_{i+1} x_{i+1} t_{i+2} \cdots x_m t_{m+1} \end{array} \right\}$ where $2 \leq i \leq m$, and x and y are new variables (i.e., $s x_1$ ends in x_i where $i \geq 2$ because of $s \neq 1$).

It suffices to show that each of these formulae is equivalent to a Boolean combination of recognizable constraints. For this let us consider a formula of the second type, the first type can be dealt with similarly. If we substitute xy for x_i we obtain the formula

$$\exists x, y \begin{Bmatrix} x_i = xy \ \wedge \ sx_1 = x_1 t_2 x_2 \cdots t_{i-1} x_{i-1} t_i x \ \wedge \\ s_2 x_2 s_3 \cdots x_{i-1} s_i x y s_{i+1} x_{i+1} \cdots s_m x_m s_{m+1} = y t_{i+1} x_{i+1} t_{i+2} \cdots x_m t_{m+1} \end{Bmatrix}, \quad (2)$$

see also the following picture.



Note that for every solution of the equation $sx_1 = x_1 t_2 \cdots x_{i-1} t_i x$, i.e., for every tuple $(u_1, u_2, \ldots, u_{i-1}, u)$ with $su_1 = u_1 t_2 \cdots u_{i-1} t_i u$, the length constraint $|s| = |t_2 u_2 \cdots t_{i-1} u_{i-1} t_i u|$ holds. But there are only finitely many tuples $(u_2, \ldots, u_{i-1}, u)$ that satisfy this constraint. Hence we can take the finite disjunction over all tuples $(u_2, \ldots, u_{i-1}, u)$ with $|s| = |t_2 u_2 \cdots t_{i-1} u_{i-1} t_i u|$ where each single disjunctive subformula results from substituting in (2) u_i for x_i and u for x. Let $(u_2, \ldots, u_{i-1}, u)$ be one of these tuples and define $v = t_2 u_2 t_3 \cdots t_{i-1} u_{i-1} t_i u$ and $w = s_2 u_2 s_3 \cdots s_{i-1} u_{i-1} s_i u$. It suffices to show that the formula

$$\exists y \left\{ \begin{array}{l} x_i = uy \ \wedge \ sx_1 = x_1 v \ \wedge \bigwedge_{2 \le j \le i-1} x_j \in \{u_j\} \ \wedge \\ \\ wy s_{i+1} x_{i+1} \cdots s_m x_m s_{m+1} = y t_{i+1} x_{i+1} t_{i+2} \cdots x_m t_{m+1} \end{array} \right\}$$

is equivalent to a Boolean combination of recognizable constraints. By Lemma 4.1 we can replace the equation $sx_1 = x_1 v$ by a formula of the form $x_1 \in L_1$ where $L_1 \in \mathrm{REC}(\Sigma^*)$. Furthermore the induction hypothesis[4] implies that the equation

$$wy s_{i+1} x_{i+1} \cdots s_m x_m s_{m+1} = y t_{i+1} x_{i+1} \cdots t_m x_m t_{m+1} \quad (3)$$

is equivalent to a finite disjunction of formulae of the form $y \in K \ \wedge \bigwedge_{i+1 \le j \le m} x_j \in L_j$ where $K, L_{i+1}, \ldots, L_m \in \mathrm{REC}(\mathbb{M})$. In this way we obtain a finite disjunction of formulae of the form

$$\exists y \left\{ x_i = uy \ \wedge \ x_1 \in L_1 \ \wedge \bigwedge_{2 \le j \le i-1} x_j \in \{u_j\} \ \wedge \ y \in K \ \wedge \bigwedge_{i+1 \le j \le m} x_j \in L_j \right\}.$$

But this formula is equivalent to the formula

$$x_1 \in L_1 \ \wedge \bigwedge_{2 \le j \le i-1} x_j \in \{u_j\} \ \wedge \ x_i \in uK \ \wedge \bigwedge_{i+1 \le j \le m} x_j \in L_j.$$

This concludes the proof of the lemma. □

[4] Note that because of $i \ge 2$, (3) contains $m - i + 1 < m$ many variables.

Next we will generalize Lemma 4.2 from words to traces. For this we need the following simple lemma.

Lemma 4.3. *Let $(\Gamma_1, \ldots, \Gamma_n)$ be a clique covering of the dependence alphabet (Σ, D). For $1 \leq i \leq n$ let $L_i \in \mathrm{REC}(\Gamma_i^*)$. Then the set of all traces $u \in \mathbb{M}$ such that $\pi_i(u) \in L_i$ for $1 \leq i \leq n$ is recognizable.*

Proof. We define a morphism $\pi : \mathbb{M} \to \Gamma_1^* \times \cdots \times \Gamma_n^*$ by $\pi(u) = (\pi_1(u), \ldots, \pi_n(u))$. Since $L_1 \times \cdots \times L_n \in \mathrm{REC}(\Gamma_1^* \times \cdots \Gamma_n^*)$ and recognizable languages are closed under inverse morphisms [18], the language $\{u \in \mathbb{M} \mid \pi_1(u) \in L_1, \ldots, \pi_n(u) \in L_n\}$ is recognizable. □

Lemma 4.4. *Let $S(x_1, \ldots, x_m)$ and $T(x_1, \ldots, x_m)$ be patterns such that $\mathrm{type}(S) \equiv_J \mathrm{type}(T)$.* [5] *Then the formula*

$$\bigwedge_{1 \leq i \leq m} x_i \in L(x_i) \ \wedge \ S(x_1, \ldots, x_m) = T(x_1, \ldots, x_m) \tag{4}$$

is equivalent in $\mathcal{M}$ to a Boolean combination of formulae $x_i \in L$ with $L \in \mathrm{REC}(\mathbb{M})$.

Proof. Let $(\Gamma_1, \ldots, \Gamma_n)$ be an arbitrary clique covering of the dependence alphabet (Σ, D). For all $i \in \{1, \ldots, m\}$ and all $j \in \{1, \ldots, n\}$ let $x_{i,j}$ be a new variable which will represent the projection of x_i onto the clique Γ_j. Since x_i is restricted to the set $L(x_i) \subseteq \mathrm{alph}(x_i)^*$, we can replace the variable $x_{i,j}$ by the empty word ϵ in case $\mathrm{alph}(x_i) \cap \Gamma_j = \emptyset$. For this we define for every $j \in \{1, \ldots, n\}$ a substitution ϑ_j by

$$\vartheta_j(a) = \pi_j(a) \quad \text{for } a \in \Sigma \qquad \text{and} \qquad \vartheta_j(x_i) = \begin{cases} \epsilon & \text{if } \mathrm{alph}(x_i) \cap \Gamma_j = \emptyset \\ x_{i,j} & \text{otherwise.} \end{cases}$$

Thus, for instance the pattern $\vartheta_j(S)$ represents the projection of the pattern S onto the clique Σ_j. We claim that $\mathrm{type}(\vartheta_j(S)) = \mathrm{type}(\vartheta_j(T))$ for all $j \in \{1, \ldots, n\}$. Since $\mathrm{type}(S) \equiv_J \mathrm{type}(T)$, it suffices to prove that $\vartheta_j(yz) = \vartheta_j(zy)$ for variables $y, z \in \{x_1, \ldots, x_m\}$ with $(y, z) \in J$. Let $(y, z) \in J$. If Γ_j is a clique then we cannot have $\mathrm{alph}(y) \cap \Gamma_j \neq \emptyset \neq \mathrm{alph}(z) \cap \Gamma_j$, since this contradicts $(y, z) \in J$. Thus either $\vartheta_j(y) = \epsilon$ or $\vartheta_j(z) = \epsilon$, which implies $\vartheta_j(yz) = \vartheta_j(zy)$.

By Lemma 3.3, the equation $S(x_1, \ldots, x_m) = T(x_1, \ldots, x_m)$ is satisfied by traces $x_i \in \mathrm{alph}(x_i)^*$ for $1 \leq i \leq m$ if and only if their projections $x_{i,j} = \pi_j(x_i) \in (\mathrm{alph}(x_i) \cap \Gamma_j)^*$ satisfy the equations $\vartheta_j(S) = \vartheta_j(T)$ for all $1 \leq j \leq n$. But since Γ_j is a clique of (Σ, D) this holds if and only if the projections $x_{i,j}$ satisfy a Boolean combination of recognizable constraints by Lemma 4.2. Hence (4) is satisfied for traces $x_1, \ldots, x_m \in \mathbb{M}$ if and only if $x_1, \ldots, x_m$ satisfy a finite disjunction of formulae of the form

$$\bigwedge_{1 \leq i \leq m} x_i \in \{u \in \mathbb{M} \mid \pi_1(u) \in L_{i,1}, \ldots, \pi_n(u) \in L_{i,n}\} \cap L(x_i)$$

[5] Note that this means that $(S, T) \in \mathbb{S}$.

where $L_{i,j} \in \mathrm{REC}(\Gamma_j^*)$. By Lemma 4.3 the sets appearing in this formula are recognizable which concludes the proof. □

5 Decidability of scattered rewriting

In this section we will prove Theorem 3.5. A remark similar to that at the beginning of the previous section applies: all existence and equivalence results in this section are effective. This fact is not stated explicitly in the following lemmas, but it can be always easily checked.

5.1 Reduction to local properties

The main tool in this section is Gaifman's locality theorem for first-order logic [20]. We will only need the special case of Gaifman's theorem where the signature contains only unary and binary relation symbols which makes the definitions a little bit easier. Let us consider a model of the form $\mathcal{A} = (A, U, B)$ where U is a set of unary relations, and B is a set of binary relations. For $a, b \in A$ we define $d_{\mathcal{A}}(a, b)$ as the length of a shortest undirected path in the graph $(A, \bigcup_{E \in B} E)$ from a to b. For $r \geq 0$ and $a \in A$, the *r-sphere* $S(r, a)$ around a is $S(r, a) = \{b \in A \mid d_{\mathcal{A}}(a, b) \leq r\}$. Note that there exists a first-order formula in the signature of $\mathcal{A}$ with two free variables that expresses the fact that $d_{\mathcal{A}}(x, y) \leq r$, i.e., the r-sphere around a is definable. Now let $\varphi(x)$ be a first-order formula in the signature of $\mathcal{A}$. Then the first-order formula $\varphi^{S(r,x)}$ results from φ by relativizing all quantifier to $S(r, x)$. It can be defined inductively, for instance $(\phi_1 \wedge \phi_2)^{S(r,x)} \equiv \phi_1^{S(r,x)} \wedge \phi_2^{S(r,x)}$, $(\exists y\, \phi)^{S(r,x)} \equiv \exists y \{d(x, y) \leq r \ \wedge \ \phi^{S(r,x)}\}$, and $P(x_1, \ldots x_n)^{S(r,x)} \equiv P(x_1, \ldots x_n)$ for atomic formulae. Now Gaifman's theorem states the following [20].

Theorem 5.1. *Let ϕ be a first-order sentence over the signature of the model $\mathcal{A}$. Then ϕ is logically equivalent to a Boolean combination of sentences of the form*

$$\exists x_1 \cdots \exists x_m \left\{ \bigwedge_{1 \leq i < j \leq m} d(x_i, x_j) > 2r \ \wedge \bigwedge_{1 \leq i \leq m} \psi^{S(r,x_i)}(x_i) \right\}$$

where ψ is a first-order sentence over the signature of $\mathcal{A}$.

Since $\mathcal{M}^{\mathrm{red}}$ contains all scattered rewrite rules, we get $d_{\mathcal{M}^{\mathrm{red}}}(u, v) \leq 1$ for any two traces u and v. Hence for the treatment of the structure $\mathcal{M}^{\mathrm{red}}$, Theorem 5.1 is of no immediate use. Note that any first-order sentence φ of the signature of $\mathcal{M}^{\mathrm{red}}$ mentions only finitely many scattered rewrite relations $\to_\rho$. Hence, $\mathcal{M}^{\mathrm{red}} \models \varphi$ if and only if $\mathcal{M}^{\mathrm{red}}_{\mathcal{R}} \models \varphi$ where $\mathcal{R} \subseteq \mathbb{S}$ is finite and contains the set of rewrite rules mentioned in φ. Thus, in order to prove the decidability of the first-order theory of $\mathcal{M}^{\mathrm{red}}$, it suffices to prove the decidability of the first-order theory of $\mathcal{M}^{\mathrm{red}}_{\mathcal{R}}$ for any finite set of rewrite rules $\mathcal{R}$. Therefore, for our further considerations, we fix some finite set $\mathcal{R} \subseteq \mathbb{S}$ of scattered rewrite rules as well as a radius $r \geq 0$.

In order to use Gaifman's locality theorem for decidability purposes, we will need a "useful" description of the set of all traces $t \in \mathbb{M}$ with $\mathcal{M}_{\mathcal{R}}^{\mathrm{red}} \models \varphi^{S(r,x)}(t)$. We will show that this set is recognizable and that it is indeed a "useful" description (cf. Section 5.4). The following two sections prepare the proof of the recognizability.

5.2 Reduction to 1-spheres

The aim of this section is to show that by enlarging the set $\mathcal{R}$ it suffices to restrict to the case $r = 1$ in Theorem 5.1. For the rest of Section 5, $\overline{x}$ (resp. $\overline{y}$) will always denote the repetition-free sequence $x_1, \ldots, x_m$ (resp. $y_1, \ldots, y_n$) of variables from Ω. We assume that $\overline{x}$ and $\overline{y}$ do not contain common variables. Furthermore we use the formula $\overline{x} \in L(\overline{x})$ as an abbreviation for the formula $\bigwedge_{1 \leq i \leq m} x_i \in L(x_i)$, and similarly for $\overline{y}$. The following lemma follows immediately from Lemma 3.2.

Lemma 5.2. *Let $T(\overline{x})$ and $U(\overline{y})$ be patterns with $\mathrm{type}(T) = x_1 \cdots x_m$ and $\mathrm{type}(U) = y_1 \cdots y_n$. Then the equation $S(\overline{x}) = T(\overline{y})$ is equivalent in $\mathcal{M}$ to a finite disjunction of formulae of the form*

$$\exists z_{1,1} \in \Sigma_{1,1}^* \cdots \exists z_{m,n} \in \Sigma_{m,n}^* \left\{ \begin{array}{l} \bigwedge\limits_{1 \leq i \leq m} x_i = U_i(z_{i,1}, \ldots, z_{i,n}) \wedge \\ \bigwedge\limits_{1 \leq j \leq n} y_j = T_j(z_{1,j}, \ldots, z_{m,j}) \end{array} \right\}$$

where $\mathrm{type}(T_j) = z_{1,j} z_{2,j} \cdots z_{m,j}$, $\mathrm{type}(U_i) = z_{i,1} z_{i,2} \cdots z_{i,n}$ and the alphabets $\Sigma_{i,j}$ satisfy $\Sigma_{i,j} \times \Sigma_{k,l} \subseteq I$ whenever $1 \leq i < k \leq m$ and $n \geq j > l \geq 1$.

Lemma 5.3. *Let $S(\overline{x})$ be a pattern, and let $L \in \mathrm{REC}(\mathbb{M})$. Then the formula $x = S(\overline{x}) \wedge x \in L$ is equivalent in $\mathcal{M}$ to a finite disjunction of formulae of the form*

$$x = S(\overline{x}) \wedge \bigwedge_{1 \leq i \leq m} x_i \in L_i$$

where $L_i \in \mathrm{REC}(\mathbb{M})$.

Proof. W.l.o.g. assume that $S = x_1 s_1 x_2 \cdots s_{m-1} x_m$. Let $h : \mathbb{M} \to Q$ be a morphism into a finite monoid Q and let $F \subseteq Q$ such that $L = h^{-1}(F)$. In order to construct the desired formula let us take a finite disjunction over all sequences $q_1, q_2, \ldots, q_m$ of elements from Q such that $q_1 h(s_1) q_2 \cdots h(s_{m-1}) q_m \in F$. For such a sequence we write down the formula

$$x = S(\overline{x}) \wedge \bigwedge_{1 \leq i \leq m} x_i \in h^{-1}(q_i).$$

It is easy to see that the resulting disjunction is equivalent in $\mathcal{M}$ to our original formula. □

Note that in Lemma 5.3 every $t \in L_i$ is a factor of some $s \in L$. Hence the set of symbols that occur in traces from L_i is a subset of the set of symbols that occur in traces from L, this fact will be used in the proof of the next lemma.

Let ρ_1 and ρ_2 be scattered rewrite rules. Then the following lemma implies easily that there exists a finite scattered rewriting system $\mathcal{P}$ such that $\exists z : x \to_{\rho_1} z \to_{\rho_2} y$ for some traces x and y is equivalent to $x \to_{\mathcal{P}} y$. In general, $\mathcal{P}$ consists of more than just one rule. The lemma will allow us to describe the r-sphere around a trace x as the 1-sphere of another, larger, but still finite, scattered rewriting system (cf. Lemma 5.5).

Lemma 5.4. *Let* $(S(\overline{x}), T(\overline{x})), (U(\overline{y}), V(\overline{y})) \in \mathbb{S}$. *The formula*

$$x = S(\overline{x}) \ \wedge \ \overline{x} \in L(\overline{x}) \ \wedge \ T(\overline{x}) = U(\overline{y}) \ \wedge \ y = V(\overline{y}) \ \wedge \ \overline{y} \in L(\overline{y}) \tag{5}$$

is equivalent in $\mathcal{M}$ *to a finite disjunction of formulae of the form*

$$\exists z_{1,1} \in L(z_{1,1}) \cdots \exists z_{m,n} \in L(z_{m,n}) \left\{ \begin{array}{l} x = P(z_{1,1}, \ldots, z_{m,n}) \wedge \\ \bigwedge\limits_{1 \le i \le m} x_i = U_i(z_{i,1}, \ldots, z_{i,n}) \wedge \\ \bigwedge\limits_{1 \le j \le n} y_j = T_j(z_{1,j}, \ldots, z_{m,j}) \wedge \\ y = R(z_{1,1}, \ldots, z_{m,n}) \end{array} \right\}$$

where $U_i(z_{i,1}, \ldots, z_{i,n})$ *and* $T_j(z_{1,j}, \ldots, z_{m,j})$ *are patterns and* (P, R) *is a scattered rewrite rule over the variables* $z_{1,1}, \ldots, z_{m,n}$.

Proof. W.l.o.g. we may assume that $\mathrm{type}(T) = x_1 \cdots x_m$ and $\mathrm{type}(U) = y_1 \cdots y_n$. By Lemma 5.2 the formulae (5) is equivalent in $\mathcal{M}$ to a finite disjunction of formulae of the form

$$\exists z_{1,1} \in \Sigma_{1,1}^* \cdots \exists z_{m,n} \in \Sigma_{m,n}^* \left\{ \begin{array}{l} x = S(\overline{x}) \ \wedge \ \overline{x} \in L(\overline{x}) \wedge \\ \bigwedge\limits_{1 \le i \le m} x_i = U_i(z_{i,1}, \ldots, z_{i,n}) \wedge \\ \bigwedge\limits_{1 \le j \le n} y_j = T_j(z_{1,j}, \ldots, z_{m,j}) \wedge \\ y = V(\overline{y}) \ \wedge \ \overline{y} \in L(\overline{y}) \end{array} \right\} \tag{6}$$

where $\mathrm{type}(T_j) = z_{1,j} z_{2,j} \cdots z_{m,j}$, $\mathrm{type}(U_i) = z_{i,1} z_{i,2} \cdots z_{i,n}$, and the alphabets $\Sigma_{i,j}$ satisfy $\Sigma_{i,j} \times \Sigma_{k,l} \subseteq I$ whenever $1 \le i < k \le m$ and $n \ge j > l \ge 1$. By Lemma 5.3 we can replace the subformulae $x_i = U_i(z_{i,1}, \ldots, z_{i,n}) \ \wedge \ x_i \in L(x_i)$ in (6) by a finite disjunction of formulae of the form

$$x_i = U_i(z_{i,1}, \ldots, z_{i,n}) \ \wedge \bigwedge_{1 \le j \le n} z_{i,j} \in K_{i,j}$$

where $K_{i,j} \in \mathrm{REC}(\mathbb{M})$. Similarly, the subformulae $y_j = T_j(z_{1,j}, \ldots, z_{m,j}) \ \wedge \ y_j \in L(y_j)$ can be replaced in (6) by a finite disjunction of formulae of the form

$$y_j = T_j(z_{1,j}, \ldots, z_{m,j}) \ \wedge \bigwedge_{1 \leq i \leq m} z_{i,j} \in L_{i,j}$$

where $L_{i,j} \in \mathrm{REC}(\mathbb{M})$. Altogether we obtain for the variable $z_{i,j}$ the recognizable constraint $z_{i,j} \in \Sigma_{i,j}^* \cap K_{i,j} \cap L_{i,j}$. We may w.l.o.g. assume that the variable $z_{i,j}$ was chosen such that $L(z_{i,j}) = \Sigma_{i,j}^* \cap K_{i,j} \cap L_{i,j}$. By Lemma 5.2 and the remark after the proof of Lemma 5.3 this implies $(z_{i,j}, z_{k,l}) \in J$ whenever ($i < k$ and $l < j$) or $(x_i, x_k) \in J$ or $(y_j, y_l) \in J$.

We may replace the pattern V by the pattern $R = V[y_j/T_j]$. Here the pattern $V[y_j/T_j]$ results from V by replacing for all j the variable y_j by the pattern T_j. Similarly, we may replace the pattern S by the pattern $P = S[x_i/U_i]$. It remains to show that $\mathrm{type}(P) \equiv_J \mathrm{type}(R)$. Since $(z_{i,j}, z_{k,l}) \in J$ if $(x_i, x_k) \in J$, we have $(\mathrm{type}(U_i), \mathrm{type}(U_k)) \in J$ whenever $(x_i, x_k) \in J$. Since $\mathrm{type}(S) \equiv_J \mathrm{type}(T) = x_1 \cdots x_m$ it follows

$$\mathrm{type}(P) \equiv_J \mathrm{type}(U_1) \cdots \mathrm{type}(U_m) = (z_{1,1} \cdots z_{1,n}) \cdots (z_{m,1} \cdots z_{m,n}).$$

Since $(z_{i,j}, z_{k,l}) \in J$ if $i < k$ and $j > l$, it follows

$$\begin{aligned}(z_{1,1} \cdots z_{1,n}) \cdots (z_{m,1} \cdots z_{m,n}) &\equiv_J (z_{1,1} \cdots z_{m,1}) \cdots (z_{1,n} \cdots z_{m,n}) \\ &= \mathrm{type}(T_1) \cdots \mathrm{type}(T_n).\end{aligned}$$

Since $(\mathrm{type}(T_j), \mathrm{type}(T_l)) \in J$, we obtain $\mathrm{type}(T_1) \cdots \mathrm{type}(T_n) \equiv_J \mathrm{type}(R)$ whenever $(y_j, y_l) \in J$ and $\mathrm{type}(V) \equiv_J \mathrm{type}(U) = y_1 \cdots y_n$. □

Lemma 5.5. *For $r \geq 0$, there exists a scattered rewriting system $\mathcal{R}_r$ over $\mathbb{M}$ such that $\mathcal{R} \subseteq \mathcal{R}_r$ and for all $s, t \in \mathbb{M}$ it holds $d_{\mathcal{M}_{\mathcal{R}}^{\mathrm{red}}}(s, t) \leq r$ if and only if $s \to_{\mathcal{R}_r} t$.*

Proof. Let $(S(\overline{x}), T(\overline{x})), \ (U(\overline{y}), V(\overline{y})) \in \mathbb{S}$. Consider the formula

$$\exists \overline{x}, \overline{y} \, \{x = S(\overline{x}) \ \wedge \ \overline{x} \in L(\overline{x}) \ \wedge \ T(\overline{x}) = U(\overline{y}) \ \wedge \ y = V(\overline{y}) \ \wedge \ \overline{y} \in L(\overline{y})\} \, .$$

It expresses the fact that $x \to_{(S,T)} z \to_{(U,V)} y$ for some z. Furthermore by Lemma 5.4 it is equivalent in $\mathcal{M}$ to a finite disjunction of formulae of the form

$$\exists z_{1,1} \in L(z_{1,1}) \cdots \exists z_{m,n} \in L(z_{m,n}) \left\{ \begin{array}{l} x = P(z_{1,1}, \ldots, z_{m,n}) \wedge \\ y = R(z_{1,1}, \ldots, z_{m,n}) \end{array} \right\},$$

i.e., $x \to_{(P,R)} y$ where $(P, R) \in \mathbb{S}$. This proves the lemma. □

Before we proceed let us first state the following simple consequence of Lemma 4.4 and Lemma 5.4.

Lemma 5.6. *Let* $(S(\overline{x}), T(\overline{x})), (V(\overline{y}), U(\overline{y})) \in \mathbb{S}$. *The formula*

$$\overline{x} \in L(\overline{x}) \ \wedge \ S(\overline{x}) \to_{(V,U)} T(\overline{x}) \tag{7}$$

is equivalent in $\mathcal{M}$ *to a Boolean combination of formulae* $x_i \in L$ *with* $L \in \mathrm{REC}(\mathbb{M})$.

Proof. First note that (7) is equivalent in $\mathcal{M}$ to the formula

$$\exists x \ \exists \overline{y} \, \{x = S(\overline{x}) \wedge \overline{x} \in L(\overline{x}) \wedge T(\overline{x}) = U(\overline{y}) \wedge x = V(\overline{y}) \wedge \overline{y} \in L(\overline{y})\} \, .$$

By Lemma 5.4 this formula is equivalent in $\mathcal{M}$ to a finite disjunction of formulae of the form

$$\exists z_{1,1} \in L(z_{1,1}) \cdots \exists z_{m,n} \in L(z_{m,n}) \left\{ \begin{array}{l} P(z_{1,1}, \ldots, z_{m,n}) = R(z_{1,1}, \ldots, z_{m,n}) \wedge \\ \bigwedge\limits_{1 \le i \le m} x_i = U_i(z_{i,1}, \ldots, z_{i,n}) \end{array} \right\}$$

where $\mathrm{type}(P) \equiv_J \mathrm{type}(R)$. Hence by Lemma 4.4 this formula is equivalent in $\mathcal{M}$ to a finite disjunction of formulae of the form

$$\exists z_{1,1} \in L(z_{1,1}) \cdots \exists z_{m,n} \in L(z_{m,n}) \bigwedge_{1 \le i \le m} \left(x_i = U_i(z_{i,1}, \ldots, z_{i,n}) \wedge \bigwedge_{1 \le j \le n} z_{i,j} \in L_{i,j} \right)$$

where $L_{i,j} \in \mathrm{REC}(\mathbb{M})$. Since in a pattern a variable occurs at most once, this formula is equivalent to

$$\bigwedge_{1 \le i \le m} x_i \in \{U_i(u_1, \ldots, u_n) \mid u_1 \in L_{i,1} \cap L(z_{i,1}), \ldots, u_n \in L_{i,n} \cap L(z_{i,n})\}.$$

By the closure properties of $\mathrm{REC}(\mathbb{M})$, the sets in the formula above are recognizable. □

5.3 Internalizing the 1-sphere

As a major tool in the further consideration we will use monadic second order logic (MSO logic) over dependence graphs. Formulae in this logic are interpreted over dependence graphs (V, E, λ) of traces. There exist first-order variables $x, y, z, \ldots$ ranging over elements of V and second-order variables $X, Y, Z, \ldots$ ranging over subsets of V. Atomic formulae are of the form $Q_a(x)$, $x \preceq y$, and $x \in X$ where x and y are first-order variables, X is a second-order variable, and Q_a is a unary relation symbol for every $a \in \Sigma$. The interpretation of $Q_a(x)$ is $\lambda(x) = a$ whereas $x \preceq y$ is interpreted as $(x, y) \in E^*$. From atomic formulae, MSO-formulae are constructed using Boolean connectives and quantification over first-order and second-order variables. For an MSO-formula ψ and a trace $t \in \mathbb{M}$ we write $D_t \models \psi$ if the dependence graph D_t satisfies the formula ψ.

We use the following terminology in order to get not confused by the different types of variables we have to deal with. First-order variables that range over elements of the model $\mathcal{M}$ are called *external first-order variables*. Second-order variables that range over subsets of an underlying dependence graph are called *internal second-order variables*. Internal first-order variables that range over single nodes of the dependence graph will not occur very much in the further consideration. Similarly, first-order formula that speak about the structure $\mathcal{M}$ (consisting of all traces) will be called external first-order formulae, whereas MSO-formulae that are interpreted over a single dependence graph are called internal MSO-formulae.

Thomas has shown in [45] that MSO logic precisely characterizes recognizable trace languages:

Theorem 5.7. *A set $L \subseteq \mathbb{M}$ is recognizable if and only if there exists an MSO-sentence Ψ such that $L = \{t \in \mathbb{M} \mid D_t \models \Psi\}$.*

This result has been extended to other logics [45, 10] and to infinite traces [17].

We will need the following internal MSO-formulae: The MSO-formula $\mathrm{conv}(X)$ defined by

$$\mathrm{conv}(X) \quad \equiv \quad \forall x, y, z\{(x, z \in X \wedge x \preceq y \preceq z) \rightarrow y \in X\}$$

holds if and only if X is a convex subset of the dependence graph. For a patter $S = x_1 s_1 x_2 \cdots s_{m-1} x_m$ we define an MSO-formula $\langle S \rangle(X, X_1, \ldots, X_m)$ which expresses for a dependence graph $D_t = (V, E, \lambda)$ the following facts:

- $X, X_1, \ldots, X_m$ are convex and $D_t{\restriction}_{X_i} \in L(x_i)$ for $1 \leq i \leq m$
- There exist convex sets $Z_1, \ldots, Z_{m-1} \subseteq X$ of D_t such that the trace represented by $D_t{\restriction}_{Z_i}$ is s_i and that the sequence $X_1, Z_1, X_2, Z_2, \ldots, Z_{m-1}, X_m$ is a factorization of $D_t{\restriction}X$ (according to the definition in Section 3.1).

Note that both points can be expressed in MSO. Finally we will use in MSO-formulae several times a constant V which denotes the whole set of nodes of the underlying dependence graph. Of course this constant can be easily defined in MSO.

Inductively, we will construct for any external first-order formula $\varphi(x)$ an internal monadic second order sentence $\mathrm{int}(\varphi)$ such that, for $t \in \mathbb{M}$, we obtain $\mathcal{M} \models \varphi^{S(1,x)}(t)$ if and only if $D_t \models \mathrm{int}(\varphi)$. For the further consideration we may w.l.o.g assume that for some number n every rule in $\mathcal{R}_r$ is defined over exactly n variables. This may be assumed since we always can add a dummy variables z with $L(z) = \{1\}$ to the end of the left- and right-hand side of a rule.

Theorem 5.8. *Let $\varphi(x, y_1, \ldots, y_\ell)$ be an external first-order formula with free variables among $\{x, y_1, \ldots, y_\ell\}$ and let $(S_k, T_k)_{1\leq k\leq \ell}$ be a tuple of rules from $\mathcal{R}_r$. There exists an internal MSO-formula*

$$\mathrm{int}(\varphi)[(S_k, T_k)_{1\leq k\leq \ell}]$$

with free second-order variables among $\{Y_k^i \mid 1 \leq k \leq \ell, 1 \leq i \leq n\}$ such that for all dependence graphs $D_t = (V, E, \lambda)$ and sets $Y_k^i \subseteq V$ $(1 \leq k \leq \ell, 1 \leq i \leq n)$ with $D_t \models \langle S_k\rangle(V, Y_k^1, Y_k^2, \ldots Y_k^n)$, the following are equivalent:

(1) $D_t \models \mathrm{int}(\varphi)[(S_k, T_k)_{1\leq k\leq \ell}]$
(2) $\mathcal{M}^{\mathrm{red}}_{\mathcal{R}_r} \models \varphi^{S(1,x)}(t, u_1, u_2 \ldots, u_\ell)$ *where* $u_k = T_k(D_t\restriction_{Y_k^1}, D_t\restriction_{Y_k^2}, \ldots, D_t\restriction_{Y_k^n})$

Note that in the theorem above, we assume that Y_k^i is a convex subset of V since $D_t \models \langle S_k\rangle(V, Y_k^1, Y_k^2, \ldots Y_k^n)$. Hence $D_t\restriction_{Y_k^i}$ is a dependence graph, i.e., there is a trace v with $D_v = D_t\restriction_{Y_k^i}$. In the definition of u_k, we identified the trace v with its dependence graph $D_t\restriction_{Y_k^i}$ (and we will continue to use this identification).

For an external first-order formula $\varphi(x)$, the tuple $(S_k, T_k)_{1\leq k\leq \ell}$, the sets $Y_k^i \subseteq D_t$, and the traces u_k disappear in the theorem above. In other words, we obtain a reduction of the external first-order formula $\varphi(x)$ to the internal monadic second-order sentence $\mathrm{int}(\varphi)$ such that for every trace t it holds $D_t \models \mathrm{int}(\varphi)$ if and only if $\mathcal{M}^{\mathrm{red}}_{\mathcal{R}_r} \models \varphi^{S(1,x)}(t)$.

Theorem 5.8 is shown by induction on the construction of the formula φ, this induction is split into the following four claims that collectively prove Theorem 5.8.

Claim 1. Thm. 5.8 holds for the formula $\varphi \equiv (y \to_\rho z)$ where $\rho \in \mathbb{S}$ is a scattered rewrite rule and $\{y, z\} \subseteq \{x, y_1, \ldots, y_\ell\}$.

Proof. First note that we may assume $y \neq z$: an atomic formula $y \to_\rho y$ can be replaced by $\exists z(y \to_\rho z \wedge y = z)$, where z is a new variable. Furthermore we may assume that $x \notin \{y, z\}$. For instance the formula $\mathrm{int}(y_1 \to_\rho x)[(S_1, T_1)]$ can be defined as the formula $\exists \overline{Y}_2\ (\langle S_2\rangle(V, \overline{Y}_2) \wedge \mathrm{int}(y_1 \to_\rho y_2)[(S_1, T_1), (S_2, T_2)])$ where $\overline{Y}_2$ is the sequence of variables $Y_2^1, \ldots, Y_2^n$ and (S_2, T_2) is a rule from $\mathcal{R}_r$ such that $u \to_{(S_2,T_2)} u$ for all $u \in \mathbb{M}$ (such a rule exists in $\mathcal{R}_r$).

It remains to define the formula $\mathrm{int}(y_1 \to_\rho y_2)[(S_1, T_1), (S_2, T_2)]$. Fix two rules $(S_k, T_k) \in \mathcal{R}_r$ $(k = 1, 2)$ and let (S_k, T_k) be defined over the variables $y_k^1, \ldots, y_k^n$. W.l.o.g. assume that $S_k = y_k^1 a_k^1 y_k^2 a_k^2 \cdots y_k^n$. For the construction of the MSO-formula $\mathrm{int}(y_1 \to_\rho y_2)[(S_1, T_1), (S_2, T_2)]$ we will use all patterns $U_i = z_{i1} d_{i1} z_{i2} d_{i2} \cdots z_{in}$ and $V_i = z_{1i} e_{1i} z_{2i} e_{2i} \cdots z_{ni}$ $(1 \leq i \leq n)$ such that

$$\begin{aligned} &d_{i,j} \text{ is a factor of } a_2^j,\ e_{i,j} \text{ is a factor of } a_1^i,\ L(z_{ij}) = \mathrm{alph}(z_{ij})^*, \text{ and} \\ &\{(z_{i,j}, z_{p,q}) \mid (i < p \text{ and } q < j) \text{ or } (y_1^i, y_1^p) \in J \text{ or } (y_2^j, y_2^q) \in J\} \subseteq J. \end{aligned} \tag{8}$$

Because of the restriction of these patterns to length n and since all the traces mentioned in U_i and V_i are factors of some a_k^j, there are only finitely many patterns satisfying (8) up to renaming of variables.

Let $t \in \mathbb{M}$ with $D_t = (V, E, \lambda)$, and let $Y_k^i \subseteq V$ $(1 \leq i \leq n,\ k = 1, 2)$. Let $u_i = T_i(D_t\restriction_{Y_i^1}, D_t\restriction_{Y_i^2}, \ldots, D_t\restriction_{Y_i^n})$. We prove the claim by successively showing the equivalence of the four statements (A) to (D).

(A) $u_1 \to_\rho u_2$ and $D_t \models \bigwedge_{k=1,2} \langle S_k \rangle (V, Y_k^1, Y_k^2, \ldots Y_k^n)$

(B) $u_1 \to_\rho u_2$ and there exist patterns U_i and V_i $(1 \leq i \leq n)$ satisfying (8) and there exist sets $Z_{i,j} \subseteq V$ such that

$$\begin{aligned} D_t \models & \bigwedge_{k=1,2} \langle S_k \rangle (V, Y_k^1, Y_k^2, \ldots Y_k^n) \wedge \\ & \bigwedge_{1 \leq i \leq n} \langle U_i \rangle (Y_1^i, Z_{i1}, Z_{i2}, \ldots Z_{in}) \wedge \\ & \bigwedge_{1 \leq i \leq n} \langle V_i \rangle (Y_2^i, Z_{1i}, Z_{2i}, \ldots Z_{ni}). \end{aligned} \tag{9}$$

The implication (B)⇒(A) is trivial. Conversely, if (A) then there exist convex sets $A_k^i \subseteq V$ such that $D_t\restriction_{A_k^i}$ represents a_k^i $(1 \leq i \leq n,\ k = 1, 2)$, and the sequence $Y_k^1, A_k^1, Y_k^2, A_k^2, \ldots, Y_k^n$ is a factorization of D_t for $k = 1, 2$. Define

$$\begin{aligned} C_{ij} &= A_1^i \cap A_2^j \text{ for } 1 \leq i, j < n, \\ D_{ij} &= Y_1^i \cap A_2^j \text{ for } 1 \leq i \leq n, 1 \leq j < n, \\ E_{ij} &= A_1^i \cap Y_2^j \text{ for } 1 \leq i < n, 1 \leq j \leq n, \\ Z_{ij} &= Y_1^i \cap Y_2^j \text{ for } 1 \leq i, j \leq n. \end{aligned}$$

The situation can be visualized by the following diagram. The rows represent the factors Y_1^i and A_1^i and the columns the factors Y_2^i and A_2^i.

Y_1^n	Z_{n1}	D_{n1}	Z_{n2}	D_{n2}	…	Z_{nn}
⋮	⋮	⋮	⋮	⋮	⋮	⋮
Y_1^2	Z_{21}	D_{21}	Z_{22}	D_{22}	…	Z_{2n}
A_1^2	E_{21}	C_{21}	E_{22}	C_{22}	…	E_{2n}
Y_1^1	Z_{11}	D_{11}	Z_{12}	D_{12}	…	Z_{1n}
A_1^1	E_{11}	C_{11}	E_{12}	C_{12}	…	E_{1n}
	Y_2^1	A_2^1	Y_2^1	A_2^2	…	Y_2^n

For $1 \leq i, j \leq n$, let z_{ij} be an external first-order variable with $\mathrm{alph}(z_{ij}) = \lambda[Z_{ij}]$ and $L(z_{ij}) = \mathrm{alph}(z_{ij})^*$. Since $\lambda[Z_{ij}] \subseteq \lambda[Y_i^1] \cap \lambda[Y_j^2]$ and $\lambda[Z_{ij}] \times \lambda[Z_{pq}] \subseteq I$ whenever $i < p$ and $j > q$ (see Lemma 3.1), we obtain

$$\{(z_{i,j}, z_{p,q}) \mid (i < p \text{ and } q < j) \text{ or } (y_1^i, y_1^p) \in J \text{ or } (y_2^j, y_2^q) \in J\} \subseteq J.$$

Furthermore let the trace $d_{ij} \in \mathbb{M}$ represent $D_t\restriction_{D_{ij}}$ and define the trace e_{ij} similarly. Then the patterns

$$\begin{aligned} U_i &= z_{i1} d_{i1} z_{i2} d_{i2} \ldots z_{in} \text{ and} \\ V_i &= z_{1i} e_{1i} z_{2i} e_{2i} \ldots z_{ni} \end{aligned}$$

for $1 \leq i \leq n$ satisfy (8). Finally from the diagram above we can easily deduce

$$\begin{aligned} D_t &\models \langle U_i \rangle (Y_1^i, Z_{i1}, Z_{i2}, \ldots Z_{in}) \text{ and} \\ D_t &\models \langle V_i \rangle (Y_2^i, Z_{1i}, Z_{2i}, \ldots Z_{ni}) \end{aligned} \tag{10}$$

for $1 \leq i \leq n$. Hence we showed the implication (A)⇒(B) and therefore the equivalence of these two statements.

Next, we show that (B) is equivalent to

(C) there exist patterns D_i and E_i for $1 \leq i \leq n$ and sets $Z_{i,j} \subseteq V$ such that (8) and (9) hold and

$$T_1[y_1^i/U_i](D_t\restriction_{Z_{11}}, \ldots D_t\restriction_{Z_{nn}}) \to_\rho T_2[y_2^i/V_i](D_t\restriction_{Z_{11}}, \ldots D_t\restriction_{Z_{nn}})$$

where $T_1[y_1^i/U_i]$ is the pattern obtained from T_1 by substituting the pattern U_i for the variable y_1^i and $T_2[y_2^i/V_i]$ is defined similarly.

From (10) it follows $D_t\restriction_{Y_1^i} = U_i(D_t\restriction_{Z_{i1}}, D_t\restriction_{Z_{i2}} \ldots D_t\restriction_{Z_{in}})$, which has to be read as an identity between traces. Since, in addition, $u_1 = T_1(D_t\restriction_{Y_1^1}, D_t\restriction_{Y_1^2}, \ldots D_t\restriction_{Y_1^n})$, we get $u_1 = T_1[y_1^i/D_i](D_t\restriction_{Z_{11}}, \ldots D_t\restriction_{Z_{nn}})$. The equation $u_2 = T_2[y_2^i/V_i](D_t\restriction_{Z_{11}}, \ldots D_t\restriction_{Z_{nn}})$ follows similarly. This implies the equivalence of (B) and (C).

Next we show that there exists an MSO-formula μ such that (C) and (D) are equivalent.

(D) there exist patterns D_i and E_i for $1 \leq i \leq n$ and sets $Z_{i,j} \subseteq V$ such that (8) and (9) hold and $D_t \models \mu$ where μ is an internal MSO formula (with free second-order variables Z_{ij}).

The definition of the patterns $T_1[y_1^i/D_i], T_2[y_2^i/E_i]$ and (8) implies $\mathrm{type}(T_1[y_1^i/U_i]) \equiv_J \mathrm{type}(T_2[y_2^i/V_i])$.[6] Because of (9), we can apply Lemma 5.6 which tells us that

$$T_1[y_1^i/U_i](D_t\restriction_{Z_{11}}, \ldots D_t\restriction_{Z_{nn}}) \to_\rho T_2[y_2^i/]V_i](D_t\restriction_{Z_{11}}, \ldots D_t\restriction_{Z_{nn}})$$

is equivalent to a Boolean combination of formulae of the form $D_t\restriction_{Z_{ij}} \in L$ for some $L \in \mathrm{REC}(\mathbb{M})$. By Theorem 5.7, this Boolean combination of external first-order formulae $D_t\restriction_{Z_{ij}} \in L$ can be translated into an equivalent Boolean combination of internal MSO-formulae $\psi\restriction_{Z_{ij}}$, where $\psi\restriction_{Z_{ij}}$ results from an MSO-sentence ψ defining L by restricting all quantifications in ψ to Z_{ij}. This Boolean combination is the formula μ whose existence we had to show. □

Claim 2. Thm. 5.8 holds for the formula $\varphi \equiv (y \in L)$ where $y \in \{x, y_1, \ldots, y_\ell\}$ and $L \in \mathrm{REC}(\mathbb{M})$.

[6] In fact this can be shown completely analogously to the equality $\mathrm{type}(P) \equiv_J \mathrm{type}(R)$ in the proof of Lemma 5.4.

Proof. Since $u \in L$ if and only if $u \to_\rho u$, where $\rho = (y, y)$ with $L(y) = L$, this is actually a special case of Claim 1.

Claim 3. Thm. 5.8 holds for the formula $\varphi \equiv (y = z)$, where $y, z \in \{x, y_1, \ldots, y_\ell\}$.

Proof. Again this is a special case of Claim 1 with $\rho = (y, y)$ and $L(y) = \mathbb{M}$. □

Claim 4. If Thm. 5.8 holds for the formulae φ and ψ, then it holds for the formulae $\varphi \wedge \psi$, $\neg\varphi$ and $\exists y\varphi$.

Proof. Since conjunction and negation are trivial, we only consider the formula $\exists y\varphi$. Let $\varphi(x, y_1, \ldots, y_\ell)$ be an external first-order formula whose free variables are contained in $\{x, y_1, \ldots, y_\ell\}$ and let w.l.o.g. $y = y_\ell$. Let $\overline{Y_\ell}$ be the sequence of variables $Y_\ell^1, \ldots, Y_\ell^n$ and set

$$\mathrm{int}(\exists y_\ell\varphi)[(S_k, T_k)_{1\leq k<\ell}] \equiv \bigvee_{(S_\ell,T_\ell)\in\mathcal{R}_r} \exists\overline{Y}_\ell(\langle S_\ell\rangle(V, \overline{Y}_\ell) \wedge \mathrm{int}(\varphi)[(S_k, T_k)_{1\leq k\leq\ell}]).$$

Suppose $Y_k^i \subseteq V$ $(1 \leq k < \ell, 1 \leq i \leq n)$ are such that $D_t \models \langle S_k\rangle(V, Y_k^1, Y_k^2, \ldots Y_k^n)$. If $D_t = (V, E, \lambda) \models \mathrm{int}(\exists y_\ell\varphi)[(S_k, T_k)_{1\leq k<\ell}]$ then there is a scattered rewrite rule $(S_\ell, T_\ell) \in \mathcal{R}_r$ and there are sets $Y_\ell^i \subseteq V$ $(1 \leq i \leq n)$ such that $D_t \models S_\ell\langle V, \overline{Y}_\ell\rangle$ and $D_t \models \mathrm{int}(\varphi)[(S_k, T_k)_{1\leq k\leq\ell}]$. Inductively we obtain $\mathcal{M}_{\mathcal{R}_r}^{\mathrm{red}} \models \varphi^{S(1,x)}(t, u_1, u_2 \ldots, u_\ell)$ where $u_k = T_k(D_t{\restriction}_{Y_k^1}, D_t{\restriction}_{Y_k^2}, \ldots, D_t{\restriction}_{Y_k^n})$. Since $u_\ell \in S(1, t)$ in the structure $\mathcal{M}_{\mathcal{R}_r}^{\mathrm{red}}$, it follows $\mathcal{M}_{\mathcal{R}_r}^{\mathrm{red}} \models (\exists y_\ell\varphi)^{S(1,x)}(t, u_1, u_2 \ldots, u_{\ell-1})$.

On the other hand if $\mathcal{M}_{\mathcal{R}_r}^{\mathrm{red}} \models (\exists y_\ell\varphi)^{S(1,x)}(t, u_1, u_2 \ldots, u_{\ell-1})$ for the traces $u_k = T_k(D_t{\restriction}_{Y_k^1}, D_t{\restriction}_{Y_k^2}, \ldots, D_t{\restriction}_{Y_k^n})$, then there is a rule $(S_\ell, T_\ell) \in \mathcal{R}_r$ and there are sets $Y_\ell^i \subseteq V$ $(1 \leq i \leq n)$ such that $D_t \models S_\ell\langle V, \overline{Y}_\ell\rangle$ and $\mathcal{M}_{\mathcal{R}_r}^{\mathrm{red}} \models \varphi^{S(1,x)}(t, u_1, u_2 \ldots, u_\ell)$, where $u_\ell = T_\ell(D_t{\restriction}_{Y_\ell^1}, D_t{\restriction}_{Y_\ell^2}, \ldots, D_t{\restriction}_{Y_\ell^n})$. Hence by induction, the dependence graph D_t satisfies the formula $\mathrm{int}(\varphi)[(S_k, T_k)_{1\leq k\leq\ell}]$, which finally implies $D_t = (V, E, \lambda) \models \mathrm{int}(\exists y_\ell\varphi)[(S_k, T_k)_{1\leq k<\ell}]$. □

This finishes the inductive proof of Theorem 5.8.

5.4 Proof of Theorem 3.5

Theorem 5.9. *Let $r \geq 0$ and let φ be a first-order sentence over the signature of $\mathcal{M}_{\mathcal{R}}^{\mathrm{red}}$. Then $\{t \in \mathbb{M} \mid \mathcal{M}_{\mathcal{R}}^{\mathrm{red}} \models \varphi^{S(r,x)}(t)\} \in \mathrm{REC}(\mathbb{M})$.*

Proof. Let $\mathcal{R}_r$ be the scattered rewriting system from Lemma 5.5 and let $t \in \mathbb{M}$ be some trace. By Lemma 5.5, $\mathcal{M}_{\mathcal{R}}^{\mathrm{red}} \models \varphi^{S(r,x)}(t)$ if and only if $\mathcal{M}_{\mathcal{R}_r}^{\mathrm{red}} \models \varphi^{S(1,x)}(t)$.[7] By Theorem 5.8, this is equivalent to $D_t \models \mathrm{int}(\varphi)$. Hence the set $\{t \in \mathbb{M} \mid \mathcal{M}_{\mathcal{R}}^{\mathrm{red}} \models \varphi^{S(r,x)}(t)\}$ is defined by the MSO-sentence $\mathrm{int}(\varphi)$; it is therefore recognizable by Theorem 5.7. □

[7] Note that $\varphi^{S(r,x)}$ is built using the signature of $\mathcal{M}_{\mathcal{R}}^{\mathrm{red}}$ while $\varphi^{S(1,x)}$ uses the larger signature of $\mathcal{M}_{\mathcal{R}_r}^{\mathrm{red}}$.

We finish this section with the proof of our main theorem.

Theorem 3.5. *There is an algorithm that, on input of an independence alphabet (Σ, I) and a first-order sentence φ in the language of the structure $\mathcal{M}^{\mathrm{red}}$, decides whether $\mathcal{M}^{\mathrm{red}} \models \varphi$.*

Proof. By Gaifman's theorem (Theorem 5.1), it suffices to decide whether a sentence of the form

$$\exists x_1 \cdots \exists x_m \left\{ \bigwedge_{1 \leq i < j \leq m} d(x_i, x_j) > 2r \ \wedge \bigwedge_{1 \leq i \leq m} \varphi^{S(r,x_i)}(x_i) \right\} \tag{11}$$

holds in $\mathcal{M}^{\mathrm{red}}_{\mathcal{R}}$. Here $\varphi(x_i)$ is a first-order formula over the signature of $\mathcal{M}^{\mathrm{red}}_{\mathcal{R}}$. By Theorem 5.9, the set $L = \{t \in \mathbb{M} \mid \mathcal{M}^{\mathrm{red}}_{\mathcal{R}} \models \varphi^{S(r,x)}(t)\}$ is recognizable. Hence we can check whether L is infinite. If this is the case than L contains traces of arbitrary length. Since the application of a rewrite rule from the finite system $\mathcal{R}$ can change the length of a trace only by some fixed amount, there exist infinitely many traces $t_i \in L$, $i \in \mathbb{N}$ such that $d_{\mathcal{M}^{\mathrm{red}}_{\mathcal{R}}}(t_i, t_j) > 2r$ for $i < j$. Hence (11) is true. On the other hand if L is finite, then we can enumerate all elements of L and calculate their r-spheres with respect to $\mathcal{R}$. In this way we can check whether there are at least m traces $t_1, \ldots, t_m \in L$ such that $d_{\mathcal{M}^{\mathrm{red}}_{\mathcal{R}}}(t_i, t_j) > 2r$ for $i < j$. Hence the decidability follows. □

5.5 An extension of first-order logic

First-order logic can be extended by modulo counting quantifiers [43]; the resulting logic is called FO+MOD. Since there is no locality theorem known for this logic,[8] our decidability proof for the first-order theory of $\mathcal{M}^{\mathrm{red}}$ does not work for this more expressive logic; but the second step of our proof, i.e., the recognizability of the set of traces satisfying some local formula in FO (cf. Theorems 5.8 and 5.7) extends to the logic FO+MOD. Thus, we obtain the decidability of local properties expressed in the logic FO+MOD. It seems that this result is new even for semi-Thue systems and, as far as we see, cannot be shown using the automata theoretic methods from [9, 23]. This extension is the topic of this section.

The only difference between FO+MOD and FO is that we now have a second type of quantifiers: if φ is a formula of FO+MOD, then $\exists^{(p,q)} x \varphi$ is a formula as well for $p, q \in \mathbb{N}$ and $p < q$. Then $\mathcal{M}^{\mathrm{red}} \models \exists^{(p,q)} x \varphi$ if the number of traces $t \in \mathbb{M}$ with $\mathcal{M}^{\mathrm{red}} \models \varphi(t)$ is finite and congruent p modulo q.

The logic MSO+MOD We will also extend the monadic second-order logic for dependence graphs MSO by modulo-counting facilities for both, set and first-order variables. MSO+MOD is the extension of MSO by the quantifiers $\exists^{(p,q)}(X_i)_{i \leq n}$ for a tuple

[8] Libkin [26, 27, 25] and Nurmonen [40] proved locality theorems for counting logics including modulo counting, but not in the form of Theorem 5.1. We could not make them work in our situation.

of first-order or second-order variables X_i and $p, q \in \mathbb{N}$ with $p < q$. This quantifier for set variables is only allowed in conjunction with the formula $\mathrm{conv}(X)$ asserting that X is convex: if φ is a formula of MSO+MOD, then $\exists^{(p,q)}(X_i)_{i\leq n}(\bigwedge_{i\leq n} \mathrm{conv}(X_i) \wedge \varphi)$ and $\exists^{(p,q)}(x_i)_{i\leq n}\varphi$ belong to MSO+MOD. A dependence graph $D_t = (V, E, \lambda)$ satisfies $\exists^{(p,q)}(X_i)_{i\leq n}\psi$ if the number of tuples of sets $\overline{X}$ with $X_i \subseteq V$ such that $D_t \models \psi(\overline{X})$ is congruent p modulo q. The interpretation of the quantifier $\exists^{(p,q)}$ applied to first-order variables is similar. Note that, differently from FO+MOD, we did not require the set of witnesses to be finite; this is not needed since any dependence graph is finite.

We show that any property expressible in MSO+MOD can also be expressed in the monadic second-order logic. The reason for introducing this extension of MSO is that the analog of Theorem 5.8 is shown much simpler in terms of MSO+MOD.

Lemma 5.10. *Let φ be a formula of MSO+MOD. Then there exists a formula ψ of MSO such that for any $t \in \mathbb{M}$ it holds $D_t \models \varphi$ if and only if $D_t \models \psi$.*

Proof. We first eliminate modulo quantifications over tuples of sets by modulo quantifications over single sets: the formula $\exists^{(p,q)}(X_i)_{1\leq i\leq n}\varphi$ holds in the dependence graph D_t if and only if for some numbers $0 \leq p_i < q$ with $\sum_{0\leq i<q} (i \cdot p_i) \equiv p \bmod q$, we have

$$D_t \models \bigwedge_{0\leq i<q} \exists^{(p_i,q)} X_1 \; \exists^{(i,q)}(X_j)_{2\leq j\leq n} \; \varphi.$$

Recall that the modular set quantification runs over convex subsets of the underlying dependence graph, only. These convex sets can be encoded by the tuples of their minimal and maximal elements. In order to count correctly, we have to rule out permutations in these antichains. This is achieved by a linear order $\sqsubseteq$ on the set Σ of letters. Let m be the number of letters in Σ. Then the convex set X is replaced by the two m-tuples $(x_i^1)_{1\leq i\leq m}$ and $(x_i^2)_{1\leq i\leq m}$ of minimal and maximal elements such that, for $1 \leq i < m$, we have either $\lambda(x_i^j) \sqsubset \lambda(x_{i+1}^j)$ or $x_i^j = x_{i+1}^j = x_{i+2}^j = \ldots x_m^j$ for $j = 1, 2$; this can be expressed by a first-order formula $\mathrm{NF}((x_i)_{1\leq i\leq m})$. Thus, the formula $\exists^{(p,q)} X \; \psi$ becomes

$$\exists^{(p,q)}(x_i^j)_{\substack{1\leq i\leq m\\ 1\leq j\leq 2}} \left\{ \begin{array}{l} \overline{\psi} \wedge (x_i^1) \text{ and } (x_i^2) \text{ form antichains} \wedge \\ \mathrm{NF}((x_i^1)_{1\leq i\leq m}) \wedge \mathrm{NF}((x_i^2)_{1\leq i\leq m}) \wedge \\ \bigwedge\limits_{1\leq i\leq m} \bigvee\limits_{1\leq k\leq m} x_i^1 \preceq x_k^2 \wedge \bigwedge\limits_{1\leq k\leq m} \bigvee\limits_{1\leq i\leq m} x_i^1 \preceq x_k^2 \end{array} \right\}$$

In this formula, $\overline{\psi}$ is obtained from ψ by replacing any subformula of the form $x \in X$ by

$$\bigvee_{1\leq i,k\leq m} x_i^1 \preceq x \preceq x_k^2$$

This eliminates all applications of modulo quantification to convex sets by modulo quantification over tuples of first-order variables. In a second step, these quantifications over tuples can be eliminated as described above for tuples of set variables. This

results in a formula from MSO+MOD where modulo quantification is applied to single first-order variables, only. These modulo quantifications can be eliminated since any dependence graph over Σ can be covered by $m = |\Sigma|$ chains: the formula $\exists^{(p,q)} x\varphi$ is equivalent to the formula

$$\exists X_1, X_2, \ldots, X_m : \left\{ \begin{array}{l} X_i \text{ are mutually disjoint chains that cover the set of vertices} \\ \wedge \bigvee \bigwedge_{i=1}^{m} \exists^{(p_i,q)} x(x \in X_i \wedge \varphi) \end{array} \right\}$$

where the disjunction runs over all tuples $p_1, p_2, \ldots, p_m$ with $0 \leq p_i < q$ and $\sum p_i \equiv p \mod q$. This formula restricts the modulo quantifiers $\exists^{(p_i,q)} x$ to elements of the chain X_i – these restricted modulo quantifications can be expressed in MSO as any introductory text on words explains (see, e.g., [46]. □

From FO+MOD to MSO+MOD Next, we will prove an analogue of Theorem 5.8 for the logic FO+MOD. By Lemma 5.10, it suffices to reduce a local formula of FO+MOD to a formula of MSO+MOD:

Theorem 5.11. *Let $\varphi(x, y_1, \ldots, y_\ell)$ be an external formula of* FO+MOD *with free variables among $\{x, y_1, \ldots, y_\ell\}$ and let $(S_k, T_k)_{1 \leq k \leq \ell}$ be a tuple of rules from $\mathcal{R}_r$. There exists an internal formula of* MSO+MOD

$$\mathrm{int}(\varphi)[(S_k, T_k)_{1 \leq k \leq \ell}]$$

with free second-order variables among $\{Y_k^i \mid 1 \leq k \leq \ell, 1 \leq i \leq n\}$ such that for all dependence graphs $D_t = (V, E, \lambda)$ and sets $Y_k^i \subseteq V$ $(1 \leq k \leq \ell, 1 \leq i \leq n)$ with $D_t \models \langle S_k \rangle (V, Y_k^1, Y_k^2, \ldots Y_k^n)$, the following are equivalent:

(1) $D_t \models \mathrm{int}(\varphi)[(S_k, T_k)_{1 \leq k \leq \ell}]$
(2) $\mathcal{M}^{\mathrm{red}}_{\mathcal{R}_r} \models \varphi^{S(1,x)}(t, u_1, u_2 \ldots, u_\ell)$ *where* $u_k = T_k(D_t\restriction_{Y_k^1}, D_t\restriction_{Y_k^2}, \ldots, D_t\restriction_{Y_k^n})$

The proof proceeds along the lines of the proof of Theorem 5.8. Actually, the proofs of Claims 1 to 4 go through verbatimly and it only remains to consider formulae of the form $\exists^{(p,q)} x\varphi$:

Claim 5. If Thm. 5.11 holds for the formula $\varphi(x, y_1, \ldots, y_\ell)$, then it holds for the formula $\exists^{(p,q)} y_\ell \varphi$.

Proof. One might be tempted to follow the proof of Claim 4 where we dealt with the usual quantifier. There, the existence of a trace y_ℓ corresponds to the existence of a tuple of sets Y_ℓ^i that factor the central trace t in a certain way. The problem we have to face here is that there is no bijection between these tuples of convex sets and the elements of the sphere around t. Here, differently from Theorem 5.8, we need a bijection since we want to count the witnesses of a formula. Therefore, we now define a linear order on (tuples of) convex sets:

Let $t \in \mathbb{M}$ and $D_t = (V, E, \lambda)$ be its dependence graph. For $X, Y \subseteq V$ and $a \in \Sigma$, we define

$$X \lhd_a^1 Y \text{ if and only if } \exists x \in \lambda^{-1}(a) \cap X \; \forall y \in \lambda^{-1}(a) \cap Y \; x \preceq y$$
$$X \lhd_a^2 Y \text{ if and only if } \exists x \in \lambda^{-1}(a) \cap X \; \forall y \in \lambda^{-1}(a) \cap Y \; x \succeq y$$

Then $\lhd_a^i$ is a linear preorder on the powerset of V for any $a \in \Sigma$ and $i = 1, 2$. Let $\Sigma = \{a_1, a_2, \ldots, a_m\}$. We order the powerset of V lexicographically: $X \lhd^1 Y$ if and only if there is $1 \leq i \leq m$ such that, for $1 \leq j < i$, we have $X \lhd_{a_j}^1 Y \lhd_{a_j}^1 X$ and $X \lhd_{a_i}^1 Y \not\lhd_{a_i}^1 X$. This, again, is a linear preorder, but not necessarily an order. The linear preorder $\lhd^2$ is defined similarly. Now the relation $\lhd$ given by

$$X \lhd Y \text{ if and only if } X \lhd^1 Y \not\lhd^1 X \text{ or } X \lhd^1 Y \lhd^1 X \text{ and } X \lhd^2 Y \not\lhd^2 X$$

is a linear preorder since it is the lexicographic product of the linear preorders $\lhd^1$ and $\lhd^2$. Its restriction to *convex* sets is a linear order: let $X, Y \subseteq V$ be different and convex. W.l.o.g. we may assume the existence of some $x \in X \setminus Y$. Let $a = \lambda(x)$. Since Y is convex, we get $x \prec \lambda^{-1}(a) \cap Y$ or $x \succ \lambda^{-1}(a) \cap Y$. In the first case, we get $X \lhd_a^1 Y \not\lhd_a^1 X$ and in the second similarly $X \lhd_a^2 Y \not\lhd_a^2 X$. Hence we cannot have $X \lhd Y \lhd X$, i.e., $\lhd$ orders the convex sets in V linearly. We denote the lexicographic extension of $\lhd$ to n-tuples of convex sets by $\lhd$, too. Note that this is again a linear order. Furthermore, there is an internal MSO-formula LE with free set variables $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ such that, for $t \in \mathbb{M}$ with $D_t = (V, E, \lambda)$ and $X_i, Y_i \subseteq V$, we have

$$D_t \models \mathrm{LE}(\overline{X}, \overline{Y}) \text{ if and only if } \overline{X} \lhd \overline{Y}.$$

Recall that a rule $(S, T) \in \mathcal{R}_r$ and a tuple $\overline{Y}$ with $D_t \models \langle V, \overline{X} \rangle$ represents a trace in the 1-sphere around t. Let $\sqsubseteq$ be a linear order of the rules in $\mathcal{R}_r$. We can write an MSO-formula $\mathrm{LEAST}_{(S,T)}(\overline{Y})$ stating that $(\overline{Y}, S, T)$ is the least representative of some trace:

$$\bigwedge_{\substack{(S',T') \in \mathcal{R}_r \\ (S',T') \sqsubset (S,T)}} \forall \overline{Z} (\langle S' \rangle (V, \overline{Z}) \to \neg \mathrm{int}(y = z)[(S, T), (S', T')])$$
$$\wedge \forall \overline{Z} ((\langle S \rangle (V, \overline{Z}) \wedge \mathrm{int}(y = z)[(S, T), (S, T)]) \Rightarrow \mathrm{LE}(\overline{Y}, \overline{Z}))$$

Now the internal equivalent $\mathrm{int}(\exists^{(p,q)} y_\ell \varphi)[(S_k, T_k)_{1 \leq k < \ell}]$ of the external FO+MOD formula $\exists^{(p,q)} y_\ell \; \varphi$ is the disjunction over all tuples $(p_\rho)_{\rho \in \mathcal{R}_r}$ with $\sum_{\rho \in \mathcal{R}_r} p_\rho \equiv p \bmod q$ of the following formula:

$$\bigwedge_{\rho = (S_\ell, T_\ell) \in \mathcal{R}_r} (\exists^{(p_\rho, q)} \overline{Y} (\langle S \rangle (V, \overline{Y}) \wedge \mathrm{LEAST}_{(S_\ell, T_\ell)}(\overline{Y}) \wedge \mathrm{int}(\varphi)[(S_k, T_k)_{1 \leq k \leq \ell}]))$$

The remaining reasoning is similar to that in Claim 4. □

After having shown Claim 5, Theorem 5.11 follows by induction similarly to Theorem 5.8.

Similarly to the main result Theorem 3.5, we can now show that the FO+MOD-theory of r-spheres is decidable:

Theorem 5.12. *There is an algorithm that, on input of an independence alphabet (Σ, I), a natural number $r \geq 0$, and a sentence φ of* FO+MOD *in the language of the structure $\mathcal{M}^{\mathrm{red}}$, decides whether there exists $t \in \mathbb{M}$ with $\mathcal{M}^{\mathrm{red}} \models \varphi^{(r,x)}(t)$.*

Proof. By Theorem 5.11 and Lemma 5.10, the set $L = \{t \in \mathbb{M} \mid \mathcal{M}_{\mathcal{R}}^{\mathrm{red}} \models \varphi^{S(r,x)}(t)\}$ can be characterized by a sentence of monadic second-order logic. Hence it is recognizable by Theorem 5.7. Since the emptiness of recognizable languages is decidable, the result follows. □

6 Complexity issues

Recall that the first step in our decidability proof was an application of Gaifman's Theorem. To the knowledge of the authors all known translations of a first-order sentence into a Boolean combination of local sentences are not elementary in the worst case, thus our decision procedure is far from efficient. In this section, we show that one cannot avoid this nonelementary complexity.

To this aim, we construct an independence alphabet (Σ, I) and a trace rewriting system $\mathcal{R}$ over $\mathbb{M}$ such that the first-order theory of $(\mathbb{M}, \to_{\mathcal{R}})$ is not elementary. Thus, our use of Gaifman's translation does not lead to an unreasonable inefficiency. We actually show a slightly stronger result, namely that the set of valid local sentences for a fixed radius is not elementary. In other words, the complexity of the decision question is already present when restricting to local sentences.

We prove the lower bound by reducing the first-order theory of finite labeled linear orders. In order to formulate this, we take the MSO-logic over dependence graphs from Section 5.3 but forbid the use of second-order variables. The resulting formulae are called first-order formulae over dependence graphs. For the further consideration we will use this logic only for dependence graphs D_t where t is in fact a word $t \in \Sigma^*$. In this case the relation symbol $\preceq$ is interpreted by the usual order on the set $\{1, \ldots, n\}$, and we speak of first-order formulae over words. For $w \in \Sigma^*$ and a first-order sentence over words φ we write $w \models \varphi$ if $D_w \models \varphi$. Throughout this section, let $\Gamma = \{\alpha, \beta\}$ be an alphabet with two elements. The *first-order theory of Γ^** is the set of all first-order sentences over words φ such that $w \models \varphi$ for all $w \in \Gamma^*$. It is known that the first-order theory of Γ^* is decidable but not elementary. This lower bound was announced in [35] where it is attributed to Stockmeyer. Stockmeyer's proof can only be found in his thesis and the same applies to the sharpening by Führer while Robertson's independent proof appeared as an extended abstract, only. The only proof that has been published seems to be [5, Example 8.1].

Let $\Sigma_1 = \Gamma \times \{0,1,2,3\}$. On this set, we define a dependence relation D_1 as follows: $((a,i),(b,j)) \in D_1$ if and only if $i = j$ or $i,j \leq 2$ or ($a = b$ and $\{i,j\} \subseteq \{1,2,3\}$). The complementary relation is denoted by I_1.

Next, we will consider the (preliminary) trace rewriting system $\mathcal{R}_1$ consisting of the following rules:

(1) $(a,0) \to (a,3)(a,1)(a,3)$ for $a \in \Gamma$, and
(2) $(a,1) \to (a,2)$ for $a \in \Gamma$.

We first reduce the first-order theory of Γ^* to the first-order theory of the structure $(\mathbb{M}(\Sigma_1, I_1), \to_{\mathcal{R}_1}, (F_t)_{t \in F})$ where $F \subseteq \mathbb{M}(\Sigma_1, I_1)$ is finite and $F_t = \mathbb{M}\ t\ \mathbb{M}$ is the set of all traces that contain the factor t:

Lemma 6.1. *The first-order theory of Γ^* can be reduced in polynomial time to the first-order theory of $(\mathbb{M}(\Sigma_1, I_1), \to_{\mathcal{R}_1}, (F_t)_{t \in F})$ for some finite set $F \subseteq \mathbb{M}(\Sigma_1, I_1)$.*

Proof. Let F be the set of traces over (Σ_1, I_1) of length at most 3. Furthermore, let φ be a first-order sentence over words. Then it belongs to the theory of Γ^* if and only if every word w satisfies φ. Since $\Gamma \times \{1,2,3\} \subseteq F$, the formula $\psi(x) = \bigwedge_{t \in \Gamma \times \{1,2,3\}} \neg F_t(x)$ is satisfied in $(\mathbb{M}(\Sigma_1, I_1), \to_{\mathcal{R}_1}, (F_t)_{t \in F})$ by a trace x if and only if x is actually a word over $\Gamma \times \{0\}$.[9] For such a trace $x = a_1 a_2 a_3 \dots a_k$, we will now express that it satisfies φ.

First note that the immediate successors of x with respect to the relation $\to_{\mathcal{R}_1}$ are of the form $a_1 a_2 \dots a_{i-1}(a_i,3)(a_i,1)(a_i,3)a_{i+1} \dots a_k$. Hence there is a bijection f from the set of positions $\{1,2,\dots,k\}$ in x to the immediate successors of x that maps i to the above trace. The position i carries the letter $(a,0)$ if and only if the immediate successor $f(i)$ of x contains a factor $(a,1)$. Since $\Gamma \times \{1\} \subseteq F$, this is first-order expressible in $(\mathbb{M}(\Sigma_1, I_1), \to_{\mathcal{R}_1}, (F_t)_{t \in F})$. It remains to be expressed that $i < j$.

For this, let us assume that $a_i = (a,0)$ and $a_j = (b,0)$. Suppose there are traces p, q, r such that

(1) $f(i) \to_{\mathcal{R}_1} p \to_{\mathcal{R}_1} r \leftarrow_{\mathcal{R}_1} q \leftarrow_{\mathcal{R}_1} f(j)$,
(2) p contains a factor $(a,2)$,
(3) r contains factors $(a,2)(a,3)(b,3)$ and $(a,3)(b,3)(b,1)$, and
(4) q contains factors $(a,1)$ and $(b,1)$, but no factors $(b,2)$ or $(c,1)$ for $c \in \Gamma \setminus \{a,b\}$.

Since $f(i)$ is an immediate successor of x and $a_i = (a,0)$, there are words $u, v \in (\Gamma \times \{0\})^*$ such that $f(i) = u(a,3)(a,1)(a,3)v$ and $|u| = i - 1$. Similarly, there are $y, z \in (\Gamma \times \{0\})^*$ such that $f(j) = y(b,3)(b,1)(b,3)z$ and $|y| = j - 1$.

The trace p is an immediate successor of $f(i)$ containing a factor $(a,2)$. Hence $p = u(a,3)(a,2)(a,3)v$. The trace q is an immediate successor of $f(j)$ not containing $(b,2)$. Hence it is of one of the following forms:

- $q = y_1(c,3)(c,1)(c,3)y_2(b,3)(b,1)(b,3)z$ or
  $q = y_1(b,3)(c,3)(c,1)(c,3)y_2(b,1)(b,3)z$ for some words $y_1, y_2 \in \Sigma_1^*$ with $y = y_1(c,0)y_2$, or

[9] For notational simplicity, we will identify the letter $a \in \Gamma$ and the pair $(a,0) \in \Sigma_1$ in the following.

– $q = y(b,3)(b,1)(b,3)z_1(c,3)(c,1)(c,3)z_2$ or
$q = y(b,3)(b,1)z_1(c,3)(c,1)(c,3)(b,3)z_2$ for some words z_1, z_2 with $z = z_1(y,0)z_2$.

Since q contains factors $(a,1)$ and $(b,1)$ and no factor $(c,1)$ for $c \notin \{a,b\}$, we have in any of these cases $c = a$. The trace r is an immediate successor of q containing a factor $(a,2) = (c,2)$. Hence r is of one of the following four forms:

– $r = y_1(a,3)(a,2)(a,3)y_2(b,3)(b,1)(b,3)z$ or
$r = y_1(b,3)(a,3)(a,2)(a,3)y_2(b,1)(b,3)z$ for some words y_1, y_2 over Σ_1 with $y = y_1(a,0)y_2$, or
– $r = y(b,3)(b,1)(b,3)z_1(a,3)(a,2)(a,3)z_2$ or
$r = y(b,3)(b,1)z_1(a,3)(a,2)(a,3)(b,3)z_2$ for some words z_1, z_2 over Σ_1 with $z = z_1(y,0)z_2$.

Note that only the first of these traces contains a factor $(a,3)(b,3)(b,1)$, i.e., $r = y_1(a,3)(a,2)(a,3)y_2(b,3)(b,1)(b,3)z$.

Since r is an immediate successor of $p = u(a,3)(a,2)(a,3)v$, we obtain $y_1 = u$ and $v = y_2(b,0)z$. Hence $i = |u| + 1 \leq |y_1(a,0)y_2| = |y| = j - 1$, i.e., $i < j$.

So far, the existence of traces p, q, r satisfying (1-4) implies $i < j$. Conversely, suppose $i < j$. Then $x = y_1(a,0)y_2(b,0)z$ with $|y_1| = i - 1$ and $|y_1(a,0)y_2| = j - 1$. Let p, q, and r be the following traces:

$$\begin{aligned} p &= y_1(a,3)(a,2)(a,3)y_2(b,0)z \\ q &= y_1(a,3)(a,1)(a,3)y_2(b,3)(b,1)(b,3)z \\ r &= y_1(a,3)(a,2)(a,3)y_2(b,3)(b,1)(b,3)z \end{aligned}$$

Then one can easily check that p, q and r satisfy the conditions (1-4). Hence the formula $\mathrm{less}_{a,b}(f(i), f(j))$ defined by

$$\exists p,q,r \left\{ \begin{array}{l} f(i) \to p \to r \leftarrow q \leftarrow f(j) \wedge \\ p \in F_{(a,2)} \wedge \\ q \in F_{(a,1)} \wedge q \in F_{(b,1)} \wedge q \notin F_{(b,2)} \wedge \bigwedge\limits_{c \in \Gamma \setminus \{a,b\}} q \notin F_{(c,1)} \wedge \\ \\ r \in F_{(b,1)(b,3)(a,3)} \wedge r \in F_{(b,3)(a,3)(a,2)} \end{array} \right\}$$

is satisfied by the traces $f(i)$ and $f(j)$ if and only if $i < j$.

Recall that φ is a first-order formula over words. Let $\psi(x)$, where x is a variable which does not occur in φ, be derived from φ by inductively

(1) replacing any subformula of the form $\exists y \gamma$ by $\exists y(x \to y \wedge \gamma)$,
(2) replacing any subformula of the form $y \leq z$ by $\mathrm{less}(y,z)$ with

$$\mathrm{less}(y,z) \quad \equiv \quad \bigvee_{a,b \in \Gamma} (y \in F_{(a,1)} \wedge z \in F_{(b,1)} \wedge \mathrm{less}_{a,b}(y,z)),$$

and

(3) replacing any subformula of the form $Q_a(y)$ by $y \in F_{(a,1)}$.

By our considerations above, a word $x \in (\Gamma \times \{0\})^*$ satisfies $\psi(x)$ in the model $(\mathbb{M}(\Sigma_1, I_1), \to_{\mathcal{R}_1}, (F_t)_{t\in F})$ if and only if $x \models \varphi$. Hence φ belongs to the first-order theory of Γ^* if and only if $(\mathbb{M}(\Sigma_1, I_1), \to_{\mathcal{R}_1}, (F_t)_{t\in F}) \models \forall x(\neg\exists y(y \to x) \Rightarrow \psi(x))$. □

Lemma 6.2. *Let $\mathcal{R}$ be a trace rewriting system such that $(\mathbb{M}(\Sigma, I), \to_{\mathcal{R}})$ is acyclic. Let $F \subseteq \mathbb{M}(\Sigma, I)$ be finite. Then there is an independence alphabet (Σ_2, I_2) and a trace rewriting system $\mathcal{R}_2$ such that the first-order theory of $(\mathbb{M}(\Sigma, I), \to_{\mathcal{R}}, (F_t)_{t\in F})$ can be reduced in polynomial time to the first-order theory of $(\mathbb{M}(\Sigma_2, I_2), \to_{\mathcal{R}_2})$.*

Proof. Let $F = \{t_1, \ldots, t_n\}$. Then define

$$\Sigma_2 = \Sigma \,\dot\cup\, \{(k, i) \in \mathbb{N}^2 \mid 1 \le i \le n, 0 \le k < i\}$$

and let $D_2 \subseteq \Sigma_2 \times \Sigma_2$ be the reflexive and symmetric closure of $D \cup \Sigma_2 \times \{(k, i) \in \mathbb{N}^2 \mid 1 \le i \le n, 0 \le k < i\}$ (where D is the complement of I). We consider the following trace rewriting system $\mathcal{R}_2$:

$$\begin{aligned}&\mathcal{R} \cup \{t_i \to (0, i) \mid 1 \le i \le n\}\\ &\quad \cup \{(k, i) \to ((k + 1) \bmod i, i) \mid 1 \le i \le n, 0 \le k < i\}.\end{aligned}$$

Intuitively, the graph $(\mathbb{M}(\Sigma_2, I_2), \to_{\mathcal{R}_2})$ is obtained from the graph $(\mathbb{M}(\Sigma, I), \to_{\mathcal{R}})$ by the addition of cycles of length i to any element of F_{t_i}. More precisely, we observe the following:

(1) A trace t in $\mathbb{M}(\Sigma_2, I_2)$ contains a letter from $\Sigma_2 \setminus \Sigma$ if and only if it lies on some cycle of length at most n. Hence the set of traces $\mathbb{M}(\Sigma, I) \subseteq \mathbb{M}(\Sigma_2, I_2)$ can be defined in the graph $(\mathbb{M}(\Sigma_2, I_2), \to_{\mathcal{R}_2})$.
(2) A trace t in $\mathbb{M}(\Sigma, I)$ belongs to F_{t_i} if and only if, in the graph $(\mathbb{M}(\Sigma_2, I_2), \to_{\mathcal{R}_2})$, it has an immediate successor that lies on a cycle of length i. Hence, also the set $F_{t_i} \subseteq \mathbb{M}(\Sigma, I)$ can be defined in the graph $(\mathbb{M}(\Sigma_2, I_2), \to_{\mathcal{R}_2})$.

□

Theorem 6.3. *There exists an independence alphabet (Σ_2, I_2) and a trace rewriting system $\mathcal{R}_2$ over $\mathbb{M}(\Sigma_2, I_2)$ such that the first-order theory of $(\mathbb{M}(\Sigma_2, I_2), \to_{\mathcal{R}_2})$ is not elementary decidable.*

Proof. By the two preceding lemmas, we can reduce the first-order theory of Γ^* to the first-order theory of $(\mathbb{M}(\Sigma_2, I_2), \to_{\mathcal{R}_2})$ in polynomial time. The result follows from [5, Example 8.1]. □

Note that the proof of the first lemma required a nontrivial independence alphabet. This allowed to relate positions i and j that are "far away" by considering a "bridge" or "shortcut" from i to j via $(a, 3)(b, 3)$. Next we want to prove a lower bound for

the first-order theory of semi-Thue systems where this trick is not available. Note that the procedure from [9, 23] is not elementary since it relies on the complementation of finite automata for any quantifier alternation. Nevertheless, we are only able to show an elementary lower bound, namely doubly exponential nondeterministic time. This is achieved by reducing the *successor theory of* Γ^*. This theory is defined in the same way as the first-order theory of Γ^*, but the order $<$ on $\{1, \dots, n\}$ is replaced by the successor-relation $\mathrm{succ} = \{(i, i+1) \mid 0 \leq i < n\}$. By [5, Example 8.6], its complexity is at least doubly exponential nondeterministic time.

Let $\Sigma_3 = \Gamma \times \{0, 1, 2\}$ and define the semi-Thue system $\mathcal{R}_3$ by the following rules:

(1) $(a, 0) \to (a, 1)$ for $a \in \Gamma$ and
(2) $(a, 1) \to (a, 2)$ for $a \in \Gamma$.

Lemma 6.4. *The successor-theory of Γ^* can be reduced in polynomial time to the first-order theory of $(\Sigma_3^*, \to_{\mathcal{R}_3}, (F_t)_{t \in F})$ for some finite set $F \subseteq \Sigma_3^*$.*

Proof. Similarly to the rewriting system $\mathcal{R}_1$, a word w contains only letters of the form $(a, 0)$ for $a \in \Gamma$ if and only if it is no target, i.e., the set $(\Gamma \times \{0\})^*$ is first-order definable in the graph $(\Sigma_3^*, \to_{\mathcal{R}_3})$.

So suppose $w \in (\Gamma \times \{0\})^*$. Then its immediate successors are of the form $u(a, 1)v$ with $u(a, 0)v = w$ and we have a bijection f from the positions in w onto its immediate successors. The position i is labeled by $(a, 0)$ if and only if the successor $f(i)$ belongs to $F_{(a,1)}$. In order to check whether $j = i + 1$, we proceed similarly to above: Suppose i is labeled by $(a, 0)$, j by $(b, 0)$ and $i \neq j$. Let p, q, r be words over Σ_3 such that

(1) $f(i) \to p \to r \leftarrow q \leftarrow f(j)$,
(2) p contains a factor $(a, 2)$,
(3) r contains a factor $(a, 2)(b, 1)$, and
(4) q contains a factor $(a, 1)(b, 1)$.

Similarly to the proof of Lemma 6.1 one can show that the existence of these three words p, q, and r is equivalent to $j = i + 1$ (in case $i \neq j$). Since all these properties are first-order expressible, the result follows. □

Theorem 6.5. *There exists an alphabet Σ_4 and a semi-Thue system $\mathcal{R}_4$ over Σ_4^* such that any decision procedure for the first-order theory of $(\Sigma_4^*, \to_{\mathcal{R}_4})$ requires at least doubly exponential nondeterministic time.*

Proof. The graph of the rewriting system $\mathcal{R}_3$ is acyclic. Hence we can apply the second lemma. In its proof, we only added letters that are dependent from any other letter. Hence we can use the same proof and construct $\mathcal{R}_4$ as required. □

7 Applications to the confluence problem

In this section we will present some applications of Corollary 3.7 to the confluence problem for trace rewriting systems. Let us fix a trace rewriting system $\mathcal{R}$ over a trace

monoid $\mathbb{M} = \mathbb{M}(\Sigma, I)$. We say that $\mathcal{R}$ is *confluent* if for all $u, v_1, v_2 \in \mathbb{M}$ with $u \xrightarrow{*}_{\mathcal{R}} v_1$ and $u \xrightarrow{*}_{\mathcal{R}} v_2$ there exists a $w \in \mathbb{M}$ with $v_1 \xrightarrow{*}_{\mathcal{R}} w$ and $v_2 \xrightarrow{*}_{\mathcal{R}} w$. If for $u \in \mathbb{M}$ there does not exist an infinite derivation $u = u_1 \to_{\mathcal{R}} u_2 \to_{\mathcal{R}} u_3 \to_{\mathcal{R}} \cdots$ then $\mathcal{R}$ is called *terminating on* u. If $\mathcal{R}$ is terminating on all $u \in \mathbb{M}$ then $\mathcal{R}$ is *terminating*. If $\mathcal{R}$ is terminating then by Newman's lemma [38] $\mathcal{R}$ is confluent if and only if $\mathcal{R}$ is locally confluent, see the definition before Lemma 3.8. We say that $\mathcal{R}$ is *length–reducing* if $|\ell| > |r|$ for every $(\ell, r) \in \mathcal{R}$. Obviously, if $\mathcal{R}$ is length reducing then $\mathcal{R}$ is also terminating. As already noted in the proof of Lemma 3.8 every semi-Thue system has only finitely many so called critical pairs. In order to check local confluence of a semi-Thue system, only these critical pairs have to be considered. By Newman's Lemma it follows easily that for terminating semi-Thue systems confluence can be decided, see e.g. [1]. Below we will generalize the notion of a critical pair to trace rewriting systems. But in contrast to semi-Thue systems, a trace rewriting system has in general infinitely many critical pairs. This is an unavoidable phenomenon, as the next theorem from [30] shows. It generalizes a previous result of [37]:

Theorem 7.1. *Let (Σ, I) be an independence alphabet. The following problem is decidable if and only if $I = \emptyset$ or $I = (\Sigma \times \Sigma) \setminus Id_\Sigma$:*
INPUT: A length-reducing trace rewriting system $\mathcal{R}$ over $\mathbb{M}(\Sigma, I)$
QUESTION: Is $\mathcal{R}$ confluent?

Hence in almost all cases confluence is undecidable for length-reducing trace rewriting systems. On the other hand, in [12, 30] several subclasses of trace rewriting systems were defined for which confluence becomes decidable. In this section we define a further class of trace rewriting systems with a decidable confluence problem. We remark that this class is not contained in the classes defined in [12, 30]. The main idea is to give sufficient conditions which imply the equivalence of confluence and α-confluence (see Section 3.3) for an $\alpha > 0$ that can be calculated effectively. This makes it possible to apply the following immediate corollary of Corollary 3.7.

Corollary 7.2. *The following problem is decidable:*
INPUT: A trace rewriting system $\mathcal{R}$ and $\alpha \in \mathbb{N}$.
QUESTION: Is $\mathcal{R}$ α-confluent?

Before we present our announced decidability result, we first have to introduce some notation. For a trace $u \in \mathbb{M}$ we define $\mathrm{D}(u) = \{a \in \Sigma \mid (a, s) \notin I\}$. In [30] we have introduced the following condition for trace rewriting systems: $\mathcal{R}$ satisfies condition (A) if the following holds:

(A1) For all $(\ell, r) \in \mathcal{R}$ and all $a \in \Sigma$ with $(a, \ell) \in I$ it holds $ar = ra$.
(A2) For all $p_0, p_1, q_0, q_1, \in \mathbb{M} \setminus \{1\}$, $r_0, r_1 \in \mathbb{M}$ with $(p_0 q_0, r_0), (p_1 q_1, r_1) \in \mathcal{R}$ and $(p_0, p_1), (q_0, q_1) \in I$ there exist $s_0, s_1, t_0, t_1 \in \mathbb{M}$ such that $r_i = s_i t_i$, $\mathrm{D}(s_i) \subseteq \mathrm{D}(p_i)$, and $\mathrm{D}(t_i) \subseteq \mathrm{D}(q_i)$ for $i = 0, 1$.

The set $\mathrm{CS}(\mathcal{R})$ of all *critical situations* of $\mathcal{R}$ is the set of all triples (t_0, t, t_1) such that there exist traces $p_0, p_1, q_0, q_1, w_0, w_1, r_0, r_1 \in \mathbb{M}$, $s \in \mathbb{M} \setminus \{1\}$ with

(CS1) $(p_0 s q_0, r_0), (p_1 s q_1, r_1) \in \mathcal{R}$,
(CS2) $(p_0, p_1), (q_0, q_1), (w_0, w_1), (s, w_0 w_1), (w_0, q_0 p_1), (w_1, p_0 q_1) \in I$
(CS3) $t = p_1 w_1 p_0 s q_0 w_0 q_1 = p_0 w_0 p_1 s q_1 w_1 q_0$,[10] $t_0 = p_1 w_1 r_0 w_0 q_1,\ t_1 = p_0 w_0 r_1 w_1 q_0$

Note that for all $(t_0, t, t_1) \in \mathrm{CS}(\mathcal{R})$ it holds $t \to_{\mathcal{R}} t_0$ and $t \to_{\mathcal{R}} t_1$. Furthermore note that a trace rewriting system has in general infinitely many critical situations.

Lemma 7.3. *Let $\mathcal{R}$ satisfy condition (A). Then $\mathcal{R}$ is α-confluent if and only if for every $(u_0, u, u_1) \in CS(\mathcal{R})$ there exists a $v \in \mathbb{M}$ with $u_0 \to_{\mathcal{R}}^{\leq \alpha} v$ and $u_1 \to_{\mathcal{R}}^{\leq \alpha} v$.*

Proof. This can be shown precisely as [30, Lemma 2.3]. There, an additional condition is present in the definition of critical situations. This condition is not needed for our purpose, and it does not influence the proof. □

For a subalphabet $\Gamma \subseteq \Sigma$ we define the rewriting system $\pi_\Gamma(\mathcal{R})$ by

$$\pi_\Gamma(\mathcal{R}) = \{(\pi_\Gamma(\ell), \pi_\Gamma(r)) \mid (\ell, r) \in \mathcal{R}\}.$$

Theorem 7.4. *The following problem is decidable.*
INPUT: A trace rewriting system $\mathcal{R}$ over the trace monoid $\mathbb{M}(\Sigma, I)$ with the following properties:

(1) $\mathcal{R}$ satisfies condition (A) and is terminating.
(2) For all $p_0, p_1, q_0, q_1, r_0, r_1 \in \mathbb{M}$, $s \in \mathbb{M}\backslash\{1\}$ with $(p_0 s q_0, r_0), (p_1 s q_1, r_1) \in \mathcal{R}$ and $(p_0, p_1), (q_0, q_1) \in I$, the rewriting system $\pi_\Gamma(\mathcal{R})$ is terminating on the traces $\pi_\Gamma(p_1 r_0 q_1)$ and $\pi_\Gamma(p_0 r_1 q_0)$ where $\Gamma = \mathrm{D}(p_0 s q_1) \cap \mathrm{D}(p_1 s q_0)$.

QUESTION: Is $\mathcal{R}$ confluent?

Proof. Let $\mathcal{R}$ be a trace rewriting system which satisfies the conditions from the theorem. We prove the theorem by effectively calculating an $\alpha > 0$ such that $\mathcal{R}$ is confluent if and only if $\mathcal{R}$ is α-confluent. This allows us to apply Corollary 7.2.

For all $p_0, p_1, q_0, q_1, r_0, r_1 \in \mathbb{M}$, $s \in \mathbb{M}\backslash\{1\}$ with $(p_0 s q_0, r_0), (p_1 s q_1, r_1) \in \mathcal{R}$ and $(p_0, p_1), (q_0, q_1) \in I$ we define a natural number $\alpha(p_0, p_1, q_0, q_1, r_0, r_1, s)$ by

$$\alpha(p_0, p_1, q_0, q_1, r_0, r_1, s) = \max\left\{ k \;\middle|\; \exists u \begin{pmatrix} \pi_\Gamma(p_1 r_0 q_1) \to_{\pi_\Gamma(\mathcal{R})}^{k} \pi_\Gamma(u) & \text{or} \\ \pi_\Gamma(p_0 r_1 q_0) \to_{\pi_\Gamma(\mathcal{R})}^{k} \pi_\Gamma(u) & \end{pmatrix} \right\}$$

where $\Gamma = \mathrm{D}(p_0 s q_1) \cap \mathrm{D}(p_1 s q_0) \subseteq \Sigma$. The assumptions on $\mathcal{R}$ imply that the number $\alpha(p_0, p_1, q_0, q_1, r_0, r_1, s)$ can be calculated effectively. Now let

$$\alpha = \max\left\{ \alpha(p_0, p_1, q_0, q_1, r_0, r_1, s) \;\middle|\; \begin{matrix} (p_0 s q_0, r_0), (p_1 s q_1, r_1) \in \mathcal{R}, \\ (p_0, p_1), (q_0, q_1) \in I,\ s \neq 1 \end{matrix} \right\}.$$

Also α can be calculated effectively. We claim that $\mathcal{R}$ is confluent if and only if $\mathcal{R}$ is α-confluent. If $\mathcal{R}$ is α-confluent then $\mathcal{R}$ is also locally confluent. Since $\mathcal{R}$ is terminating, $\mathcal{R}$ must be confluent. Now assume that $\mathcal{R}$ is confluent. We claim that $\mathcal{R}$

[10] Note that the equality of these two factorizations of t follows from the independencies listed in point (1).

is α-confluent. By Lemma 7.3 it suffices to consider all critical situations of $\mathcal{R}$. Let $(t_0, t, t_1) \in \mathrm{CS}(\mathcal{R})$. By the definition of critical situations there exist traces p_0, p_1, q_0, q_1, w_0, w_1, r_0, r_1 from $\mathbb{M}$, and $s \in \mathbb{M}\backslash\{1\}$ such that (CS1), (CS2), and (CS3) hold. Since $t \to_{\mathcal{R}} t_0$, $t \to_{\mathcal{R}} t_1$, and $\mathcal{R}$ is confluent, there exist $\alpha_0, \alpha_1 \geq 0$ and a trace $u \in \mathbb{M}(\Sigma, I)$ such that

$$t_0 = p_1 w_1 r_0 w_0 q_1 \to_{\mathcal{R}}^{\alpha_0} u \quad \text{and} \quad t_1 = p_0 w_0 r_1 w_1 q_0 \to_{\mathcal{R}}^{\alpha_1} u. \tag{12}$$

Let $\Gamma = \mathrm{D}(p_0 s q_1) \cap \mathrm{D}(p_1 s q_0)$. ¿From (CS2) we obtain $(w_0, p_1 s q_0), (w_1, p_0 s q_1) \in I$. Thus $\pi_\Gamma(w_0) = 1 = \pi_\Gamma(w_1)$. Hence (12) implies

$$\pi_\Gamma(p_1 r_0 q_1) \to_{\pi_\Gamma(\mathcal{R})}^{\alpha_0} \pi_\Gamma(u) \quad \text{and} \quad \pi_\Gamma(p_0 r_1 q_0) \to_{\pi_\Gamma(\mathcal{R})}^{\alpha_1} \pi_\Gamma(u).$$

The definition of α and (CS1) and (CS2) implies that $\alpha_0, \alpha_1 \leq \alpha$. Finally Lemma 7.3 implies that $\mathcal{R}$ is α-confluent. □

The decidability criterion from Theorem 7.4 is quite technical. In the following we will present two less technical decidability criterions which can be deduced from Theorem 7.4. The following corollary can also be found in [28].

Corollary 7.5. *The following problem is decidable:*
INPUT: A trace rewriting system $\mathcal{R}$ over $\mathbb{M}(\Sigma, I)$ such that there exists a clique covering $(\Gamma_1, \ldots, \Gamma_n)$ of the dependence alphabet $(\Sigma, (\Sigma \times \Sigma)\backslash I)$ with the following properties:

(1) For all rules $(\ell, r) \in \mathcal{R}$ and all $i, j \in \{1, \ldots, n\}$ with $i \neq j$ and $\pi_i(\pi_j(\ell)) = \epsilon$, we have $\pi_i(\pi_j(r)) = \epsilon$.
(2) For all $i \in \{1, \ldots, n\}$, the semi-Thue system $\pi_i(\mathcal{R})$ is terminating.

QUESTION: Is $\mathcal{R}$ confluent?

Proof. Let $\mathcal{R}$ be a trace rewriting system over $\mathbb{M}(\Sigma, I)$, and let $(\Gamma_1, \ldots, \Gamma_n)$ be a clique covering of the dependence alphabet $(\Sigma, (\Sigma \times \Sigma)\backslash I)$ which satisfies the conditions from the corollary. We will show that $\mathcal{R}$ satisfies the conditions on the input from the problem presented in Theorem 7.4.

Condition (2) from the corollary obviously implies that $\mathcal{R}$ is terminating. Next we show that $\mathcal{R}$ satisfies condition (A). Let ℓ be a left-hand side of a rule in $\mathcal{R}$. Condition (2) from the corollary implies $\pi_i(\ell) \neq \epsilon$ for all $i \in \{1, \ldots, n\}$, because otherwise $\pi_i(\mathcal{R})$ would contain a rule of the form $\epsilon \to \pi_i(r)$ and therefore would not be terminating. Thus there cannot exist an $a \in \Sigma$ with $(a, \ell) \in I$. Thus $\mathcal{R}$ satisfies condition (A1). Condition (A2) can be verified as follows. Let $(p_0 q_0, r_0), (p_1 q_1, r_1) \in \mathcal{R}$ such that $\{1\} \notin \{p_0, p_1, q_0, q_1\}$ and $(p_0, p_1), (q_0, q_1) \in I$. We have to find factorizations $r_0 = s_0 t_0$ and $r_1 = s_1 t_1$ such that $\mathrm{D}(s_j) \subseteq \mathrm{D}(p_j)$ and $\mathrm{D}(t_j) \subseteq \mathrm{D}(q_j)$ for $j = 0, 1$. First note that for arbitrary traces u, v, if $(u, v) \in I$ then $\pi_i(u) = \epsilon$ or $\pi_i(v) = \epsilon$ for every clique Γ_i. Since furthermore $\pi_i(p_0 q_0) = \pi_i(\ell_0) \neq \epsilon \neq \pi_i(\ell_1) = \pi_i(p_1 q_1)$ holds, for all $i \in \{1, \ldots, n\}$ exactly one of the following two cases holds:

– case 1. $\pi_i(p_0) \neq \epsilon \neq \pi_i(q_1)$ and $\pi_i(p_1) = \epsilon = \pi_i(q_0)$
– case 2. $\pi_i(p_1) \neq \epsilon \neq \pi_i(q_0)$ and $\pi_i(p_0) = \epsilon = \pi_i(q_1)$

Now we define for every $i \in \{1, \ldots, n\}$ the factorization $\pi_i(r_0) = u_i v_i$ of the word $\pi_i(r_0)$ as follows:

– $u_i = \pi_i(r_0)$ and $v_i = \epsilon$ if case 1 holds for i.
– $u_i = \epsilon$ and $v_i = \pi_i(r_0)$ if case 2 holds for i.

We claim that the tuples $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ are reconstructible. It suffices, by Lemma 3.4, to show that $\pi_i(u_j) = \pi_j(u_i)$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$. So let $i, j \in \{1, \ldots, n\}$. If i and j both satisfy case 1 or i and j both satisfy case 2, than this holds obviously. Thus w.l.o.g. let us assume that i satisfies case 1 and j satisfies case 2. This implies $u_i = \pi_i(r_0)$, $u_j = \epsilon$ and

$$\begin{aligned} \pi_i(p_0) \neq \epsilon \neq \pi_i(q_1), \qquad & \pi_i(p_1) = \epsilon = \pi_i(q_0), \\ \pi_j(p_1) \neq \epsilon \neq \pi_j(q_0), \qquad & \pi_j(p_0) = \epsilon = \pi_j(q_1). \end{aligned}$$

Therefore $\pi_i(\pi_j(\ell_0)) = \pi_i(\pi_j(p_0 q_0)) = \epsilon$ and $\pi_i(\pi_j(\ell_1)) = \pi_i(\pi_j(p_1 q_1)) = \epsilon$. Condition (1) for $\mathcal{R}$ from the corollary implies that $\pi_i(\pi_j(r_0)) = \epsilon = \pi_i(\pi_j(r_1))$. Thus $\pi_i(u_j) = \pi_i(\epsilon) = \epsilon = \pi_j(\pi_i(r_0)) = \pi_j(u_i)$. We have shown that the tuples $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ are reconstructible. Thus there exist unique traces s_0 and t_0 with $\pi_i(s_0) = u_i$ and $\pi_i(t_0) = v_i$ for all $i \in \{1, \ldots, n\}$. Lemma 3.3 implies $r_0 = s_0 t_0$.

Now assume that $a \in \mathrm{D}(s_0)$. Thus there exist a $j \in \{1, \ldots, n\}$ such that $a \in \Gamma_j$ and $u_j = \pi_j(s_0) \neq \epsilon$. Thus case 1 from the construction of $(u_1, \ldots, u_n)$ holds for j which implies that $\pi_j(p_0) \neq \epsilon$, i.e., $a \in D(p_0)$. Analogously it can be show that $\mathrm{D}(t_0) \subseteq \mathrm{D}(q_0)$. Thus we have constructed the desired factorization of r_0. The desired factorization $r_1 = s_1 t_1$ of r_1 can be constructed analogously. Thus $\mathcal{R}$ satisfies condition (A).

In order to show the second condition from Theorem 7.4, let us consider rules $(p_0 s q_0, r_0), (p_1 s q_1, r_1) \in \mathcal{R}$ with $(p_0, p_1), (q_0, q_1) \in I$ and $s \neq 1$. Let $\Gamma = \mathrm{D}(p_0 s q_1) \cap \mathrm{D}(p_1 s q_0)$. Because of $s \neq 1$ there exists a clique Γ_i with $\Gamma_i \subseteq \mathrm{D}(s) \subseteq \Gamma$. Since $\pi_i(\mathcal{R})$ is terminating, also $\pi_\Gamma(\mathcal{R})$ is terminating which concludes the proof of the corollary. □

Example 7.6. Let $\mathcal{R}$ be a trace rewriting system over a direct product $\prod_{i=1}^{n} \Gamma_i^*$ of free monoids. If each of the semi-Thue systems $\pi_{\Gamma_i}(\mathcal{R})$ is terminating, $i \in \{1, \ldots, n\}$, then we can apply Corollary 7.5 in order to check whether $\mathcal{R}$ is confluent. In particular this is true if $\mathcal{R}$ is length-reducing in each of its n components. On the other hand, confluence is undecidable for the class of all rewriting systems over $\{a, b\}^* \times \{c\}^*$ for which all rules are length-preserving in one component but length-reducing in the other component (this follows immediately from the construction given in the proof of [30, Lemma 3.4]). This gives a very sharp borderline between decidability and undecidability for the case of direct products of free monoids.

The statement of the following corollary is similar to that of Corollary 7.5. A trace $t \in \mathbb{M}$ is connected if there does not exist a factorization $t = t_1 t_2$ such that $t_1 \neq 1 \neq t_2$ and $(t_1, t_2) \in I$.

Corollary 7.7. *The following problem is decidable:*
INPUT: A trace rewriting system $\mathcal{R}$ over $\mathbb{M}(\Sigma, I)$ such that

(1) for all $(\ell, r) \in \mathcal{R}$ the trace ℓ is connected, and
(2) there exists a clique covering $(\Gamma_1, \ldots, \Gamma_n)$ of $(\Sigma, (\Sigma \times \Sigma) \backslash I)$ such that for all $i \in \{1, \ldots, n\}$ the semi-Thue system $\pi_i(\mathcal{R})$ is terminating.

QUESTION: Is $\mathcal{R}$ confluent?

Proof. The proof is quite similar to the proof of Corollary 7.5. Let $\mathcal{R}$ be a trace rewriting system which satisfies condition (1) and (2) from the corollary. Let $(\Gamma_1, \ldots, \Gamma_n)$ be a clique covering of the dependence alphabet $(\Sigma, (\Sigma \times \Sigma) \backslash I)$ such that for all $i \in \{1, \ldots, n\}$ the semi-Thue system $\pi_i(\mathcal{R})$ is terminating. Thus $\mathcal{R}$ must be terminating. Analogously to the proof of Corollary 7.5 it can be shown that $\mathcal{R}$ satisfies the second condition from Theorem 7.4. Furthermore $\pi_i(\ell) \neq \epsilon$ holds for all left-hand sides ℓ of $\mathcal{R}$ and all $i \in \{1, \ldots, n\}$. Hence $\mathcal{R}$ also satisfies condition (A1). It remains to show that $\mathcal{R}$ satisfies condition (A2). Assume that there exist rules $(p_0 q_0, r_0), (p_1 q_1, r_1) \in \mathcal{R}$ such that $1 \notin \{p_0, p_1, q_0, q_1\}$ and $(p_0, p_1), (q_0, q_1) \in I$. We will deduce a contradiction. Assume that $\pi_i(p_0) \neq \epsilon$ for some $i \in \{1, \ldots, n\}$. Then $(p_0, p_1) \in I$ implies $\pi_i(p_1) = \epsilon$. From $\pi_i(p_1 q_1) = \pi_i(\ell_1) \neq \epsilon$ it follows $\pi_i(q_1) \neq \epsilon$. Finally $(q_0, q_1) \in I$ implies $\pi_i(q_0) = \epsilon$. We have shown that $\pi_i(p_0) \neq \epsilon$ implies $\pi_i(q_0) = \epsilon$ for all $i \in \{1, \ldots, n\}$. It follows $(p_0, q_0) \in I$, because otherwise there would exist a clique Γ_j with $\pi_j(p_0) \neq \epsilon \neq \pi_j(q_0)$. But because of $p_0 \neq 1 \neq q_0$ this contradicts the assumption that $\ell_0 = p_0 q_0$ is connected. □

Example 7.8. Let $\Sigma = \{a, b, c, d\}$ and $I = \{(a, d), (d, a), (b, c), (c, b)\}$. Let $\mathcal{R}$ be the trace rewriting system $\{bdc \to a\}$ over the trace monoid $\mathbb{M}(\Sigma, I)$. A clique covering of the corresponding dependence alphabet is $(\{a, b\}, \{b, d\}, \{c, d\}, \{a, c\})$. The trace bdc is connected and the projection of the rule $bdc \to a$ onto any of the four cliques is terminating. Therefore we can apply Corollary 7.7 in order to check whether $\mathcal{R}$ is confluent (it is easy to see that $\mathcal{R}$ is confluent). Note that here we cannot apply Corollary 7.5 because $\pi_{\{a,b\}}(\pi_{\{a,c\}}(bdc)) = \epsilon$ and $\pi_{\{a,b\}}(\pi_{\{a,c\}}(a)) \neq \epsilon$. On the other hand, in general Corollary 7.7 cannot be applied to trace rewriting systems over direct products of free monoids which we have considered in Example 7.6.

8 Open questions

In Section 6, we proved a lower bound for the complexity of the first-order theory of the one-step rewriting by a semi-Thue system. There is a huge gap between this doubly exponential lower and the nonelementary upper bound that follows immediately from the proofs in [9, 23].

Although our decidability result is very similar to corresponding results in [9, 23], our technique is new. It could provide a means to identify term rewriting systems whose rewrite graph has a decidable first-order theory. So far, this property has been shown for ground term rewriting systems, only. On the other hand, Treinen's result [48] shows that not much more can be expected.

Semi-Thue systems can be seen as term rewriting modulo associativity (it is a very simple case since there are no further symbols). Similarly, trace rewriting is term rewriting modulo associativity and partial commutativity. Is it possible to use the technique developed in this paper to handle other "term rewriting modulo ..." theories?

In Section 5.5, we considered a counting extension of first-order logic. So far, our technique can deal with local properties of this logic, only. Is the FO+MOD theory of one-step rewriting in general decidable? In order to answer this question positively, one would have to develop another Gaifman theorem for this counting logic.

The heart of our proof asserts that the set of centres of spheres having a certain first-order property is a recognizable set of traces. In general, any recognizable set of traces can be described in this way with a scattered rewrite system: the local formula $x \to_\rho x$ with $\rho = (x, x)$ describes the recognizable language $L(x)$. Using trace rewriting systems, one can describe only aperiodic languages (this follows from an analysis of our proof). It would be interesting to have a characterization of all aperiodic languages described in this way - a question that is open for semi-Thue systems as well.

References

1. R. Book and F. Otto. *String–Rewriting Systems*. Springer, 1993.
2. William W. Boone. The word problem. *Annals of Mathematics (2)*, 70:207–265, 1959.
3. P. Cartier and D. Foata. *Problèmes combinatoires de commutation et réarrangements*. Lecture Notes in Mathematics vol. 85. Springer, Berlin - Heidelberg - New York, 1969.
4. D. Caucal. On the regular structure of prefix rewriting. *Theoretical Computer Science*, 106:61–86, 1992.
5. K. Compton and C. Henson. A uniform method for proving lower bounds on the computational complexity of logical theories. *Annals of Pure and Applied Logic*, 48:1–79, 1990.
6. R. Cori and Y. Métivier. Recognizable subsets of some partially abelian monoids. *Theoretical Computer Science*, 35:179–189, 1985.
7. R. Cori and D. Perrin. Automates et commutations partielles. *R.A.I.R.O. — Informatique Théorique et Applications*, 19:21–32, 1985.
8. B. Courcelle. The monadic second-order logic of graphs, II: Infinite graphs of bounded width. *Mathematical Systems Theory*, 21:187–221, 1989.
9. M. Dauchet and S. Tison. The theory of ground rewrite systems is decidable. In *Proceedings of the 5th Annual IEEE Symposium on Logic in Computer Science (LICS '90)*, pages 242–256. IEEE Computer Society Press, 1990.
10. M. Droste and D. Kuske. Trace languages definable with modular quantifiers. In *Developments in Language Theory*, pages 386-395, World Scientific Publ. Co., 1995.
11. V. Diekert. On the Knuth-Bendix completion for concurrent processes. In Th. Ottmann, editor, *Proceedings of the14th International Colloquium on Automata, Languages and Programming (ICALP 87), Karlsruhe (FRG) 1987*, number 267 in Lecture Notes in Computer Science, pages 42–53, Berlin-Heidelberg-New York, 1987. Springer.
12. V. Diekert. *Combinatorics on Traces*. Number 454 in Lecture Notes in Computer Science. Springer, 1990.
13. V. Diekert, Y. Matiyasevich, and A. Muscholl. Solving word equations modulo partial commutations. *Theoretical Computer Science*, 224(1–2):215–235, 1999.
14. V. Diekert and G. Rozenberg, editors. *The Book of Traces*. World Scientific, Singapore, 1995.

15. C. Duboc. On some equations in free partially commutative monoids. *Theoretical Computer Science*, 46:159–174, 1986.
16. V. G. Durnev. Undecidability of the positive $\forall\exists^3$-theory of a free semi-group. *Sibirsky Matematicheskie Jurnal*, 36(5):1067–1080, 1995. (English translation).
17. W. Ebinger and A. Muscholl. Logical definability on infinite traces. *Theoretical Computer Science*, 154:67-84, 1996.
18. S. Eilenberg. *Automata, Languages and Machines vol. A*. Academic Press, New York, 1974.
19. C. Frougny and J. Sakarovitch. Synchronized rational relations of finite and infinite words. *Theoretical Computer Science*, 108(1):45–82, 1993.
20. H. Gaifman. On local and nonlocal properties. In J. Stern, editor, *Logic Colloquium '81*, pages 105-135, 1982, North Holland.
21. Y. Gurevich. Modest theory of short chains I. *J. of Symb. Logic*, 44:481–490, 1979.
22. W. Hodges. *Model Theory*. Cambridge University Press, 1993.
23. F. Jacquemard. *Automates d'arbres et Réécriture de termes*. PhD thesis, Université de Paris-Sud, 1996.
24. M. Jantzen. Confluent string rewriting. *EATCS Monographs on Theoretical Computer Science*, volume 14. Springer, 1988.
25. L. Libkin. Logics capturing local properties. *ACM Transactions on Computational Logic*. To appear.
26. L. Libkin. On counting logics and local properties. In *LICS*, pages 501–512. IEEE, 1998.
27. L. Libkin. Logics with counting and local properties. *ACM Transactions on Computational Logics*, 1:33–59, 2000.
28. M. Lohrey. On the confluence of trace rewriting systems. In V. Arvind and R. Ramanujam, editors, *Proceedings of the 18th Conference on Foundations of Software Technology and Theoretical Computer Science, (FSTTCS'98), Chennai (India)*, number 1530 in Lecture Notes in Computer Science, pages 319–330. Springer, 1998.
29. M. Lohrey. *Das Konfluenzproblem für Spurersetzungssysteme*. PhD thesis, Universität Stuttgart, 1999.
30. M. Lohrey. Confluence problems for trace rewriting systems. *Information and Computation*. 170:1–25, 2001.
31. S. S. Marchenkov. Unsolvability of positive $\forall\exists$-theory of a free semi-group. *Sibirsky Matematicheskie Jurnal*, 23(1):196–198, 1982.
32. A. Markov. On the impossibility of certain algorithms in the theory of associative systems. *Doklady Akademii Nauk SSSR*, 55, 58:587–590, 353–356, 1947.
33. Y. Matiyasevich. Some decision problems for traces. In S. Adian and A. Nerode, editors, *Proceedings of the 4th International Symposium on Logical Foundations of Computer Science (LFCS'97), Yaroslavl, Russia, July 6–12, 1997*, number 1234 in Lecture Notes in Computer Science, pages 248–257, Berlin-Heidelberg-New York, 1997. Springer.
34. A. Mazurkiewicz. Concurrent program schemes and their interpretation. Technical report, DAIMI Report PB-78, Aarhus University, 1977.
35. A. Meyer. Weak monadic second order theory of one successor is not elementary recursive. In *Proc. Logic Colloquium*, Lecture Notes in Mathematics vol. 453, pages 132–154. Springer, 1975.
36. D.E. Muller and P.E. Schupp. The theory of ends, pushdown automata, and second-order logic. *Theoretical Computer Science*, 37(1):51–75, 1985.
37. P. Narendran and F. Otto. Preperfectness is undecidable for Thue systems containing only length-reducing rules and a single commutation rule. *Information Processing Letters*, 29:125–130, 1988.
38. M. H. A. Newman. On theories with a combinatorial definition of "equivalence". *Annals Mathematics*, 43:223–243, 1943.
39. P. S. Novikov. On the algorithmic unsolvability of the word problem in group theory. *American Mathematical Society, Translations Series 2*, 9:1–122, 1958.
40. J. Nurmonen. Counting modulo quantifiers on finite structures. *Information and Computation*, 160:62–87, 2000. LICS 1996, Part I (New Brunswick, NJ).
41. E. Post. Recursive unsolvability of a problem of Thue. *Journal of Symbolic Logic*, 12(1):1–11, 1947.
42. D. Seese. The structure of models of decidable monadic theories of graphs. *Annals of Pure and Applied Logic*, 53:169–195, 1991.
43. H. Straubing, D. Thérien, and W. Thomas. Regular languages defined with generalized quantifiers. *Information and Computation*, 118:289–301, 1995.
44. Terese. Term Rewriting Systems. To appear with Cambridge University Press, 2001.
45. W. Thomas. On logical definability of trace languages. In V. Diekert, editor, *Proceedings of a workshop of the ESPRIT Basic Research Action No 3166: Algebraic and Syntactic Methods in Computer Science (ASMICS), Kochel am See (Germany)*, Report TUM-I9002, Technical University of Munich, pages 172–182, 1990.

46. W. Thomas. Automata on infinite objects. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science*, pages 133-191, 1990, Elsevier Science Publ.
47. A. Thue. Probleme über die Veränderungen von Zeichenreihen nach gegebenen Regeln. *Skr. Vid. Kristiania, I Math. Natuv. Klasse*, No. 10, 34 S., 1914.
48. R. Treinen. The first-order theory of linear one-step tewriting is undecidable. *Theoretical Computer Science*, 208(1-2):149–177, 1998.