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Another step towards a theory of regular MSC languages

Dietrich Kuske

Department of Mathematics and Computer Science
University of Leicester
LEICESTER
LE1 7RH, England
d.kuske@mcs.le.ac.uk

Abstract. This paper resumes the study of regular sets of Message Sequence Charts initiated by Henriksen, Mukund, Narayan Kumar & Thiagarajan [HMKT99]. Differently from their results, we consider infinite MSCs. It is shown that for bounded sets of infinite MSCs, the notions of recognizability, axiomatizability in monadic second order logic, and acceptance by a deterministic Message Passing Automaton with Muller acceptance condition coincide. We furthermore characterize the expressive power of first order logic and of its extension by modulo-counting quantifiers over bounded infinite MSCs.

1 Introduction

Message sequence charts (MSCs) form a popular visual formalism used in the software development. In its simplest incarnation, an MSC depicts the desired exchange of messages and corresponds to a single partial-order execution of the system. Several methods to specify sets of MSCs have been considered, among them MSC-graphs or High-level MSCs (HMSCs) that generate sets of MSCs by concatenating "building blocks", (Büchi-)automata that accept the linear extensions of MSCs, and logics. In general, these formalisms have different expressive power.

In [AY99], Alur & Yannakakis show that the collection of MSCs generated by a "bounded" ("locally synchronized" in the terminology of [HMKT99]) MSC-graph can be represented as a string language recognizable by a finite deterministic automaton. Based on this observation, Henriksen *et al.* [HMKT99] study sets of MSCs whose linear extensions form a regular string language. I will call these sets of MSCs "recognizable". The notion of recognizability has proven to be robust and fruitful in different settings like strings, trees, Mazurkiewicz traces and other classes of partial orders (both finite and infinite). The robustness is reflected by the fact that there are alternative definitions of recognizable sets of objects using combinatorial, algebraic, or logical methods: in all these settings, recognizable sets can be presented by finite-state devices, by congruences of finite index, or by sentences of monadic second order logic. Even more, natural subclasses of monadic second order logic correspond to natural classes of congruences and finite-state devices (*e.g.*, first-order logic corresponds to groupfree syntactic monoids and to counterfree automata). The main results in [HMKT99] show similar equivalences for sets of bounded finite MSCs. In particular, they prove the equivalence of the following three concepts for bounded sets K of finite MSCs:

1. The set of linear extensions of K can be accepted by a finite deterministic automaton.
2. There is a sentence φ of the monadic second order logic such that K is the set of bounded MSCs that satisfy φ.
3. The set K can be accepted by a nondeterministic message passing automaton.

This result was sharpened in [MKS00] who showed that deterministic message passing automata suffice.

The main focus of this paper is the extension of these results to sets of infinite MSCs. These infinite MSCs occur naturally as executions of systems that are not meant to stop like distributed operating systems or telecommunication networks. In the first part, we will extend the equivalence between the first and the second statement. We will also consider two fragments of monadic second order logic, namely first-order logic FO and its extension by modulo-counting quantifiers FO+MOD(n) [STT95]. We describe the expressive power of these logics in the spirit of Büchi's theorem: for a set of bounded possibly infinite MSCs K, the following statements are equivalent (Theorems 3.5, 4.10, and 4.11)

1. The set of linear extensions of K is recognizable (n-solvable, aperiodic, resp.).
2. The set K is axiomatizable by a sentence of monadic second order logic (of the logic FO+MOD(n), first-order logic, resp.) relative to all possibly infinite MSCs.

The second part of the paper gives a characterization of recognizable sets of infinite MSCs in terms of message passing automata. To this aim, we extend the model from [HMKT99] by a Muller-acceptance condition. It is shown that for a set of bounded possibly infinite MSCs K, the following statements are equivalent (Theorem 5.14).

1. The set of linear extensions of K is recognizable.
3. The set K is accepted by a finite deterministic message passing automaton with Muller-acceptance condition.

The proof of the implication 1$\rightarrow$2 relies on a first-order interpretation of a bounded MSC in any of its linearisations. This allows us to use results from [Büc60b,Lad77] and [STT95] that characterize the expressive power of the logics in question for infinite words. The proof 2$\rightarrow$1 for *finite* MSCs from [HMKT99] uses a first-order interpretation of the lexicographically least linear extension of t in the finite MSC t. This proof method does not extend to the current setting since in general no linear extension of order type ω can be defined in an infinite MSC. To overcome this problem, we use ideas from [TW97] by choping an infinite MSC into its finite and its infinite part. It turns out that the infinite part is the disjoint union of infinite posets to which the "classical" method from [HMKT99] is applicable. The proof of the implication 3→1 is an obvious variant of similar proofs for finite automata for words (*cf.* [Tho90a]), asynchronous automata for traces [Tho90b], or asynchronous cellular automata for pomsets without autoconcurrency [DGK00]. Mukund *et al.* proved the implication 1→3 for finite MSCs. In order to do so, they had to reprove several results from the theory of Mazurkiewicz traces in the more complex realm of MSCs. Differently, my proof for infinite MSCs uses asynchronous mappings for Mazurkiewicz traces directly (*cf.* [DR95] for surveys on this theory). These mappings are applicable since any recognizable set of MSCs can be represented as a set of traces up to an easy relabeling. This observation has in my opinion several nice aspects: (1) it simplifies the proof, (2) it also results in smaller message passing automata for finite MSCs, and (3) it highlights the similarity of MSCs and Mazurkiewicz traces and the unifying role that Mazurkiewicz traces can play in the theory of distributed systems. This last point is also stressed by the fact that similar proof techniques have been used, *e.g.*, in [Arn91,DG96,Kus98,DGK00,KM00,Mor01].

2 Notation

Let $\mathcal{P}$ be a finite set of processes (or agents) which communicate with each other through messages via reliable FIFO-channels. Let Σ be the set of communication actions $p!q$ and $q?p$ for $p, q \in \mathcal{P}$ distinct. The action $p!q$ is to be read as "p sends to q" and $q?p$ is to be read as "q receives from p". Hence $p\theta q$ is performed by the process p, denoted $\mathrm{proc}(p\theta q) = p$. Following [HMKT99], we shall not be concerned with the internal actions of the agents which is no essential restriction since the results of this paper can be extended to deal with internal actions. We will also not consider the actual messages that are sent and received.

A Σ-labeled poset is a structure $t = (V, \leq, \lambda)$ where $(V, \leq)$ is a partially ordered set, $\lambda : V \to \Sigma$ is a mapping, $\lambda^{-1}(\sigma) \subseteq V$ is linearly ordered for any $\sigma \in \Sigma$, and any $v \in V$ dominates a finite set. For $v \in V$, we write $\mathrm{proc}(v)$ as a shorthand for $\mathrm{proc} \circ \lambda(v)$. The ideal ${\downarrow}v$ generated by v is defined by ${\downarrow}v = \{u \in V \mid u \leq v\}$. The covering relation $-\!\!\!< \subseteq V^2$ is defined by $u -\!\!\!< w$ iff $u < w$ and, for any $v \in V$ with $u \leq v < w$, we have $u = v$. In order to define message sequence charts, for a Σ-labeled poset t, we define the relations $\sqsubseteq_{\mathcal{P}}$ and $\sqsubseteq$ as follows:

- $v \sqsubseteq_{\mathcal{P}} w$ iff $\mathrm{proc}(v) = \mathrm{proc}(w)$ and $v \leq w$.
- $v \sqsubseteq w$ iff $\lambda(v) = p!q$, $\lambda(w) = q?p$, and $|{\downarrow}v \cap \lambda^{-1}(p!q)| = |{\downarrow}w \cap \lambda^{-1}(q?p)|$ for some $p, q \in \mathcal{P}$ distinct.

Definition 2.1. *A* message sequence chart or MSC *for short is a Σ-labeled poset $t = (V, \leq, \lambda)$ satisfying*

- $\leq = (\sqsubseteq_{\mathcal{P}} \cup \sqsubseteq)^\star$,
- $\mathrm{proc}^{-1}(p) \subseteq V$ *is linearly ordered for any $p \in \mathcal{P}$, and*

– $|\lambda^{-1}(p!q)| = |\lambda^{-1}(q?p)|$ *for any* $p, q \in \mathcal{P}$ *distinct.*

The MSC t *is* B-bounded *for some* $B \in \mathbb{N}$ *if, for any* $v \in V$, *we have* $|{\downarrow}v \cap \lambda^{-1}(p!q)| - |{\downarrow}w \cap \lambda^{-1}(q?p)| \leq B$. *By* MSC^∞, *we denote the set of all message sequence charts while* MSC *denotes the set of finite MSCs. Furthermore,* MSC_B *and* MSC_B^∞ *denote the sets of* B*-bounded (finite) MSCs. Finally,* ${\downarrow}\mathrm{MSC}$ *denotes the set of order ideals in finite MSCs, and* ${\downarrow}\mathrm{MSC}_B^\infty$ *etc are defined similarly.*

We call a set K of ideals in MSCs *bounded* if there exists $B \in \mathbb{N}$ with $K \subseteq {\downarrow}\mathrm{MSC}_B^\infty$. Let $w = w_0 w_1 w_2 \ldots$ be a finite or infinite word over Σ. We write $w{\restriction}_i = w_0 w_1 \ldots w_i$ for the prefix of w of length $i+1$ where $0 \leq i < |w|$. The word w is *proper* if, for any $0 \leq i < |w|$ and any $p, q \in \mathcal{P}$ distinct, we have $|w{\restriction}_i|_{p!q} \geq |w{\restriction}_i|_{q?p}$. From a proper word w, we can define the Σ-labeled poset $\mathrm{MSC}(w) = (\mathrm{dom}(w), \leq, \lambda_w)$ as follows:

– $V = \mathrm{dom}(w) = \{i \in \mathbb{N} \mid 0 \leq i < |w|\}$,
– $\lambda_w(i) = w_i$, and
– $\leq = R^\star$ where $(i,j) \in R$ iff $i < j$ and either $\mathrm{proc}(w_i) = \mathrm{proc}(w_j)$ or $w_i = p!q$, $w_j = q?p$ and $|w{\restriction}_i|_{p!q} = |w{\restriction}_j|_{q?p}$.

It is easily seen that $\mathrm{MSC}(w)$ is an element of ${\downarrow}\mathrm{MSC}^\infty$, *i.e.*, an ideal in an MSC. The proper word w is B-bounded if, for any $0 \leq i < |w|$, we have $|w{\restriction}_i|_{p!q} - |w{\restriction}_j|_{q?p} \leq B$. Note that an MSC t is B-bounded iff any proper word w with $\mathrm{MSC}(w) = t$ is B-bounded. On the other hand, there are proper B-bounded words w for which $\mathrm{MSC}(w)$ is not B-bounded (*e.g.*, $w = (p!q\, q?p)^\omega$). We call a proper word w *complete* provided $|w|_{p!q} = |w|_{q?p}$ for $p, q \in \mathcal{P}$. A proper word w is complete iff $\mathrm{MSC}(w)$ is an MSC.

For an ideal in a MSC t, let $\mathrm{Lin}_\omega(t)$ denote the set of all proper words w with $\mathrm{MSC}(w) = t$. For $K \subseteq {\downarrow}\mathrm{MSC}^\infty$, let $\mathrm{Lin}_\omega[K]$ denote the union of all sets $\mathrm{Lin}_\omega(t)$ for $t \in K$. Using this set of words, we define below recognizable sets of MSCs.

Recall that a set L of finite words over Σ is recognizable (by a finite deterministic automaton) iff there exists a finite monoid S and a homomorphism $\eta : \Sigma^\star \to S$ such that $L = \eta^{-1}\eta[L]$. The set L is aperiodic if S can be assumed to be aperiodic (*i.e.*, groupfree). Finally, we call L n-solvable (for some $n \in \mathbb{N}$) if we can assume that any group in S is solvable and has order dividing some power of n. Similarly, one can define recognizable, n-solvable, and aperiodic sets of words in Σ^∞: $L \subseteq \Sigma^\infty$ is recognizable iff there exists a finite monoid S and a homomorphism $\eta : \Sigma^\star \to S$ such that, for any $u_i, v_i \in \Sigma^\star$ with $\eta(u_i) = \eta(v_i)$ and $u_0 u_1 u_2 \cdots \in L$, we get $v_0 v_1 v_2 \cdots \in L$. The set L is said to be n-solvable or aperiodic if the monoid S can be assumed to satisfy the corresponding conditions.

Definition 2.2. *A set* $K \subseteq {\downarrow}\mathrm{MSC}^\infty$ *is* recognizable*,* n-solvable*, or* aperiodic *if* $\mathrm{Lin}_\omega[K]$ *is recognizable,* n*-solvable, or aperiodic, respectively.*

The sets of recognizable, n-solvable, and aperiodic word languages have been characterized in terms of fragments of monadic second order logic. In the first part of this paper, these results will be extended to subsets of ${\downarrow}\mathrm{MSC}^\infty$.

Formulas of the monadic second order language MSO over Σ involve first order variables $x, y, z...$ for nodes and set variables $X, Y, Z...$ for sets of nodes. They are built up from the atomic formulas $\lambda(x) = \sigma$ for $\sigma \in \Sigma$, $x \leq y$, and $x \in X$ by means of the boolean connectives $\neg, \wedge$ and the quantifier $\exists$ (both for first order and for set variables). A first order formula is a formula without set variables. Formulas without free variables are called sentences. The satisfaction relation $t \models \varphi$ between Σ-labeled posets t and formulas φ is defined canonically. Let X be a set of Σ-labeled posets. A set $K \subseteq X$ is *axiomatizable relative to* X iff there is a sentence φ such that $K = \{t \in X \mid t \models \varphi\}$.

Let $\varphi(x)$ be a first-order formula with one free variable. Since Σ-labeled posets have width at most $|\Sigma|$, for any $m, n \in \mathbb{N}$ with $m < n$, one can express in monadic second order logic that the set of witnesses of φ consists of $m \mod n$ elements. If we add only this modulo-counting ability to first-order logic, we obtain the logic FO+MOD(n): formulas of the logic FO+MOD(n) are built up from the atomic formulas $\lambda(x) = a$ and $x \leq y$ be the connectives $\wedge$ and $\neg$ and the first-order quantifier $\exists$ and $\exists^m$ for $0 \leq m < n$. A Σ-labeled poset $t = (V, \leq, \lambda)$ satisfies $\exists^m \varphi(x)$ if the number of nodes $v \in V$ such that $t \models \varphi(v)$ is congruent $m \mod n$.

To stress that a set of Σ-labeled posets K is axiomatizable relative to X by a sentence of FO, FO+MOD(n), or MSO, we will speak of first-order, FO+MOD(n)-, or monadically axiomatizable sets.

3 From logics to monoids

Words over the alphabet Σ can be considered as Σ-labeled (linear) posets in the natural way: the word $w \in \Sigma^\infty$ determines the structure $(\mathrm{dom}(w), \leq, \lambda_w)$ and *vice versa* where $\leq$ is the usual linear order on the natural numbers. In this sense, we can say "$w \models \varphi$" where φ is a sentence.

Lemma 3.1. *For $p, q \in \mathcal{P}$ and $B, n \in \mathbb{N}$, let $L_{p,q,B,n}$ be the set of all finite words $w \in \Sigma^\star$ satisfying*

(1) any prefix v of w satisfies $0 \leq |v|_{p!q} - |v|_{q?p} \leq B$, and
(2) $|w|_{p!q} - |w|_{q?p} = n$.

There exists a first order sentence $\beta_{p,q,B,n}$ such that, for $w \in \Sigma^\star$,

$$w \models \beta_{p,q,B,n} \iff w \in L_{p,q,B,n}.$$

Proof. The set $L = L_{p,q,B,n}$ is easily seen to be recognizable by an automaton with $B+1$ states that keeps track of the difference $|v|_{p!q} - |v|_{q?p}$. Now let $u, v, w \in \Sigma^\star$ with $uv^{B+1}w \in L$. By (1), $|v|_{p!q} = |v|_{q?p}$ which implies that $uv^{B+2}w$ satisfies (1) and (2), *i.e.*, belongs to L. If, conversely, $uv^{B+2}w \in L$, we can argue similarly to infer $uv^{B+1}w \in L$. Hence L is aperiodic and therefore first-order axiomatizable by [MP71]. □

Proposition 3.2. *The sets of B-bounded and proper (complete, resp.) words in $\Sigma^\star$ and Σ^ω are first-order axiomatizable relative to $\Sigma^\star$ and Σ^ω, resp.*

Proof. A finite word $u \in \Sigma^\star$ is B-bounded and proper iff, for any $p, q \in \mathcal{P}$, there exists $n \leq B$ such that $u \in L_{p,q,n,B}$. Hence the set of B-bounded and proper finite words is

$$K_1 = \bigcap_{p,q\in\mathcal{P}} \bigcup_{0\leq n\leq B} L_{p,q,B,n}$$

which is first-order axiomatizable by Lemma 3.1. The set of B-bounded and complete finite words is

$$K_2 = \bigcap_{p,q\in\mathcal{P}} L_{p,q,B,0}$$

and hence first-order axiomatizable. An infinite word is B-bounded and proper iff any of its finite prefixes is in K_1, *i.e.*, iff it satisfies $\forall x(\varkappa_1{\restriction_{\leq x}})$ where $\varkappa_1$ is a first-order sentence that axiomatizes K_1. An infinite word u is B-bounded and complete iff it is B-bounded, proper, and satisfies for any $p, q \in \mathcal{P}$:

- there are infinitely many occurrences of $q?p$ in u or
- there is a finite prefix $v \in L_{p,q,B,0}$ of u and the corresponding suffix does not contain any occurrence of $p!q$.

Since, by Lemma 3.1 and the first statement of this proposition, this can be expressed in first-order logic, the set of B-bounded and complete words in Σ^ω is first-order axiomatizable. $\square$

Lemma 3.3. *Let $B \in \mathbb{N}$. There exists a first-order formula $\chi(x, y)$ such that for all B-bounded and proper words $w \in \Sigma^\infty$ and all $0 \leq i, j < |w|$, we have: $w \models \chi(i, j)$ iff $w_i = p!q$, $w_j = q?p$, and $|w{\restriction_{\leq i}}|_{p!q} = |w{\restriction_{\leq j}}|_{q?p}$ for some $p, q \in \mathcal{P}$.*

Proof. For $0 \leq n \leq B$, $p, q \in \mathcal{P}$, let

$$\begin{aligned}\chi_{p,q,n}(x, y) = \;& (\lambda(x) = p!q) \wedge (\lambda(y) = q?p) \wedge x \leq y \wedge (\beta_{p,q,B,n}{\restriction_{\leq x}})\wedge \\ & \text{“the interval } [x, y] \text{ contains precisely } n \text{ positions labelled } q?p\text{”}\end{aligned}$$

Then a B-bounded and proper word w satisfies $\chi_{p,q,n}(i, j)$ iff

1. the letters w_i and w_j are $p!q$ and $q?p$, respectively,
2. $|w{\restriction_{\leq i}}|_{p!q} - |w{\restriction_{\leq i}}|_{q?p} = n$, and
3. $|w{\restriction_{\leq j}}|_{q?p} = |w{\restriction_{\leq i}}|_{q?p} + n$.

The combination of the second and third statement is equivalent to $|w{\restriction_{\leq i}}|_{p!q} = |w{\restriction_{\leq j}}|_{q?p}$. Setting $\chi(x, y) = \bigvee_{n=0}^{B} \bigvee_{p,q\in\mathcal{P}} \chi_{p,q,n}(x, y)$ finishes the proof. $\square$

Proposition 3.4. *Let $B \in \mathbb{N}$. There exists a first-order formula $\varphi(x, y)$ such that for any B-bounded and proper word $w \in \Sigma^\infty$, we have*

$$(\mathrm{dom}(w), \varphi^w, \lambda_w) \cong \mathrm{MSC}(w)$$

where $\varphi^w = \{(i, j) \mid 0 \leq i, j < |w|, w \models \varphi(i, j)\}$.

Proof. The result follows since $R^\star = R^{2|\mathcal{P}|}$ (*cf.* page 4) and since R is first-order definable by the preceding lemma. $\square$

Theorem 3.5. *Let* $K \subseteq {\downarrow}\mathrm{MSC}^\infty$ *be axiomatizable in* $\left\{\begin{array}{l}\textit{(1)}\ \mathrm{MSO}\\ \textit{(2)}\ \mathrm{FO{+}MOD}(n)\\ \textit{(3)}\ \mathrm{FO}\end{array}\right\}$ *and bounded. Then* $\mathrm{Lin}_\omega K \subseteq \Sigma^\infty$ *is* $\left\{\begin{array}{l}\textit{(1) recognizable}\\ \textit{(2) } n\textit{-solvable}\\ \textit{(3) aperiodic}\end{array}\right\}$.

Proof. By Proposition 3.2, the set of B-bounded and complete words is axiomatizable in first-order logic relative to Σ^∞. Let $\varkappa$ be the sentence that axiomatizes K, and let φ be the first-order formula from Proposition 3.4. In the sentence $\varkappa$, replace any occurrence of a subformula $x \leq y$ by $\varphi(x, y)$ and let $\varkappa'$ be the resulting sentence. Then, for a B-bounded and complete word $u \in \Sigma^\infty$, we get $u \models \varkappa'$ iff $\mathrm{MSC}(u) \models \varkappa$ by Proposition 3.4. Hence $\mathrm{Lin}_\omega(K)$ is axiomatizable in MSO/FO+MOD(n)/FO. Now [Büc60a,STT95,Lad77] imply the statement of the theorem. $\square$

4 From monoids to logics

Henriksen *et al.* [HMKT00b] showed that any recognizable set K of *finite* MSCs is axiomatizable in monadic second order logic relative to MSC_B for some B. Their proof strategy follows an idea from [EM96]: The set of linear extensions of K is a recognizable subset of $\Sigma^\star$. Hence, by [Büc60b], there is a monadic sentence φ that axiomatizes the linear extensions relative to $\Sigma^\star$. A canonical linear extension of a finite MSC t can be defined in t by a first-order formula, *i.e.*, this canonical linear extension can be interpreted in t. Hence there is a monadic sentence φ' that is satisfied by t iff the canonical linear extension of t satisfies φ, *i.e.*, K is monadically axiomatizable. Since the canonical linear order is defined by a first-order formula, this proof can be used to show that any aperiodic/n-solvable set of finite MSCs is first-order/FO+MOD(n) axiomatizable.

Let $t = (V, \leq, \lambda)$ be a Σ-labeled poset. If V is finite, then any linear extension of t can be identified with a finite word over Σ. Now suppose V is infinite, *e.g.*, the disjoint union of two ω-chains. Then there is a linear extension that cannot be seen as an ω-word, since it is order-isomorphic to $\omega + \omega$. Note that $\mathrm{Lin}_\omega(t)$ is the set of all finite or ω-words that are linear extensions of t. Clearly, since all principal ideals in t are finite, $\mathrm{Lin}_\omega(t)$ is nonempty. Nevertheless, the proof idea from [EM96,HMKT99] cannot be used for infinite MSCs since there is no *definable* linear extension of the MSC $((p!q)(q?p)(p'!q')(q'?p'))^\omega$ of order type ω.[1]

[1] Suppose the first-order formula $\varphi(x, y)$ defines a linear extension of this partial order of order type ω. Then there exist arbitrary large natural numbers k and ℓ such that the k^{th} $p!q$-event precedes the ℓ^{th} $p'!q'$-event. But then for all $m > k$, the m^{th} $p!q$-event precedes the ℓ^{th} $p'!q'$-event since first order logic cannot distinguish between these $p!q$-events.

To overcome this problem, we will chop an MSC into pieces, consider these pieces independently, and combine the results obtained for them.

Before we start this programme, we need an extension of Mezei's theorem to languages of infinite words and therefore some more notation. By $u \sqcup\!\sqcup v$ we denote the set of all shuffles of the words u and v. Then $K \sqcup\!\sqcup L$ is the set of shuffles of elements of K and L. A pair (s, e) of elements of a finite monoid S is *linked* if $s \cdot e = s$ and $e \cdot e = e$. Two linked pairs (s, e) and (s', e') are *conjugated* if there exist $x, y \in S$ with $e = xy$, $e' = yx$, and $s' = sx$. Since conjugacy is an equivalence relation [PP99, Prop. II.3.6], we can speak on conjugacy classes. Now let $\eta : \Sigma^\star \to S$ be a surjective homomorphism. Then two linked pairs (s, e) and (s', e') are conjugated iff $\eta^{-1}(s)(\eta^{-1}(e))^\omega \cap \eta^{-1}(s)(\eta^{-1}(e))^\omega \neq \emptyset$ (*cf.* [PP99, Cor. II.3.8]). A set $L \subseteq \Sigma^\infty$ is *weakly recognized by* η if $L = \bigcup \eta^{-1}(s)(\eta^{-1}(e))^\omega$ where the union is taken over all linked pairs (s, e) in S with $\eta^{-1}(s)(\eta^{-1}(e))^\omega \cap L \neq \emptyset$. If L is recognized by the morphism η, then η recognizes L weakly [PP99, Prop. VI.5.2]. The converse is false in general, but the following weaker assertion holds. A finite monoid S' is an *aperiodic extension* of a monoid S if there is a surjective homomorphism $\eta : S' \to S$ such that $\eta^{-1}(f)$ is an aperiodic semigroup for any idempotent element $f \in S$. Note that aperiodic extensions of finite/n-solvable/aperiodic monoids are finite/n-solvable/aperiodic. If η recognizes L weakly, then there exists a homomorphism into some aperiodic extension of S that recognizes L [PP99, Thm. VI.5.10, Prop. VI.5.18].

Theorem 4.1. *Let $\Sigma = \Sigma_1 \dot\cup \Sigma_2$ be an alphabet. Let $L \subseteq \Sigma^\infty$ be recognized by a surjective homomorphism onto $(S, \cdot)$ and suppose that $u_1 \sqcup\!\sqcup u_2 \cap L \neq \emptyset$ implies $u_1 \sqcup\!\sqcup u_2 \subseteq L$ for any $u_i \in \Sigma_i^\infty$ $(i = 1, 2)$. Then L is a finite union of sets $L_1 \sqcup\!\sqcup L_2$ where $L_i \subseteq \Sigma_i^\infty$ is recognized by an aperiodic extension of $(S, \cdot)$.*

Proof. Let $\eta : \Sigma^\star \to (S, \cdot)$ be a homomorphism that recognizes L. Then $\eta_i = \eta\restriction_{\Sigma_i^\star}$ is a homomorphism from $\Sigma_i^\star$ into $(S, \cdot)$. For a linked pair $(s, e) \in S$, let $\overline{\eta}(s, e) = \eta^{-1}(s)\eta^{-1}(e)^\omega$ and define $\overline{\eta}_i(s, e)$ similarly. Furthermore, let $P_\eta(L) = \{(s, e) \in S^2 \mid (s, e)$ is a linked pair, and $\overline{\eta}(s, e) \cap L \neq \emptyset\}$. We call a nonempty set $X \subseteq \Sigma^\infty$ *η-minimal* if it is a minimal set weakly recognized by the homomorphism η.

Now we prove the theorem by a sequence of claims

Claim 1 A nonempty set $X \subseteq \Sigma^\infty$ is η-minimal iff $P_\eta(X)$ is a conjugacy class and X is weakly recognized by η.

Proof of Claim 1 Suppose X is η-minimal. Let $(s, e), (s', e')$ be conjugated linked pairs with $(s, e) \in P_\eta(X)$. Since X is weakly recognized by η, this implies $\overline{\eta}(s, e) \subseteq X$. Since (s, e) and (s', e') are conjugated, we get $\emptyset \neq \overline{\eta}(s, e) \cap \overline{\eta}(s', e') \subseteq X \cap \overline{\eta}(s', e')$. Hence $P_\eta(X)$ is closed under conjugation.

Now let C be some conjugacy class with $Y = \overline{\eta}[C]$. We show that Y is weakly recognized by η: let $(s', e') \in P_\eta(Y)$. Then there exists $(s, e) \in C$ with $\overline{\eta}(s, e) \cap$

$\overline{\eta}(s'e') \neq \emptyset$. Hence (s,e) and (s',e') are conjugated, *i.e.*, $(s,e) \in C$. In other words, $P_\eta(Y) = C$, *i.e.*, Y is weakly recognized by η.

Hence, for any η-minimal set X, there is a conjugacy class C with $C \subseteq P_\eta(X)$ and $\emptyset \neq \overline{\eta}[C] \subseteq \overline{\eta}[P_\eta(X)] = X$. Since $\overline{\eta}[C]$ is weakly recognized by η, this implies $X = \overline{\eta}[C]$, *i.e.*, $P_\eta(X) = C$.

Let, conversely, C be a conjugacy class and $\emptyset \neq X \subseteq \overline{\eta}[C]$ be weakly recognized by η. Since $P_\eta(X)$ is closed under conjugation, this implies $P_\eta(X) = C$, *i.e.*, $\overline{\eta}[C] = X$. Hence $\overline{\eta}[C]$ is η-minimal. **q.e.d.**

Claim 2 Let (s_i, e_i) be linked pairs s.t. $(\overline{\eta}_1(s_1,e_1) \sqcup\sqcup \overline{\eta}_2(s_2,e_2)) \cap L \neq \emptyset$. Then $\overline{\eta}_1(s_1,e_1) \sqcup\sqcup \overline{\eta}_2(s_2,e_2) \subseteq L$.

Proof of Claim 2 Since $(\overline{\eta}_1(s_1,e_1) \sqcup\sqcup \overline{\eta}_2(s_2,e_2)) \cap L \neq \emptyset$, there are $u_i \in \overline{\eta}_i(s_i,e_i)$ with $u_1 \sqcup\sqcup u_2 \cap L \neq \emptyset$ and therefore $u_1 \sqcup\sqcup u_2 \subseteq L$ by our assumption on L. Since $u_i \in \overline{\eta}_i(s_i,e_i)$, there are $u_i^j \in \Sigma_i^\star$ for $j \in \omega$ such that $u_i = u_i^0 u_i^1 u_i^2 \ldots$, $\eta_i(u_i^0) = s_i$, and $\eta_i(u_i^j) = e_i$ for $j > 0$. Then $\prod_{j\in\omega}(u_1^j u_2^j) \in u_1 \sqcup\sqcup u_2 \subseteq L$.

Now let $v_i^j \in \Sigma_i^\star$ for $j \in \omega$ such that $\eta_i(v_i^0) = s_i$ and $\eta_i(v_i^j) = e_i$ for $j > 0$, *i.e.*, $\eta_i(v_i^j) = \eta_i(u_i^j)$ for $j \in \omega$. Since η recognizes L and $\prod_{j\in\omega}(u_1^j u_2^j) \in L$, we obtain $\prod_{j\in\omega}(v_1^j v_2^j) \in L$, *i.e.*, $v_1 \sqcup\sqcup v_2 \cap L \neq \emptyset$ for $v_i = \prod_{j\in\omega} v_i^j$. Hence, by the assumption on L, we have $v_1 \sqcup\sqcup v_2 \subseteq L$. Since $v_i \in \overline{\eta}(s_i,e_i)$ are arbitrary elements, this implies $\overline{\eta}_1(s_1,e_1) \sqcup\sqcup \overline{\eta}_2(s_2,e_2) \subseteq L$. **q.e.d.**

Claim 3

$$L = \bigcup \{L_1 \sqcup\sqcup L_2 \mid L_i \subseteq \Sigma_i^\infty \ \eta_i\text{-minimal}, L_1 \sqcup\sqcup L_2 \cap L \neq \emptyset\}.$$

Proof of Claim 3 Let $u \in L$ and u_i be the projection of u to Σ_i. Then, by [PP99, Prop. II.3.3], $P_{\eta_i}(\{u_i\})$ is not empty. Hence, by claim 1, there are η_i-minimal sets L_i such that $u_i \in L_i$. But then $u \in L_1 \sqcup\sqcup L_2$ and $L_1 \sqcup\sqcup L_2 \cap L \neq \emptyset$. Hence we showed the inclusion $\subseteq$.

Now let $L_i \subseteq \Sigma_i^\infty$ be η_i-minimal and suppose $u_i, w_i \in L_i$ with $u_1 \sqcup\sqcup u_2 \cap L \neq \emptyset$. By claim 1, there exist conjugated linked pairs (s_i,e_i) and (s_i',e_i') such that $u_i \in \overline{\eta}_i(s_i,e_i)$ and $w_i \in \overline{\eta}_i(s_i',e_i')$. Hence $\overline{\eta}_1(s_1,e_1) \sqcup\sqcup \overline{\eta}_2(s_2,e_2) \cap L \neq \emptyset$. Then claim 2 implies $\overline{\eta}_1(s_1,e_1) \sqcup\sqcup \overline{\eta}_2(s_2,e_2) \subseteq L$. Furthermore, there exist $v_i \in \overline{\eta}_i(s_i,e_i) \cap \overline{\eta}_i(s_i',e_i')$ by [PP99, Cor. II.3.8]. Since $\overline{\eta}_1(s_1,e_1) \sqcup\sqcup \overline{\eta}_2(s_2,e_2) \subseteq L$, in particular, $v_1 \sqcup\sqcup v_2 \subseteq L$ and therefore $\overline{\eta}_1(s_1',e_1') \sqcup\sqcup \overline{\eta}_2(s_2',e_2') \cap L \neq \emptyset$. By claim 2, this implies $\overline{\eta}_1(s_1',e_1') \sqcup\sqcup \overline{\eta}_2(s_2',e_2') \subseteq L$ and therefore in particular $w_1 \sqcup\sqcup w_2 \subseteq L$. **q.e.d.**

Now we can prove the theorem: Since $(S,\cdot)$ is finite, there are only finitely many sets weakly recognizable by η_i. Hence, by Claim 3, L is a finite union of sets $L_1 \sqcup\sqcup L_2$ with $L_i \subseteq \Sigma_i^\infty$ weakly recognizable by $(S,\cdot)$. By [PP99, VI.5.10 and VI.5.18], L_i is recognizable by a finite aperiodic extension of $(S,\cdot)$. $\square$

In [HMKT00b], the following proposition is shown for sets of finite MSCs. The proof for ideals in possibly infinite MSCs goes through verbatim.

Proposition 4.2 (*cf.* [HMKT00b, Prop. 3.2]). *Let $K \subseteq {\downarrow}\mathrm{MSC}^\infty$ be recognizable. Then K is bounded, i.e., there exists a positive integer B such that $L \subseteq {\downarrow}\mathrm{MSC}_B^\infty$.*

Above, I explained that in general the linear extensions of an MSC need not be of order type ω. We now start to characterize some Σ-labeled posets for which we need not worry about this.

Let $t = (V, \leq, \lambda)$ be a Σ-labeled poset and $X \subseteq V$. Then $t{\restriction}_X = (X, \leq \cap X \times X, \lambda{\restriction}_X)$ is the restriction of t to X. For $A \subseteq \Sigma$, we will abbreviate $t{\restriction}_{\lambda^{-1}(A)}$ by $t{\restriction}_A$. The poset t is *directed* if it does not contain two disjoint filters.

Now let $t = (V, \leq, \lambda) \in {\downarrow}\mathrm{MSC}^\infty$. Then $\mathrm{alph}(t) = \lambda[V]$ and $\mathrm{alphInf}(t) = \{\sigma \in \Sigma \mid \lambda^{-1}(\sigma) \text{ is infinite}\}$. Let Y be the largest filter in t with $\lambda[Y] \subseteq \mathrm{alphInf}(t)$ and let $X = V \setminus Y$. Then the *finitary part of* t is defined by $\mathrm{Fin}(t) = t{\restriction}_X$ and the *infinitary part of* t by $\mathrm{Inf}(t) = t{\restriction}_Y$. Note that $\mathrm{Fin}(t)$ is an ideal in a finite MSC while in general $\mathrm{Inf}(t)$ is only a Σ-labeled poset.

Let $E \subseteq \Sigma^2$ contain all pairs $\sigma, \tau \in \Sigma$ with $\mathrm{proc}(\sigma) = \mathrm{proc}(\sigma)$ or $\{\sigma, \tau\} = \{p!q, q?p\}$ for some $p, q \in \mathcal{P}$. Then (Σ, E) is an undirected graph.

Lemma 4.3. *Let $t = (V, \leq, \lambda) \in {\downarrow}\mathrm{MSC}_B^\infty$, and $A = \mathrm{alphInf}(t)$. Let $(A_i)_{1 \leq i \leq n}$ be the connected components of the graph (A, E). Then the Σ-labeled posets $t_i = \mathrm{Inf}(t){\restriction}_{A_i}$ are directed for $1 \leq i \leq n$, and $\mathrm{Inf}(t) = \dot{\bigcup}_{1 \leq i \leq n} t_i$.*

Proof. We first show that t_i is directed: By contradiction, assume X and Y to be disjoint filters in t_i. Then they involve disjoint instances, *i.e.*, $\mathrm{proc}[\lambda[X]] \cap \mathrm{proc}[\lambda[Y]] = \emptyset$. Since $A_i = \mathrm{alphInf}(t_i)$ is connected, there exist $p, q \in \mathcal{P}$ with $p!q \in \lambda[X]$ and $q?p \in \lambda[Y]$ (or vice versa). Since Y is a filter and $\lambda^{-1}(q?p)$ is a linear order of order type at most ω, we obtain that $\lambda^{-1}(q?p) \setminus Y$ is finite. Furthermore ${\downarrow}x \cap \lambda^{-1}(q?p) \subseteq \lambda^{-1}(q?p) \setminus Y$ for $x \in X$ since X and Y are disjoint. Since, on the other hand, X contains infinitely many nodes x with $\lambda(x) = p!q$, there is one $x \in X$ with $|{\downarrow}x \cap \lambda^{-1}(p!q)| - |{\downarrow}x \cap \lambda^{-1}(q?p)| > B$, contradicting our assumption that t be B-bounded. Hence, indeed, t_i is directed.

Next we show that $\mathrm{Inf}(t)$ is the disjoint union of the pomsets t_i: Since $A = \bigcup_{1 \leq i \leq n} A_i$, the carrier set of $\mathrm{Inf}(t)$ is the disjoint union of the carrier sets of t_i. Now let x be a node in t_{i_1} and z a node in t_{i_2} such that $x \leq z$. Since $t \in {\downarrow}\mathrm{MSC}^\infty$, we get $x(\sqsubseteq_{\mathcal{P}} \cup \sqsubseteq)^\star z$. Hence there exist $y_1, y_2, \ldots, y_m$ with $x = y_1$, $(y_i, y_{i+1}) \in \sqsubseteq_{\mathcal{P}} \cup \sqsubseteq$, and $y_m = z$. Since $x \leq y_i$ and x is a node in $\mathrm{Inf}(t)$, the label $\lambda(y_i)$ belongs to $\mathrm{alphInf}(t)$. Furthermore, $(\lambda(y_i), \lambda(y_{i+1})$ is an edge in (Σ, E). Hence $\lambda(x)$ and $\lambda(z)$ belong to the same connected component of (A, E) implying $i_1 = i_2$. $\square$

There are directed MSCs with linear extensions not of order type ω (*e.g.* $((p!q)(q?p))^\omega$). This cannot happen for MSCs with "connected communication graph", a result, slightly stronger than the following that suffices for our needs:

Lemma 4.4. *Let $t = (V, \leq, \lambda) \in {\downarrow}\mathrm{MSC}_B^\infty$ be directed. Then any linear extension of t is of order type at most ω.*

Proof. By contradiction, assume there is a linear extension $\preceq$ of t of order type $> \omega$. Then there are $x_i, z_0 \in V$ with $x_1 \prec x_2 \prec x_3 \cdots \prec z_0$. Since ${\downarrow}x_0$ is finite in t, there is $i_1 \in \mathbb{N}$ such that, for $j \geq i_1$, we have $z_0 \parallel x_j$. Since t is directed, there exists $z_1 \in V$ with $z_0, x_{i_1} \leq z_1$. Since $\preceq$ is an order extension of $\leq$, this implies $x_i \prec z_0 \prec z_1$ for all $i \in \mathbb{N}$. Since ${\downarrow}z_1$ is finite in t, we again find $i_2 \in \mathbb{N}$ with $x_j \parallel z_1$ for $j \geq i_2$. From $x_{i_1} \leq z_2$, it follows $i_1 < i_2$. Inductively, we find an increasing sequence $i_1 < i_2 < i_3 \dots$ of natural numbers and an increasing sequence $z_1 < z_2 < z_3 \dots$ of elements of V such that, for all $i, j \in \mathbb{N}$, $z_i \not\leq x_{i_j}$, and $x_{i_j} \leq z_i$ iff $j < i$. For $j \in \mathbb{N}$, let $y_j = x_{i_j}$.

Since ${\downarrow}z_0$ is finite and $z_0 \parallel y_i$ for $i > 0$, there is a minimal element b_0 satisfying $y_0 \leq b_0 \leq z_0$ that is incomparable with all y_i for $i > 0$. Since $y_0 < y_1$, we get $y_0 < b_0$. Hence there is a_0 with $y_0 \leq a_0 \prec\!\!\cdot\, b_0 \leq z_0$. Since b_0 was chosen minimal, a_0 lies below one (and therefore almost all) y_i. Choose $j_1 > 0$ with $a_0 < y_{j_1}$. We can inductively choose a_i, b_i and $j_i \in \mathbb{N}$ satisfying $y_{j_i} \leq a_i \prec\!\!\cdot\, b_i \leq z_{j_i}$, $a_i \leq y_{j_{i+1}}$, and $b_i \parallel a_j$ for all $j > i$. We can, without loss of generality, assume $\lambda(a_i) = \sigma$ and $\lambda(b_i) = \tau$ for all j since Σ is finite. Since $b_2 \parallel a_1$, we get $\mathrm{proc}(\sigma) \neq \mathrm{proc}(\tau)$. Hence there are $p, q \in \mathcal{P}$ such that $\sigma = p!q$, $\tau = q?p$, and $|{\downarrow}a_i \cap \lambda^{-1}(\sigma)| = |{\downarrow}b_i \cap \lambda^{-1}(\tau)|$ for $i \in \mathbb{N}$. But this implies $|{\downarrow}a_{B+1} \cap \lambda^{-1}(\sigma)| - |{\downarrow}b_{B+1} \cap \lambda^{-1}(\tau)| > B$, contradicting our assumption on t to be B-bounded. $\square$

Let $\preceq$ be a fixed linear order on Σ. Let furthermore X be a subset of an MSC t. Then $\preceq$ defines the lexicographic order on the linear extensions of $t{\restriction}_X$. Since these linear extensions are well orders, the lexicographic order is a well order, too. Hence there is a lexicographically least linear extension of $t{\restriction}_X$ that we denote by $\mathrm{lexNF}(t{\restriction}_X)$.

Lemma 4.5. *There exists a first-order formula φ with two free variables such that for any $t = (V, \leq, \lambda) \in {\downarrow}\mathrm{MSC}^\infty$ and $X \subseteq V$, we have*

$$(X, \varphi^{t\restriction X}, \lambda{\restriction}_X) \cong \mathrm{lexNF}(t{\restriction}_X).$$

Proof. The existence of such a formula was shown independently in [EM96,DK96] in more restricted settings, but the proofs go through in the current setting as well (*cf.* also [DM97] for a comprehensive proof). $\square$

Now suppose $t \in {\downarrow}\mathrm{MSC}_B^\infty$ and let $\mathrm{alphInf}(t) = A$. Then, by Lemma 4.3, $\mathrm{Inf}(t)$ is the disjoint union of the Σ-labeled posets $t_i = \mathrm{Inf}(t){\restriction}_{A_i}$ for A_i a connected component of A. We define $\mathrm{NFInf}(t) = \sqcup\!\!\sqcup_{1 \leq i \leq n} \mathrm{lexNF}(t_i)$.

Lemma 4.6. *Let $t \in {\downarrow}\mathrm{MSC}_B^\omega$. Then $\mathrm{NFInf}(t) \subseteq \Sigma^\omega$.*

Proof. It suffices to show that $\mathrm{lexNF}(t_i)$ is of order type ω for $t_i = (V_i, \leq, \lambda) = \mathrm{Inf}(t){\restriction}_{A_i}$ where A_i is a connected component of $A = \mathrm{alphInf}(t)$. Let $X = {\downarrow}V_i = \{x \in V \mid \exists y \in V_i : x \leq y\}$. Then $t{\restriction}_X \in {\downarrow}\mathrm{MSC}_B^\infty$. Since, by Lemma 4.3, t_i is directed, so is $t{\restriction}_X$. Hence, by Lemma 4.4, any linear extension of X is of order type at most ω. Since any order extension of t_i can be extended to an order extension of X, the result follows. $\square$

Hence we have in particular $\mathrm{NFInf}(t) \subseteq \mathrm{Lin}_\omega(\mathrm{Inf}(t))$.

Definition 4.7. *Let* $t \in {\downarrow}\mathrm{MSC}_B^\infty$, $A = \mathrm{alphInf}(t)$ *and let* $\$ \notin \Sigma$. *Then* $\mathrm{NF}(t) = \mathrm{lexNF}(\mathrm{Fin}(t)) \cdot \{\$\} \cdot \mathrm{NFInf}(t)$ *is the set of normal forms of* t. *For* $K \subseteq {\downarrow}\mathrm{MSC}_B^\infty$, *we set* $\mathrm{NF}[K] = \bigcup_{t \in K} \mathrm{NF}(t)$.

Since $\mathrm{NFInf}(t)$ is a subset of Σ^ω, we get $\mathrm{NF}(t) \subseteq (\Sigma \cup \$)^\infty$. Furthermore, the restriction of any word in $\mathrm{NF}(t)$ to Σ is a linear extension of t, *i.e.*, $\mathrm{NF}(t) \subseteq \mathrm{Lin}_\omega(t) \sqcup\!\sqcup \{\$\}$. Next we show that $\mathrm{NF}[{\downarrow}\mathrm{MSC}_B^\infty]$ is first-order axiomatizable:

Lemma 4.8. *The set* $\mathrm{NF}[{\downarrow}\mathrm{MSC}_B^\infty]$ *is first-order axiomatizable relative to* $(\Sigma \cup \{\$\})^\infty$.

Proof. The set $\Sigma^\infty \sqcup\!\sqcup \{\$\}$ is first-order axiomatizable relative to $(\Sigma \cup \{\$\})^\infty$. By Proposition 3.2, the set L of B-bounded and proper words in Σ^∞ is first-order axiomatizable. Hence, it suffices to show that $\mathrm{NF}[{\downarrow}\mathrm{MSC}_B^\infty]$ is first-order axiomatizable relative to $L \sqcup\!\sqcup \{\$\}$. By Proposition 3.4, there is a first-order formula $\varphi(x, y)$ such that $(\mathrm{dom}(w), \varphi^w, \lambda_w) \cong \mathrm{MSC}(w)$ for any word $w \in L$. In $t = \mathrm{MSC}(w)$, we can define the nodes that belong to $\mathrm{Fin}(t)$ and the connected components t_i of $\mathrm{Inf}(t)$. Furthermore, by [EM96], we can define the lexicographic normal forms of $\mathrm{Fin}(t)$ and t_i that belong to $\Sigma^\star$ and Σ^ω, respectively. Now let $u_1\$u_2$ be a word in $L \sqcup\!\sqcup \{\$\}$. By what we saw so far, we can express that u_1 is the lexicographic normal form of $\mathrm{Fin}(\mathrm{MSC}(u_1u_2))$ and that u_2 is an element of $\mathrm{NFInf}(\mathrm{MSC}(u_1u_2))$. Hence, indeed, $\mathrm{NF}[{\downarrow}\mathrm{MSC}_B^\infty]$ is first-order axiomatizable. $\square$

Let $u \in \Sigma^\infty$ and $k \in \mathbb{N}$. Then the set of all first-order sentences φ of quantifier depth at most k is the *k-first-order theory of u*. A set of first-order sentences of quantifier depth at most k is a *complete k-first-order theory* if it is the k-first-order theory of some word $u \in \Sigma^\infty$. Since, up to logical equivalence, there are only finitely many first-order sentences of quantifier depth at most k, there are only finitely many complete k-first-order theories. Furthermore, each complete k-first-order theory T is characterized by one first-order sentence γ_T of quantifier depth k, *i.e.*, for any word $u \in \Sigma^\infty$, we have $u \models \gamma$ for all $\gamma \in T$ iff $u \models \gamma_T$ (*cf.* [Hod93, Thm. 3.3.2]). For notational convenience, we will identify the characterizing sentence γ_T and the complete k-first-order theory T.

Let $K \subseteq {\downarrow}\mathrm{MSC}_B^\infty$ be aperiodic. Then $\mathrm{Lin}_\omega[K]$ is first-order axiomatizable relative to Σ^∞. Let $k \geq 2$ be the least integer such that $\mathrm{Lin}_\omega[K] \sqcup\!\sqcup \{\$\}$ and $\mathrm{NF}[{\downarrow}\mathrm{MSC}_B^\infty]$ are first-order axiomatizable relative to $(\Sigma \cup \{\$\})^\infty$ by a sentence of quantifier depth at most k. Let T be a complete k-first-order theory and $A \subseteq \Sigma$. Then

$$K_{T,A} := \{t \in K \mid \mathrm{lexNF}(\mathrm{Fin}(t)) \models T, \mathrm{alphInf}(t) = A\}$$

and

$$X_{T,A} := \mathrm{NF}[K_{T,A}].$$

Lemma 4.9. *In the first order language of $(\Sigma \cup \{\$\})$-labeled linear orders with one constant c, there exists a sentence φ of quantifier depth k such that $(v, c) \models \varphi$ iff $v \in X_{T,A}$ and $\lambda(c) = \$$ for $v \in (\Sigma \cup \{\$\})^\infty$.*

Proof. The sentence $\forall x(\lambda(x) = \$ \iff x = c)$ is satisfied by v iff $v = u\$u'$ with $u \in \Sigma^\star$ and $u' \in \Sigma^\infty$ and $c = |u|$. But then $u\$u'$ belongs to $X_{T,A}$ iff $u \models T$, $\mathrm{alphInf}(u\$u') = A$, $u\$u' \in \mathrm{NF}[{\downarrow}\mathrm{MSC}_B^\infty]$, and $uu' \in \mathrm{Lin}_\omega[K]$. Note that $u \models T$ iff $(u\$u', |u|) \models T{\restriction}_{<c}$. Since A is finite, $\mathrm{alphInf}(uu') = A$ can be expressed by a sentence of quantifier depth 2. By Lemma 4.8 and by our choice of k, $\mathrm{NF}[{\downarrow}\mathrm{MSC}_B^\infty]$ is first-order axiomatizable by a sentence of quantifier depth k. Similarly, $\mathrm{Lin}_\omega[K] \sqcup\!\sqcup \{\$\}$ is first-order axiomatizable by a sentence of quantifier depth k. ◻

Theorem 4.10. *Let $K \subseteq {\downarrow}\mathrm{MSC}^\infty$ be aperiodic. Then K is first-order axiomatizable relative to ${\downarrow}\mathrm{MSC}^\infty$ and bounded.*

Proof. Let T be a complete k-first-order theory and let $A \subseteq \Sigma$. In order to prove the theorem, it suffices to show that $K_{T,A}$ is first-order axiomatizable. By Lemma 4.9, the set $\{(u\$u', |u|) \mid u\$u' \in X_{T,A}\}$ is axiomatizable by a sentence of quantifier depth k in the extended language. Hence $X_{T,A}$ is first-order axiomatizable (by a sentence of quantifier depth $k+1$) and therefore aperiodic [Lad77]. Let X denote the set of finite words $u \in \Sigma^\star$ with $u \models T$ and let $Y = \bigcup_{u \in X} (u\$)^{-1} X_{T,A}$. This set is aperiodic since $X_{T,A}$ is aperiodic. We show $X_{T,A} = X\$Y$: the inclusion $\subseteq$ is trivial. So let $u \in X$ and $u' \in Y$. Then there is $v \in X$ with $v\$u' \in X_{T,A}$. Since $v \in X$, we have $v \models T$, *i.e.*, the k-first-order theories of v and u coincide. Hence, by a standard application of Ehrenfeucht-Fraïssé-games (*cf.* [Hod93]) the k-first-order theories of $(v\$u', |v|)$ and $(u\$u', |u|)$ coincide as well. Hence, by Lemma 4.9, $v\$u' \in X_{T,A}$ implies $u\$u' \in X_{T,A}$, *i.e.*, $X_{T,A} = X\$Y$.

By the definition of $X_{T,A}$, this implies $X = \{\mathrm{lexNF}(\mathrm{Fin}(t)) \mid t \in K_{T,A}\}$ and $Y = \bigcup_{t \in K_{T,A}} \mathrm{NFInf}(t)$. Let $(A_i)_{1 \le i \le n}$ be the connected components of (A, E), let $u \in Y$, and let u_i be the restriction of u to A_i. Then there exists $t \in K_{T,A}$ such that $\mathrm{Inf}(t) = \dot{\bigcup}_{1 \le i \le n} t_i$ and $u_i = \mathrm{lexNF}(t_i)$. Hence $\sqcup\!\sqcup_{1 \le i \le n} u_i \subseteq Y$. Hence we can apply Theorem 4.1 which implies

$$Y = \bigcup_{1 \le j \le m} \sqcup\!\sqcup_{1 \le i \le n} Y_i^j$$

with $Y_i^j \subseteq A_i^\omega$ aperiodic. Let Y_i^j be axiomatized by the first-order sentence ν_i^j relative to Σ^∞ [Lad77]. Furthermore, let φ axiomatize X relative to $\Sigma^\star$. There are certainly first-order formulas that define $\mathrm{Fin}(t)$ and $\mathrm{Inf}(t)$ for $t \in {\downarrow}\mathrm{MSC}_B^\infty$. By Lemma 4.5, we can interpret its lexicographically least linear extension in $\mathrm{Fin}(t)$ and therefore express whether $\mathrm{lexNF}(\mathrm{Fin}(t)) \models \varphi$. Similarly, we can define the connected component t_i of $\mathrm{Inf}(t)$. Since t_i is directed, any linear extension of t_i is of order type at most ω by Lemmas 4.3 and 4.4. Since we can interpret the lexicographically least linear extension in t_i, we can express whether this lexicographical normal form satisfies ν_i^j. $\square$

There is an obvious definition of a complete k-FO+MOD(n)- and of a complete k-MSO-theory. Using results from [STT95,Büc60b] in place of [Lad77], one obtains the following result along the lines of the above proofs:

Theorem 4.11. *Let $K \subseteq \downarrow\mathrm{MSC}^\infty$ be recognizable (n-solvable). Then K is monadically axiomatizable (*FO+MOD(n)*-axiomatizable, resp.) relative to $\downarrow\mathrm{MSC}^\infty$ and bounded.*

5 Deterministic message passing automata

So far, we used the algebraic characterization of recognizable word languages. There is a well (and presumably even better) known combinatorial characterization using finite automata. The extension of these automata-theoretic characterizations is the topic of this section. Message passing automata, the automata model that we consider, reflect the concurrent behavior of an MSC. It was introduced by Henriksen *et al.* [HMKT99] and is similar to asynchronous cellular automata from the theory of Mazurkiewicz traces. We will extend results from [HMKT99,MKS00] to infinite MSCs. Since the proofs rely on the theory of Mazurkiewicz traces, we first investigate the relation between these traces and MSCs.

5.1 The key observation

A *dependence alphabet* is a pair (Γ, D) where Γ is a finite set and D is a reflexive and symmetric dependence relation. A *trace over* (Γ, D) is a Γ-labeled partial order $(V, \leq, \lambda')$ such that

- $\downarrow x$ is finite for any $x \in V$,
- $(\lambda'(x), \lambda'(y)) \notin D$ whenever $x, y \in V$ are incomparable, and
- $(\lambda'(x), \lambda'(y)) \in D$ whenever $x \mathrel{-\!\!\!<} y$.

The set of all traces over (Γ, D) is denoted by $\mathbb{R}(\Gamma, D)$, the set $\mathbb{M}(\Gamma, D)$ comprises the finite traces.

The key observation that is announced by the title of this section is that any recognizable set of MSCs is the "relabeling" of a monadically axiomatizable set of traces over a suitable dependence alphabet. Recall that any recognizable set of MSCs is bounded by Proposition 4.2. The bound B influences the chosen dependence alphabet as defined in the following paragraph.

For a positive integer $B \in \mathbb{N}$, let $\Gamma = \Sigma \times \{0, 1, \ldots, B-1\}$. On this alphabet, we define a dependence relation D as follows: (σ_1, n_1) and (σ_2, n_2) are dependent iff

1. $\mathrm{proc}(\sigma_1) = \mathrm{proc}(\sigma_2)$, or
2. $\{(\sigma_1, n_1), (\sigma_2, n_2)\} = \{(p!q, n), (q?p, n)\}$ for some $p, q \in \mathcal{P}$ and $0 \leq n < B$.

For $t = (V, \leq, \lambda) \in {\downarrow}\mathrm{MSC}_B^\infty$, we define a new Γ-labeling λ' by

$$\lambda'(v) = (\lambda(v), |{\downarrow}v \cap \lambda^{-1}\lambda(v)| \mod B),$$

i.e. the first component of the label is the old label and the second counts modulo B the number of occurrences of the same action in the past of v. We then define $\mathrm{tr}(t) = (V, \leq, \lambda')$. Our first aim is to show that $\{\mathrm{tr}(t) \mid t \in {\downarrow}\mathrm{MSC}_B^\infty\}$ is a first-order axiomatizable set of traces in $\mathbb{R}(\Gamma, D)$.

Lemma 5.1. *Let* $t = (V, \leq, \lambda) \in {\downarrow}\mathrm{MSC}_B^\infty$ *be an ideal in an MSC. Then* $\mathrm{tr}(t)$ *is a trace over* (Γ, D).

Proof. Let $v_1, v_2 \in V$ with $\lambda(v_i) = p_i\theta_i q_i$. Suppose $v_1 \prec\!\!\cdot\, v_2$. Since the partial order $\leq$ is the transitive and reflexive closure of $\sqsubseteq_{\mathcal{P}} \cup \sqsubseteq$, there are two possibilities:

1. If $p_1 = p_2$, we obtain immediately $(\lambda'(v_1), \lambda'(v_2)) \in D$.
2. If $p_1 \neq p_2$, then $v_1 \sqsubseteq v_2$, *i.e.*, $\lambda(v_1) = p_1!p_2$, $\lambda(v_2) = p_2?p_1$, and $|{\downarrow}v_1 \cap \lambda^{-1}(p_1!p_2)| = |{\downarrow}v_2 \cap \lambda^{-1}(p_2?p_1)|$. Hence $(\lambda'(v_1), \lambda'(v_2)) \in D$.

Now suppose that v_1 and v_2 are incomparable. Then $p_1 \neq p_2$. By contradiction, we assume $(\lambda'(v_1), \lambda'(v_2)) \in D$. Then, w.l.o.g., $\lambda(v_1) = p!q$, $\lambda(v_2) = q?p$, and

$$|{\downarrow}v_1 \cap \lambda^{-1}(p!q)| \equiv |{\downarrow}v_2 \cap \lambda^{-1}(q?p)| \mod B. \qquad (\star)$$

Since $v_1 \not\leq v_2$ and $\lambda(v_1) = p!q$, we have $|{\downarrow}v_1 \cap \lambda^{-1}(p!q)| > |{\downarrow}v_2 \cap \lambda^{-1}(p!q)| \geq |{\downarrow}v_2 \cap \lambda^{-1}(q?p)|$. By $(\star)$, this implies $|{\downarrow}v_1 \cap \lambda^{-1}(p!q)| - |{\downarrow}v_2 \cap \lambda^{-1}(q?p)| \geq B$. Since, on the other hand, $v_2 \not\leq v_1$ and $\lambda(v_2) = q?p$, we also have $|{\downarrow}v_1 \cap \lambda^{-1}(q?p)| < |{\downarrow}v_2 \cap \lambda^{-1}(q?p)|$. Hence

$$\begin{aligned} B &\leq |{\downarrow}v_1 \cap \lambda^{-1}(p!q)| - |{\downarrow}v_2 \cap \lambda^{-1}(q?p)| \\ &< |{\downarrow}v_1 \cap \lambda^{-1}(p!q)| - |{\downarrow}v_1 \cap \lambda^{-1}(q?p)|. \end{aligned}$$

But this contradicts our assumption that t is B-bounded. ☐

We will consider the following three properties that can be expressed in first-order logic:

I. $s{\restriction}_{\{(p!q,n),(q?p,n)\}}$ is a prefix of $((p!q, n)(q?p, n))^\omega$ for $p, q \in \mathcal{P}$ and $0 \leq n < B$.
II. $s{\restriction}_{\{(\sigma,n) \mid 0 \leq n < B\}}$ is a prefix of $((\sigma, 1)(\sigma, 2) \ldots (\sigma, B-1)(\sigma, 0))^\omega$ for $\sigma \in \Sigma$.
III. If $v, w \in V$ with $v \prec\!\!\cdot\, w$, then $\mathrm{proc} \circ \lambda'(v) = \mathrm{proc} \circ \lambda'(w)$ or $\lambda'(v) = (p!q, n)$ and $\lambda'(w) = (q?p, n)$ for some $p, q \in \mathcal{P}$ and $0 \leq n < B$.

Lemma 5.2. *Let* $t = (V, \leq, \lambda) \in {\downarrow}\mathrm{MSC}_B^\infty$. *Then* $s = \mathrm{tr}(t)$ *satisfies I, II, and III.*

Proof. Let $t \in \mathrm{MSC}_B^\infty$ and let s be an ideal in t. Then $\mathrm{tr}(s)$ is an ideal in $\mathrm{tr}(t)$. Hence it suffices to prove the lemma for B-bounded MSCs.

Property II is immediate by the definition of $\mathrm{tr}(t)$. Next we show property I by contradiction. Since the n^{th} sending precedes the n^{th} receiving on the channel from p to q in t, the word $s\restriction_{\{(p!q,n),(q?p,n)\}}$ starts with $(p!q,n)$. Assume $s\restriction_{\{(p!q,n),(q?p,n)\}}$ contains a factor $(q?p,n)(q?p,n)$. Then there are $v_1, v_2 \in V$ with $\lambda'(v_i) = (q?p,n)$, $v_1 < v_2$, and there is no $v \in V$ with $v_1 < v < v_2$ and $\lambda'(v) = (p!q,n)$. Since t is an MSC, there exists $u \in V$ with $\lambda(u) = p!q$ and $|{\downarrow}u \cap \lambda^{-1}(p!q)| = |{\downarrow}v_2 \cap \lambda^{-1}(q?p)|$. Hence in particular $v_1 \not< u < v_2$, *i.e.*, $u < v_1$ since $\lambda'(u)$ and $\lambda'(v_1)$ are dependent. Now we consider the ideal ${\downarrow}u$:

$$\begin{aligned}|{\downarrow}u \cap \lambda^{-1}(p!q)| - |{\downarrow}u \cap \lambda^{-1}(q?p)| &= |{\downarrow}v_2 \cap \lambda^{-1}(q?p)| - |{\downarrow}u \cap \lambda^{-1}(q?p)| \\ &> |{\downarrow}v_2 \cap \lambda^{-1}(q?p)| - |{\downarrow}v_1 \cap \lambda^{-1}(q?p)|\end{aligned}$$

since $u < v_1$ and $\lambda(v_1) = q?p$. Since $|{\downarrow}v_2 \cap \lambda^{-1}(q?p)| > |{\downarrow}v_1 \cap \lambda^{-1}(q?p)|$ are congruent modulo B, we get $|{\downarrow}u\cap\lambda^{-1}(p!q)|-|{\downarrow}u\cap\lambda^{-1}(q?p)| > B$, a contradiction.

We also have to consider the case

$$s\restriction_{\{(p!q,n),(q?p,n)\}} \in ((p!q,n)(q?p,n))^\star(p!q,n)(p!q,n)W$$

for some word W: Similarly to above, there are $v_1, v_2 \in V$ with $\lambda'(v_i) = (p!q,n)$, $v_1 < v_2$, and there is no $v \in V$ with $v_1 < v < v_2$ and $\lambda'(v) = (q?p,n)$. Since t is an MSC, there exists $w \in V$ with $\lambda(w) = q?p$ and $|{\downarrow}w \cap \lambda^{-1}(q?p)| = |{\downarrow}v_1 \cap \lambda^{-1}(p!q)|$. Hence in particular $v_1 < w \not< v_2$, *i.e.*, $v_2 < w$. Now we consider the ideal ${\downarrow}v_2$:

$$\begin{aligned}&|{\downarrow}v_2 \cap \lambda^{-1}(p!q)| - |{\downarrow}v_2 \cap \lambda^{-1}(q?p)| \\ &\quad > |{\downarrow}v_2 \cap \lambda^{-1}(p!q)| - |{\downarrow}w \cap \lambda^{-1}(q?p)| \text{ since } v_2 < w \text{ and } \lambda(w) = q?p \\ &\quad = |{\downarrow}v_2 \cap \lambda^{-1}(p!q)| - |{\downarrow}v_1 \cap \lambda^{-1}(p!q)|\end{aligned}$$

again contradicting the B-boundedness of t.

It remains to be shown that s satisfies property III: Let $v, w \in V$ with $v \multimap w$ and $\mathrm{proc} \circ \lambda'(v) \neq \mathrm{proc} \circ \lambda'(w)$. Since $\leq = (\sqsubseteq_{\mathcal{P}} \cup \sqsubseteq)^\star$, this implies $v \sqsubseteq w$ and therefore $\lambda'(v) = (p!q,n)$ and $\lambda'(w) = (q?p,n)$ for some $p, q \in \mathcal{P}$ and $0 \leq n < B$. $\square$

Lemma 5.3. *Let $s \in \mathbb{R}(\Gamma, D)$ be a trace satisfying I, II and III. Then $s \in \mathrm{tr}[{\downarrow}\mathrm{MSC}_B^\infty]$.*

Proof. Let $s = (V, \leq, \lambda')$ and define $\lambda(v) = \pi_1 \circ \lambda'(v)$ for $v \in V$ where π_1 is the projection to the first component. Then $t = (V, \leq, \lambda)$ is a Σ-labeled poset. From II, we obtain immediately $\mathrm{tr}(t) = s$. So it remains to be shown that t is an ideal in a B-bounded MSC:

- Since $\gamma, \delta \in \Gamma$ are dependent whenever $\mathrm{proc}(\gamma) = \mathrm{proc}(\delta)$, the set $\lambda'^{-1} \circ \mathrm{proc}^{-1}(p)$ is linearly ordered in the trace s. Hence $\sqsubseteq_{\mathcal{P}}$ is a linear order on $\lambda^{-1}(\{p\theta q \mid q \in \mathcal{P}, \theta \in \{?,!\}\})$ for any $p \in \mathcal{P}$.

– Let $v \sqsubseteq w$, *i.e.*, $\lambda(v) = p!q$, $\lambda(w) = q?p$, and $|{\downarrow}v \cap \lambda^{-1}(p!q)| = |{\downarrow}w \cap \lambda^{-1}(q?p)|$. Then, by I, $v < w$. Hence $(\sqsubseteq_{\mathcal{P}} \cup \sqsubseteq)^\star \subseteq \leq$.
– Let $v \longrightarrow\!\!\!< w$. Then $(\lambda'(v), \lambda'(w)) \in D$. If $\mathrm{proc} \circ \lambda'(v) = \mathrm{proc} \circ \lambda'(w)$, then $(v, w) \in \sqsubseteq_{\mathcal{P}}$. Otherwise, by III, $\lambda'(v) = (p!q, n)$ and $\lambda'(w) = (q?p, n)$ for some $p, q \in \mathcal{P}$ distinct and $0 \leq n < B$. Hence, by I, $|{\downarrow}v \cap \lambda^{-1}(p!q)| = |{\downarrow}w \cap \lambda^{-1}(q?p)|$, *i.e.*, $v \sqsubseteq w$. Thus, we showed $\leq \subseteq (\sqsubseteq_{\mathcal{P}} \cup \sqsubseteq)^\star$.
– Suppose, by contradiction, that t is not B-bounded. Then there exists $v \in V$ with $|{\downarrow}v \cap \lambda^{-1}(p!q)| - |{\downarrow}v \cap \lambda^{-1}(q?p)| > B$. Hence, by II, there is n with $0 \leq n < B$ such that $|{\downarrow}v \cap \lambda'^{-1}(p!q, n)| - |{\downarrow}v \cap \lambda'^{-1}(q?p, n)| > 1$. But this contradicts I. □

Proposition 5.4. *Let $K \subseteq {\downarrow}\mathrm{MSC}_B^\infty$ by monadically axiomatizable relative to ${\downarrow}\mathrm{MSC}_B^\infty$. Then $\mathrm{tr}[K] \subseteq \mathbb{R}(\Gamma, D)$ is monadically axiomatizable relative to $\mathbb{R}(\Gamma, D)$.*

Proof. By Lemma 5.1, $\mathrm{tr}[{\downarrow}\mathrm{MSC}_B^\infty]$ is a set of traces. Note that the properties I, II, and III are expressible by a first-order sentence. Hence the set $\mathrm{tr}[{\downarrow}\mathrm{MSC}_B^\infty]$ is first-order axiomatizable relative to $\mathbb{R}(\Gamma, D)$ by Lemmas 5.2 and 5.3. Since the labeling of $t \in {\downarrow}\mathrm{MSC}_B^\infty$ can be recovered from the labeling of $\mathrm{tr}(t)$, we can interpret t in $\mathrm{tr}(t)$. □

So far, we transformed any monadically axiomatizable set of bounded MSCs into a monadically axiomatizable set of traces. In order to make use of this transformation, we need the following definitions and results from the theory of Mazurkiewicz traces.

Let $s = (V, \leq, \lambda')$ be a trace over (Γ, D) and let $A \subseteq \Gamma$. Then $\partial_A(t)$ is the least ideal in t such that the corresponding filter does not contain any A-labeled vertex. Let $a \in \Gamma$. Then $D(a) = \{b \in \Gamma \mid (a, b) \in D\}$. Furthermore, ta is the unique trace $(V \dot\cup \{\star\}, \leq', \rho)$ with $ta{\restriction}_V = t$, $\rho(\star) = a$, and $\star \in \max(ta)$. A mapping $\mu : \mathbb{M}(\Gamma, D) \to A$ is *asynchronous* if, for any $\Delta_1, \Delta_2 \subseteq \Gamma$, any $\gamma \in \Gamma$, and any $t \in \mathbb{M}(\Gamma, D)$,

1. $\mu(\partial_{\Delta_1 \cup \Delta_2}(t))$ is completely determined by $\mu(\partial_{\Delta_1}(t))$, $\mu(\partial_{\Delta_2}(t))$, and the sets Δ_1 and Δ_2, and
2. $\mu(\partial_\gamma(t\gamma))$ is completely determined by $\mu(\partial_{D(\gamma)}(t))$ and the letter γ.

Theorem 5.5 ([Tho90b,CMZ93]). *Let (Γ, D) be a dependence alphabet and $L \subseteq \mathbb{M}(\Gamma, D)$. Then L is monadically axiomatizable if, and only if, there exists an asynchronous mapping μ into some finite set such that $L = \mu^{-1}\mu[L]$.*

This result was used to construct a deterministic asynchronous cellular automaton that accepts a given recognizable language of finite traces. Diekert & Muscholl [DM94] use the same concept of an asynchronous mapping to construct a deterministic asynchronous cellular automaton with Muller acceptance condition that accepts a given recognizable set of infinite traces. In order to state their result, we need some more notations:

Let (Γ, D) be a dependence alphabet, $t = (V, \leq, \lambda') \in \mathbb{R}(\Gamma, D)$ a trace, and $\gamma \in \Gamma$. If $\lambda'^{-1}(\gamma)$ is infinite, then $\mu_\gamma^\infty(t) \subseteq A$ is the set of all $a \in A$ for which there are infinitely many nodes $v \in V$ with $\lambda'(v) = \gamma$ and $\mu(t\restriction_{\downarrow v}) = a$. If $\lambda'^{-1}(\gamma)$ is finite, $\mu_\gamma^\infty(t) = \{\mu(\partial_\gamma(t)\}$.

Theorem 5.6 ([DM94,EM96]). *Let (Γ, D) be a dependence alphabet and $L \subseteq \mathbb{R}(\Gamma, D)$ be monadically axiomatizable. Then there exists a finite set A, a set $\mathcal{T} \subseteq \prod_{\gamma\in\Gamma} 2^A$ of Γ-tuples of subsets of A, and an asynchronous mapping $\mu : \mathbb{M}(\Gamma, D) \to A$ such that for $t \in \mathbb{R}(\Gamma, D)$, we have:*

$$t \in L \iff (\mu_\gamma^\infty(t))_{\gamma\in\Gamma} \in \mathcal{T}.$$

This theorem is not stated in this form in [DM94], but it can be extracted from the proof of their Theorem 4.2.

5.2 The construction of deterministic message passing automata

A *message passing automaton* is a structure $\mathcal{A} = ((\mathcal{A}_p)_{p\in\mathcal{P}}, \Delta, s_{in}, F)$ where

1. Δ is a finite set of messages,
2. each component $\mathcal{A}_p$ is of the form $(S_p, \to_p)$ where
   - S_p is a finite set of local states,
   - $\to_p \subseteq S_p \times \Sigma_p \times \Delta \times S_p$ where $\Sigma_p = \{\sigma \in \Sigma \mid \mathrm{proc}(\sigma) = p\}$ is a local transition relation,
3. $s^{in} \in \prod_{p\in\mathcal{P}} S_p$ is the global initial state, and
4. $F \subseteq \prod_{p\in\mathcal{P}} S_p$ is a set of accepting global states.

Let $(s, a, m, s') \in \to_p$ be a local transition of process p. Suppose a is a send event, *i.e.*, $a = p!q$ for some process q. Then the transition (s, a, m, s') denotes that the process p can perform the action $a = p!q$ in state s; it changes its local state to s' and sends a message m into the FIFO-channel from process p to process q. By enlarging the set of messages and local states (if necessary), we can assume that $m = s'$ for any send action (in particular, $\Delta = \bigcup_{p\in\mathcal{P}} S_p$). Now suppose that $a = p?q$ is a receive action. Then the transition (s, a, m, s') denotes that the process p can change its local state from s to s' when reading the message m from the channel that connects p and q.

A message passing automaton is *deterministic* if

- $(s, p!q, m_1, s_1), (s, p!q, m_2, s_2) \in \to_p$ imply $s_1 = s_2$ (and therefore $m_1 = m_2$ by our assumption $m_i = s_i$)
- $(s, p?q, m, s_1), (s, p?q, m, s_2) \in \to_p$ imply $s_1 = s_2$.

The message passing automaton is *complete*, if

- there exists a transition $(s, p!q, m, s')$ for any $s \in S_p$ and $q \in \mathcal{P}$
- there exists a transition $(s, p?q, m, s')$ for any $s \in S_p$, $m \in \Delta$, and $q \in \mathcal{P}$

Let $t = (V, \leq, \lambda)$ be an ideal in an MSC and let $\mathcal{A}$ be a message passing automaton. Let furthermore $r : V \to \bigcup_{p \in \mathcal{P}} S_p$ be a mapping and $v \in V$. We define a second mapping $r^- : V \to \bigcup_{p \in \mathcal{P}} S_p$: if there is $u < v$ with $\mathrm{proc}(u) = \mathrm{proc}(v)$, let u be maximal with this property and let $r^-(v)$ denote $r(u)$. If v is the minimal event performed by the process $\mathrm{proc}(v)$, let $r^-(v) = s^{in}_{\mathrm{proc}(v)}$.

A *run* of $\mathcal{A}$ on t is a mapping $r : V \to \bigcup_{p \in \mathcal{P}} S_p$ that satisfies the following for any $v \in V$:

1. If $\lambda(v) = p!q$, then there is a transition $(r^-(v), p!q, m, r(v))$ in $\to_p$ for some message m (which turns out to be $r(v)$ by our assumption).
2. Now let $\lambda(v) = p?q$. Since t is an ideal in an MSC, there is a unique matching node $u \in V$ with $u \sqsubseteq v$. We require that $(r^-(v), p?q, r(u), r(v)) \in \to_p$.

Let $t = (V, \leq, \lambda)$ be a finite ideal in an MSC and let r be a run of the message passing automaton $\mathcal{A}$ on t. For $p \in \mathcal{P}$ let $f_p = s^{in}_p$ if the process p does not perform any action in t. Otherwise, let $f_p = r(v)$ where v is the last action performed by the process p. The run r is *successful* if the tuple f is an accepting state in F. A finite ideal t in an MSC is *accepted* by a message passing automaton $\mathcal{A}$ if there is a successful run of $\mathcal{A}$ on t. A set K is *accepted by $\mathcal{A}$ relative to $X \subseteq {\downarrow}\mathrm{MSC}$* if, for any $t \in X$, $t \in L$ iff t is accepted by $\mathcal{A}$.

Above, we associated to any monadically axiomatizable subset of ${\downarrow}\mathrm{MSC}_B$ a monadically axiomatizable set of traces. By Theorem 5.5, we therefore get an asynchronous mapping. Next, we construct a message passing automaton from an asynchronous mapping.

Proposition 5.7. *Let $\mu : \mathbb{M}(\Gamma, D) \to A$ be some asynchronous mapping into a finite set A. Then there exists a complete deterministic message passing automaton $\mathcal{A}$ with local state space S and a function $f : S \to A$ such that for the run r of $\mathcal{A}$ on $t = (V, \leq, \lambda) \in {\downarrow}\mathrm{MSC}^\infty_B$, we have $f(r(v)) = \mu(\mathrm{tr}(t\restriction_{\downarrow v}))$ for any $v \in V$.*

Proof. We construct the MPA with local state space $S = A \times (\{0, 1, \ldots, B - 1\}^{|\mathcal{P}|})^2 \times \Sigma$ and message set A. Then $f(a, \mathrm{snd}, \mathrm{rcv}, \sigma) = a$ for $(a, \mathrm{snd}, \mathrm{rcv}, \sigma) \in S$. A run r on $t = (V, \leq, \lambda) \in {\downarrow}\mathrm{MSC}^\infty_B$ will satisfy:

$$\begin{aligned} r(v) &= (\mu(\mathrm{tr}(t\restriction_{\downarrow v})), \mathrm{snd}, \mathrm{rcv}, \lambda(v)) \text{ where} \\ \mathrm{snd}(q) &= |{\downarrow}v \cap \lambda^{-1}(p!q)| \mod B \text{ and} \\ \mathrm{rcv}(q) &= |{\downarrow}v \cap \lambda^{-1}(p?q)| \mod B \text{ for } q \in \mathcal{P} \end{aligned}$$

for $v \in V$ with $\mathrm{proc}(v) = p$.

The second, third and fourth components can easily be computed (even without sending any messages as they only refer to the current process). For the first component, *i.e.* the image of the asynchronous mapping, observe the following:

1. If the current action $\lambda(v)$ is a send action, say $p!q$, the ideal ${\downarrow}v \setminus \{v\}$ is a principal ideal whose maximal node v' is the p-predecessor of the node v.

Hence $\mu(\mathrm{tr}(t\lceil_{\downarrow v}))$ is uniquely given by $\mu(\mathrm{tr}(t\lceil_{\downarrow v'}))$, the letter $p!q$, and the number of $p!q$-events that have occurred so far (modulo B). As all this is known when executing v, we can compute $\mu(\mathrm{tr}(t\lceil_{\downarrow v}))$. This value is send by the action to the process q.

2. Now suppose $\lambda(v) = p?q$. Then there exists a unique node v_1 with $\lambda(v_1) = q!p$ and $v_1 \sqsubseteq_{Ch} v$. By the consideration above, v reads from the channel $q \to p$ the value $\mu(\mathrm{tr}(t\lceil_{\downarrow v_1}))$. Before executing v, the process p is in some state $(\mu(\partial_{\mathrm{proc}^{-1}(p)}(\mathrm{tr}(t\lceil_{\downarrow v\setminus\{v\}})), \mathrm{snd}, \mathrm{rcv}, \sigma)$. Let v_2 be the maximal node of $\partial_{\mathrm{proc}^{-1}(p)}(\mathrm{tr}(t\lceil_{\downarrow v\setminus\{v\}}))$. Then $\downarrow v \setminus \{v\} = \downarrow v_1 \cup \downarrow v_2$. Hence we can compute $\mu(\mathrm{tr}(\downarrow v) \setminus \{v\})$ and therefore $\mu(\mathrm{tr}(\downarrow v))$.

□

5.3 The finite case

Example 5.8. Let K denote the set of all MSCs of the form

$$(p!q)^n(p!q')(q'?p)(q'!q)(q?q')(q?p)^m$$

for some $m, n \in \mathbb{N}$ with $m \leq n$. Due to the factor $(p!q')(q'?p)(q'!q)(q?q')$, K consists of linear orders, only. This set can be accepted by an MPA relative to $\downarrow$MSC: the process p performs any sequence from $(p!q)^\star(p!q')$ and stops in an accepting state. The process q' performs $(q'?p)(q'!q)$ before moving to an accepting state. Finally, the process q performs any sequence from $(q?q')(q?p)^\star$ and accepts. The only ideals in MSCs that allow a run of this MPA correspond to the words $(p!q)^n(p!q')(q'?p)(q'!q)(q?q')(q?p)^m$ with $m \leq n$ (since the read actions $q?p$ can only be performed if their is a message in the channel from p to q). But this set is not monadically axiomatizable.

The example above shows that in general the expressive power of MPAs and that of monadic second order logic differ. The reason is that an MPA is actually a device with infinitely many internal states since a priori the channels are not bounded. The situation changes if we restrict to sets of bounded MSCs.

Proposition 5.9. *Let $K \subseteq \downarrow\mathrm{MSC}_B$ be accepted by the message passing automaton $\mathcal{A}$ relative to $\downarrow\mathrm{MSC}_B$. Then K is monadically axiomatizable relative to $\downarrow\mathrm{MSC}_B$.*

Proof. The proof is an obvious variant of similar proofs for finite automata for words (*cf.* [Tho90a]), asynchronous automata for traces [Tho90b], or asynchronous cellular automata for pomsets without autoconcurrency [DGK00]. The only difficulty is to monadically define the relation $\sqsubseteq$ in $t = (V, \leq, \lambda) \in \downarrow\mathrm{MSC}_B$. Let $u, v \in V$ be nodes with $\lambda(u) = p!q$ and $\lambda(v) = q?p$. Then $u \sqsubseteq v$ iff $|\downarrow u \cap \lambda^{-1}(p!q)| = |\downarrow v \cap \lambda^{-1}(q?p)|$. Since t is B-bounded, this is equivalent to $|\downarrow u \cap \lambda^{-1}(p!q)| \equiv |\downarrow v \cap \lambda^{-1}(q?p)| \mod B$, $u < v$, and there are at most B actions labeled $p!q$ in the interval $[u,v]$. Since this is expressible in monadic second order logic, the relation $\sqsubseteq$ can be defined. The rest of the proof are the standard arguments. $\square$

Lemma 5.10. *Let $K \subseteq {\downarrow}\mathrm{MSC}_B$ be monadically axiomatizable. Then there is a deterministic message passing automaton that accepts K relative to ${\downarrow}\mathrm{MSC}_B$.*

Proof. By Proposition 5.4, $L = \mathrm{tr}[K] \subseteq \mathbb{M}(\Gamma, D)$ is monadically axiomatizable relative to $\mathbb{M}(\Gamma, D)$. By Theorem 5.5, there exists an asynchronous mapping $\mu : \mathbb{M}(\Gamma, D) \to A$ into some finite set A such that $L = \mu^{-1}\mu[L]$. Let $\mathcal{A}$ denote the deterministic message passing automaton from Proposition 5.7. Let $t \in {\downarrow}\mathrm{MSC}_B$. Then there is precisely one run of $\mathcal{A}$ on t. From the final state reached by this automaton, we can infer $\mu(\mathrm{tr}(t))$, *i.e.* whether $\mathrm{tr}(t)$ belongs to L or not. But this is the case iff $t \in K$. $\square$

Lemma 5.11. *The set ${\downarrow}\mathrm{MSC}_B$ is first-order axiomatizable relative to ${\downarrow}\mathrm{MSC}$.*

Proof. Let $t = (V, \leq, \lambda) \in {\downarrow}\mathrm{MSC}$. Then $t \notin {\downarrow}\mathrm{MSC}_B$ iff there exists $v \in V$ with $\lambda(v) = p!q$ such that the number of $p!q$-events in ${\downarrow}v$ exceeds that of $q?p$-events by more that B. Hence $t \notin {\downarrow}\mathrm{MSC}_B$ iff there exist $v \in V$ and $p, q \in \mathcal{P}$ such that the lexicographic normal form of $t\restriction_{\downarrow v}$ does not belong to $\bigcup_{0 \leq n \leq B} L_{p,q,B,n}$ (*cf.* Lemma 3.1). Since this is expressible in first-order logic by Lemmas 4.5 and 3.1, the result follows. $\square$

Lemma 5.12. *The set ${\downarrow}\mathrm{MSC}_{B-1}$ can be accepted by a deterministic message passing automaton for any $B > 1$ relative to ${\downarrow}\mathrm{MSC}$.*

Proof. Let

$$H = \{t \in {\downarrow}\mathrm{MSC} \mid \text{ any proper ideal in } t \text{ belongs to } {\downarrow}\mathrm{MSC}_B, \text{ but } t \notin {\downarrow}\mathrm{MSC}_B\} .$$

By Lemma 5.11, H is a first-order axiomatizable subset of ${\downarrow}\mathrm{MSC}_{B+1}$ and any of its elements has a largest node. Hence $\mathrm{tr}[H] \subseteq \mathbb{M}(\Gamma, D)$ is monadically axiomatizable, *i.e.*, there exists an asynchronous mapping $\mu : \mathbb{M}(\Gamma, D) \to A$ into some finite set that recognizes $\mathrm{tr}[H]$ (Theorem 5.5). Let $\mathcal{A}$ be the deterministic MPA from Proposition 5.7. Then, for the run r of $\mathcal{A}$ on $t = (V, \leq, \lambda) \in {\downarrow}\mathrm{MSC}$, the value $r(v)$ determines whether $t\restriction_{\downarrow v} \in H$. Since ${\downarrow}\mathrm{MSC} \setminus {\downarrow}\mathrm{MSC}_{B-1} = \{t \in {\downarrow}\mathrm{MSC} \mid \forall v \in V : t\restriction_{\downarrow v} \notin H\}$, the set ${\downarrow}\mathrm{MSC} \setminus {\downarrow}\mathrm{MSC}_{B-1}$ can be accepted by a deterministic MPA relative to ${\downarrow}\mathrm{MSC}$. But this implies that ${\downarrow}\mathrm{MSC}_{B-1}$ can be accepted by a deterministic MPA relative to ${\downarrow}\mathrm{MSC}$. $\square$

Recall that Mukund, Narayan Kumar, & Sohoni showed that any recognizable set of MSCs K is B-bounded for some B and can be accepted by a deterministic MPA relative to MSC_B. A preliminary version of it, claiming the existence of a *non*deterministic MPA was shown in [HMKT00b]. The following theorem strenghtens the result by Mukund *et al.* since it implies that K can be accepted relative to all MSCs.

Theorem 5.13 (*cf.* [MKS00, Thm. 3.2]). *Let $K \subseteq {\downarrow}\mathrm{MSC}$. Then K is recognizable iff K is bounded and there exists a deterministic message passing automaton that accepts L relative to ${\downarrow}\mathrm{MSC}$.*

Proof. By Theorem 4.11, K is monadically axiomatizable. By Proposition 4.2, K is bounded, *i.e.*, there is $B \in \mathbb{N}$ with $K \subseteq {\downarrow}\mathrm{MSC}_B$. Hence, by Lemma 5.10, there exists a deterministic message passing automaton $\mathcal{A}_1$ such that $t \in {\downarrow}\mathrm{MSC}_B$ is accepted by $\mathcal{A}_1$ iff it belongs to K.

By Lemma 5.12, ${\downarrow}\mathrm{MSC}_B$ can be accepted relative to ${\downarrow}\mathrm{MSC}$ by a deterministic MPA. Hence K is the intersection of two sets that are acceptable by deterministic MPAs relative to ${\downarrow}\mathrm{MSC}$. □

5.4 The infinite case

A *Muller message passing automaton* is a structure $\mathcal{A} = ((\mathcal{A}_p)_{p\in\mathcal{P}}, \Delta, s_{in}, \mathcal{S})$ where

1. $((\mathcal{A}_p)_{p\in\mathcal{P}}, \Delta, s_{in}, \emptyset)$ is a deterministic message passing automaton and
2. $\mathcal{S} \subseteq \prod_{p\in\mathcal{P}} 2^{S_p}$ is a Muller acceptance condition.

Let $r : V \to \bigcup_{p\in\mathcal{P}} S_p$ be a run of a Muller message passing automaton on $t = (V, \leq, \lambda) \in {\downarrow}\mathrm{MSC}^\infty$. For $p \in \mathcal{P}$, let $X_p \subseteq S_p$ be the set of all $s \in S_p$ such that, for any $v \in V$ with $\mathrm{proc}(v) = p$, there exists $w \in V$ with $v \leq w$, $\mathrm{proc}(w) = p$, and $r(w) = s$. The run r is *successful* provided $(X_p)_{p\in\mathcal{P}} \in \mathcal{S}$.

By Lemma 5.12, ${\downarrow}\mathrm{MSC}_{B-1}$ can be accepted by a deterministic message passing automaton for any $B > 1$. Using the same arguments, one can also show that ${\downarrow}\mathrm{MSC}^\infty_{B-1}$ can be accepted by a Muller MPA, a fact that we will use in the following proof.

Theorem 5.14. *Let $K \subseteq {\downarrow}\mathrm{MSC}^\infty$. Then K is recognizable iff it is bounded and there exists a Muller message passing automaton that accepts K relative to ${\downarrow}\mathrm{MSC}^\infty$.*

Proof. Similarly to Proposition 5.9, one can show that any Muller MPA corresponds to a sentence of MSO. Hence it accepts a recognizable set.

By Theorem 5.6, there exists an asynchronous mapping $\mu : \mathbb{M}(\Gamma, D) \to A$ into some finite set A and a set $\mathcal{T} \subseteq \prod_{p\in\mathcal{P}} 2^A$ that accept $\mathrm{tr}[K] \subseteq \mathbb{R}(\Gamma, D)$. Let $\mathcal{A}$ be the MPA from Proposition 5.7. For a local state $(a, \mathrm{snd}, \mathrm{rcv}, \sigma) \in S_p$, an action $\tau \in \Sigma$, and an integer $0 \leq i < B$, we define

$$\pi_{(\tau,i)}(a, \mathrm{snd}, \mathrm{rcv}, \sigma) = \begin{cases} q & \text{if } \tau = \sigma \text{ and } (\mathrm{snd}, \mathrm{rcv})(\sigma) = i \\ \perp & \text{otherwise} \end{cases}$$

where $(\mathrm{snd}, \mathrm{rcv})(a) = \mathrm{snd}(p')$ if $a = p!p'$ and $\mathrm{rcv}(p')$ if $a = p?p'$. Let $\mathcal{S}$ comprise all tuples $(S_p)_{p\in\mathcal{P}}$ satisfying $(\pi_\gamma[S_{\mathrm{proc}(\gamma)}])_{\gamma\in\Gamma} \in \mathcal{T}$.

Let $t = (V, \leq, \lambda) \in {\downarrow}\mathrm{MSC}_B^\infty$, $a \in A$, and $(\sigma, i) \in \Gamma$. Then $q \in \mu^\infty_{(\sigma,i)}(\mathrm{tr}(t))$ iff $\mathrm{tr}(t)$ has infinitely many (σ, i)-prefixes u with $\mu(u) = a$. But this is the case iff for infinitely many $v \in V$ we have $\pi_1 \circ r(v) = a$, $\lambda(v) = \sigma$, and $i = |{\downarrow}v \cap \lambda^{-1}\lambda(v)|$ mod B. This last statement is equivalent to saying that the process $\mathrm{proc}(\sigma)$ is infinitely often in a state $(a, \mathrm{snd}, \mathrm{rcv}, \sigma)$ with $(\mathrm{snd}, \mathrm{rcv})(\sigma) = i$, *i.e.*, in a state s with $\pi_{(\sigma,i)}(s) = a$. Hence, relative to ${\downarrow}\mathrm{MSC}_B^\infty$, the Muller MPA accepts K. Since, as we remarked above, ${\downarrow}\mathrm{MSC}_B^\infty$ can be accepted relative to ${\downarrow}\mathrm{MSC}^\infty$, the theorem is proved. $\square$

Concluding remark Let $K \subseteq {\downarrow}\mathrm{MSC}_B$ be recognizable. We want to compute the number of local states of the deterministic message passing automaton that accepts K: Let $n = |\mathcal{P}|$ be the number of processes. Let m be the number of states of the minimal automaton accepting $\mathrm{Lin}_\omega[K]$.

First, we determine the size of the syntactic monoid of $\mathrm{tr}[{\downarrow}\mathrm{MSC}_B]$: To check property I for a given triple (p, q, n) with $p, q \in \mathcal{P}$ and $0 \leq i < B$, a monoid of size 6 suffices. To check property II for $\sigma \in \Sigma$, we can use a monoid of size $B^2 + 2$. Finally, to check property III, we have to keep track of the minimal and maximal nodes in a trace, *i.e.*, $4^{|\Gamma|} + 1 = 4^{n^2 B} + 1$ monoid elements suffice. Hence, alltogether, the syntactic monoid of $\mathrm{tr}[{\downarrow}\mathrm{MSC}_B]$ contains at most

$$6^{n^2 B} \cdot 2n^2(B^2 + 2) \cdot (4^{n^2 B} + 1)$$

elements, and that for $\mathrm{tr}[K]$ at most

$$m^m \cdot 6^{n^2 B} \cdot 2n^2(B^2 + 2) \cdot (4^{n^2 B} + 1).$$

By [CMZ93,DM94], the set of traces $\mathrm{tr}[K]$ can be accepted by an asynchronous mapping into a set of size

$$\begin{aligned} &(|\Gamma| + 1)^{|\Gamma|^2} \cdot m^m \cdot 6^{n^2 B} \cdot 2n^2(B^2 + 2) \cdot (4^{n^2 B} + 1) \cdot 2^{|\Gamma|} \\ = &(n^2 B + 1)^{(n^2 B)^2} \cdot m^m \cdot 6^{n^2 B} \cdot 2n^2(B^2 + 2) \cdot (4^{n^2 B} + 1) \cdot 2^{n^2 B} \end{aligned}$$

Hence, the deterministic message passing automaton that we constructed has

$$\begin{aligned} &(n^2 B + 1)^{(n^2 B)^2} \cdot m^m \cdot 6^{n^2 B} \cdot 2n^2(B^2 + 2) \cdot (4^{n^2 B} + 1) \cdot 2^{n^2 B} \cdot B^{2n^2} \cdot |\Sigma| \\ = &(n^2 B + 1)^{(n^2 B)^2} \cdot m^m \cdot 6^{n^2 B} \cdot 2n^2(B^2 + 2) \cdot (4^{n^2 B} + 1) \cdot 2^{n^2 B} \cdot B^{2n^2} \cdot 2n^2 \end{aligned}$$

local states. But this is in $2^{m \log m + O((n^2 B)^2 \log(n^2 B))}$. Recall that the construction from [MKS00] needed $2^{2^{O(n^2 B)} m \log m}$ local states. Thus, with growing buffer size B, their construction is less efficient than the one we presented.

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