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On random relational structures

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Abstract

Erdős & Rényi gave a probabilistic construction of the countable universal homogeneous graph. We extend their result to more general structures of first order predicate calculus. Our main result shows that if a class of countable relational structures $\mathcal{C}$ contains an infinite ω-categorical universal homogeneous structure $\mathbb{U}$, then $\mathbb{U}$ can be constructed probabilistically.

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1 Introduction

The investigation of universal structures dates back at least to Cantor [Can95] who showed that in the class of countable linear orderings, the chain $(\mathbb{Q}, <)$ is universal, meaning that every countable linear ordering $(L, <)$ can be order-embedded into $(\mathbb{Q}, <)$. Fraïssé [Fra54] gave a model-theoretic method for more general classes $\mathcal{C}$ of countable relational structures to show that $\mathcal{C}$ contains a universal object $\mathbb{U}$. By this construction, $\mathbb{U}$ then turns out to be also homogeneous, i.e., each isomorphism between two finite substructures of $\mathbb{U}$ extends to an automorphism of $\mathbb{U}$, and, moreover , such universal and homogeneous objects $\mathbb{U} \in \mathcal{C}$ are unique up to isomorphism. For a wealth of applications of this result in model theory and infinite permutation group theory, we refer the reader to Hodges [Hod93], Cameron [Cam90], and Macpherson & Kaye [KM94].

Fraïssé's result applies, in particular, to the classes of all countable undirected graphs and countable partial orders, respectively; both contain a universal homogeneous object. Erdős & Rényi [ER63] gave a probabilistic construction of countable graphs which, with probability 1, produces the universal homogeneous graph, therefore also called the random graph. This method decides independently and randomly for each pair of nodes whether they are connected by an edge. This amazing result spurred much further research on probabilistic laws

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in finite model theory and also on the algebraic structure of the automorphism group of the random graph [Tru85, Tru89, MSST93]. However, it applied only to graph- or hypergraph-like structures, and several authors (e.g., Hodges [Hod93, p. 352] and Cameron [Cam90]) asked for an extension to other classes of relational structures. In [BR90], Bankston & Ruitenburg considered a probabilistic method to construct arbitrary relational structures that differs from the one by Erdős & Rényi: They construct successive one-point extensions of finite structures, employing probabilities assigned to these one-point extensions. Hence their probability measure can be seen as an 'outer' one since it takes into account complete structures.

In this paper, we wish to present an 'inner' method of probabilistic constructions on classes $\mathcal{C}$ of relational structures, and we show, that if $\mathcal{C}$ contains an infinite ω-categorical universal homogeneous object $\mathbb{U}$, then such a construction produces $\mathbb{U}$ with probability 1. Recall that Erdős & Rényi's probabilistic method chooses independently and randomly for any pair of natural numbers whether they are joined by an edge. Similar to this, for instance here in the case of partial orders, we choose the order and non-order (i.e., incomparability) edges for pairs of elements randomly and successively, subject only to staying inside the class of partial orders. Then, using a natural enumeration of the random choices, with probability 1 we obtain the universal homogeneous poset. For linear orders, the same procedure produces the dense linear ordering without endpoints. Although our construction is obviously similar to Erdős & Renyi's given for graphs, we point out a difference occurring already for the case of linear orders: here, the outcome of the procedure might depend on the enumeration of random choices; we show that a dynamic enumeration results in another linear order. In the case of more general relational structures, our method decides randomly and successively whether a relation holds for a given tuple of elements or not, and the method also extends to structures of first order predicate calculus with function symbols.

For the random graph, there is also an explicit number-theoretic description [Rad64]. Here, we present such a description for the universal homogeneous poset.

2 Probabilistic constructions: the random poset

For the convenience of the reader and the sake of comparison, we first recall a construction of the random graph (Erdős & Rényi [ER63], cf. Hodges [Hod93, pp. 351/2], Cameron [Cam90, pp. 86 ff.]). As underlying set, we take $\mathbb{N}$, the natural numbers. Choose an enumeration of all 2-subsets $S_i = \{a_i, b_i\}$ $(i \in \mathbb{N})$ of $\mathbb{N}$. Then toss a fair coin to decide whether a_i and b_i become connected by an edge or not. Since the choices are completely independent of each other, they can be made by following the given enumeration, any other order, or even simultaneously and in parallel. Then, in any case, with probability 1 we obtain the universal homogeneous graph.

Next we wish to deal with partial orders to illustrate our method. It is an adaptation to partial orders and simplification of the more general construction for arbitrary relational structures presented later. As is well-known, since the class of finite partial orders (as the class of finite graphs) satisfies the amalgamation property, by Fraïssé's Theorem [Fra54] (see Proposition 2 below), there is a countable universal homogeneous poset $\mathbb{P}$, which is unique up to isomorphism with these properties. Our probabilistic construction of a partial order is similar to the above, but we proceed successively as prescribed by the chosen enumeration of the 2-subsets. So, at stage j, we decide randomly whether $a_j < b_j$ or $b_j < a_j$ or a_j and b_j are incomparable. Since we have to ensure that we obtain a partial ordering, this choice might be restricted by the choices done at earlier stages $i < j$. But this (obviously necessary) requirement that we stay inside the class of partial orders is our only restriction. The details are as follows, where we just note that the precise definition of the employed probability space on $\{0,1\}^\omega$ is the standard one as used for the random graph, cf. [Cam90, Hod93].

Probabilistic construction of partial orders As underlying set, we take again $\mathbb{N}$. Let $s = (S_i)_{i\in\omega}$ be an enumeration of the 2-subsets $S_i = \{a_i, b_i\}$ of $\mathbb{N}$. We will construct a strict partial order $\prec$ on $\mathbb{N}$ with associated incomparability relation $\|$ (i.e., $a \parallel b$ iff neither $a \preceq b$ nor $b \preceq a$ for $a, b \in \mathbb{N}$). We wish to decide successively (and randomly) for $i = 0, 1, 2, \ldots$ whether $a_i \prec b_i$ or $a_i \succ b_i$ or $a_i \parallel b_i$; "in the end", after ω steps, we obtain our partial order $(\mathbb{N}, \prec)$.

Let $j \in \omega$ and assume that we already have defined two binary relations $\prec_j$ and $\|_j$ on $P_j := \bigcup_{i<j} S_i$ (with $P_0 = \emptyset$) such that

(a) $(P_j, \prec_j)$ is a strict partial order,

(b) $\|_j$ is irreflexive and symmetric, and $\prec_j \cap \|_j = \emptyset$,

(c) for $w, x, y, z \in P_k$:

- if $x \prec_j y \prec_j z$, $w \parallel_j x$ and $w \parallel_j z$, then $w \parallel_j y$,
- if $x \prec_j y$, $w \prec_j z$ and $w \parallel_j y$ and $x \parallel_j z$, then $w \parallel_j x$ and $y \parallel_j z$,

(d) for each $0 \leq i < j$, either $a_i \prec_j b_i$ or $b_i \prec_j a_i$, or $a_i \parallel_j b_i$.

We wish to define $\prec_{j+1}$ and $\|_{j+1}$ on P_{j+1} such that $\prec_j \subseteq \prec_{j+1}$, $\|_j \subseteq \|_{j+1}$, and (a)-(d) hold correspondingly for $(P_{j+1}, \prec_{j+1}, \|_{j+1})$.

Case 1. If $a_j \notin P_j$ or $b_j \notin P_j$, we may put either $a_j \prec_{j+1} b_j$ or $b_j \prec_{j+1} a_j$ or $a_j \parallel_{j+1} b_j$, and we decide on this choice fairly with probability $\frac{1}{3}$. In the first two cases, we let $\prec_{j+1}$ be the transitive closure of $\prec_j \cup \{(a_j, b_j)\}$ resp. of $\prec_j \cup \{(b_j, a_j)\}$, and we put $\|_{j+1} = \|_j$. If $a_j \parallel_{j+1} b_j$, we put $\prec_{j+1} = \prec_j$ and $\|_{j+1} = \|_j \cup \{(a_j, b_j), (b_j, a_j)\}$. This ensures (a)-(d) for the structure $(P_{j+1}, \prec_{j+1}, \|_{j+1})$.

Case 2. Now assume $a_j, b_j \in P_j$. Then we will distinguish four subcases.
Subcase A Either $a_j \prec_j b_j$ or $b_j \prec_j a_j$ or $a_j \parallel b_j$. In this case, there is no freedom: we put $\prec_{j+1}=\prec_j$ and $\parallel_{j+1}=\parallel_j$.

Assume we are not in subcase A. There could be one of the following side-restrictions:

(i) There is $x \in P_j$ with $x \prec_j a_j$ and $x \parallel_j b_j$, or there is $y \in P_j$ with $b_j \prec_j y$ and $y \parallel_j a_j$. This prevents $a_j \prec_{j+1} b_j$.

(ii) The dual of (i), which prevents $b_j \prec_{j+1} a_j$.

Assumption (c) implies that (i) and (ii) cannot both hold, since then we would have $a_j \parallel_j b_j$. Hence there are three cases:
Subcase B If (i) holds, we may put $a_j \parallel_{j+1} b_j$ or $b_j \prec_{j+1} a_j$, and we decide on this with a fair coin.
Subcase C If (ii) holds, we decide on $a_j \parallel_{j+1} b_j$ or $a_j \prec_{j+1} b_j$.
Subcase D If neither (i) nor (ii) holds, we choose fairly between $a_j \prec_{j+1} b_j$, $b_j \prec_{j+1} a_j$, and $a_j \parallel_{j+1} b_j$.

In each case, we let $\prec_{j+1}$ be the transitive closure of $\prec_j$ and possibly the new pair, and we let $\parallel_{j+1}$ be the smallest relation containing $\parallel_j$ and possibly (a_j, b_j) satisfying conditions (b) and (c).

Then let $\prec=\bigcup_{j\in\omega} \prec_j$. It follows that the associated incomparability relation $\parallel$ equals $\bigcup_{j\in\omega} \parallel_j$ since, for each $j \in \omega$, either $a_j \prec_{j+1} b_j$ or $a_j \succ_{j+1} b_j$ or $a_j \parallel_{j+1} b_j$.

The results Note that the above procedure depends on the enumeration s of the 2-subsets of $\mathbb{N}$. Next we show how the chosen enumeration influences the outcome of the procedure.

We call the enumeration $s = s_2 s_3 s_4 \ldots$ with $s_j = (\{(1, j), (2, j), \ldots, (j-1, j)\}$ the *natural enumeration* of the 2-subsets of $\mathbb{N}$. We will show

Theorem 1 *Applied to the natural enumeration of the 2-subsets of* $\mathbb{N}$*, the above probabilistic construction of the partial order* $(\mathbb{N}, \prec)$ *yields with probability 1 the universal homogeneous poset.*

Before turning to the proof, let us recall some well-known model-theoretic facts which will be useful also later on. Let L be a language of first-order predicate calculus containing possibly infinitely many relation and function symbols. If $\mathbb{B}$ is an L-structure and $A \subseteq B$, we denote by $\langle A\rangle$ the L-substructure of $\mathbb{B}$ generated by A in $\mathbb{B}$. The structure $\mathbb{B}$ is called *finitely generated* if $\mathbb{B} = \langle A\rangle$ for some finite set $A \subseteq B$, and *locally finite* if each finitely generated substructure of $\mathbb{B}$ is finite. Note that any relational structure is locally finite.

Let $\mathcal{C}$ be a class of countable locally finite L-structures. A structure $\mathbb{U} \in \mathcal{C}$ is called *universal*, if each structure $\mathbb{A} \in \mathcal{C}$ embeds into $\mathbb{U}$, and $\mathbb{U}$ is *homogeneous*, if each isomorphism between two finite substructures of $\mathbb{U}$ extends to an automorphism of $\mathbb{U}$. If $\mathbb{A}, \mathbb{B} \in \mathcal{C}$ are two finite structures with $\mathbb{A} \subseteq \mathbb{B}$ and $\mathbb{B} = \langle A \cup \{x\}\rangle$ for some $x \in B$ (this includes possibly $A = \emptyset$), we say that B is a *one-point extension* of $\mathbb{A}$ in $\mathcal{C}$. We say that a structure $\mathbb{U} \in \mathcal{C}$ *realizes all one-point extensions of finite substructures*, if whenever $\mathbb{A} \subseteq \mathbb{B}$ is a one-point extension in $\mathcal{C}$ and $\mathbb{A}$ is a substructure of $\mathbb{U}$, then there is an embedding $h : \mathbb{B} \to \mathbb{U}$ which is the identity on A. Now the following is well-known:

Proposition 2 *[Fraïssé [Fra54], cf. [Hod93, Cam90, Fra00]] Assume $\mathcal{C}$ is a class of countable locally finite L-structures which is closed under isomorphisms and under taking substructures and unions of ascending chains of finite structures in $\mathcal{C}$.*

1. *A structure $\mathbb{U} \in \mathcal{C}$ is universal and homogeneous iff $\mathbb{U}$ realizes all one-point extensions of finite substructures.*

2. *(Fraïssé's Theorem) The following are equivalent:*

   (a) There exists a universal homogeneous structure $\mathbb{U}$ in $\mathcal{C}$.

   (b) $\mathcal{C}$ contains, up to isomorphism, at most countably many finite structures, and the finite structures in $\mathcal{C}$ satisfy the amalgamation property.

   Moreover, in this case, $\mathbb{U}$ is unique up to isomorphism.

Now we can give the

Proof of Theorem 1 First we remark that due to the choice of the enumeration s, our construction produces a partial order $\prec$ successively on the sets $\{1, 2, \ldots, j\}$ for $j = 2, 3, \ldots$. Moreover, suppose we have obtained a strict partial order $(\{1, 2, \ldots, j-1\}, \prec)$. Then in order to extend $\prec$ to $\{1, 2, \ldots, j\}$, we decide on the order resp. incomparability relation between j and i successively for $i = 1, 2, 3, \ldots, j-1$.

Now, by Proposition 2(1), it suffices to show that with probability 1, $(\mathbb{N}, \prec)$ realizes all one-point extensions of finite substructures. Let $(A, \prec) \subseteq (B, \prec)$ be a one-point extension. Let $B = A\dot{\cup}\{x\}$. It suffices to show that with probability 1 we obtain an element $j \in \mathbb{N}$ such that the mapping $h : B \to A \cup \{j\}$, given by $h(x) = j$ and $h(a) = a$ for $a \in A$, is an isomorphism.

Let m be the maximal element of A with respect to the natural linear order on $\mathbb{N}$. Let $j > m$ be any greater number. Assume we have constructed the poset $(\{1, 2, \ldots, j-1\}, \prec)$. Then $(A, \prec) \subseteq (\{1, 2, \ldots, j-1\}, \prec)$. The probability, when extending $\prec$ to $\{1, 2, \ldots, j\}$, that j has precisely the same order-relations to elements of A as does x in $(B, \prec)$ depends only on the structure of $(\{1, 2, \ldots, m\}, \prec)$ and of $(B, \prec)$. In particular, it is strictly positive and independent of $m+1, m+2, \ldots, j-1$. Consequently, the probability that *no* $j > m$ realizes x in $(\mathbb{N}, \prec)$ as prescribed by $(B, \prec)$ is zero. $\square$

Note that the above proof would also work for many other enumerations, e.g., for enumerations where for any natural number m, the set $\{1, 2, \ldots, m\}$ is related to a 'fresh' natural number infinitely often (j is 'fresh' if its relations are not determined yet). Albert & Burris [AB86] showed that any poset realizing all one-point extensions of 5-element substructures is already universal and homogeneous. Using this, one can show that any enumeration s satisfying the following condition yields with probability 1 the universal homogeneous partial order:

> For any $n_1, n_2, \ldots, n_5 \in \mathbb{N}$, there exist infinitely many prefixes s_0 of s and natural numbers $j \in \mathbb{N}$ such that j does not occur in s_0 and $s_0.\{n_1, j\}.\{n_2, j\}.\ldots, \{n_5, j\}$ is a prefix of s.

To deal with linear orders, we replace conditions (b)-(c) by the requirement that $\|_j = \emptyset$. Clearly, the natural enumeration produces in this case with probability 1 a dense linear ordering without endpoints, i.e., $(\mathbb{N}, \prec) \cong (\mathbb{Q}, <)$. However, we do not know at present whether this construction works for *any* enumeration. In fact, this is open even for the 'reverse' natural enumeration $s = s_2^{-1} s_3^{-1} s_4^{-1} \ldots$ where $s_j^{-1} = (\{j-1, j\}, \{j-2, j\}, \{j-3, j\}, \ldots, \{1, j\})$ and even for the case of linear orders. We contribute the following conjecture which we strongly believe to be true but could verify by elementary explicit calculations only for small values of n:

Conjecture Let the above probabilistic construction be applied with the reverse natural enumeration to linear orders. Then the probability that 1 is minimal in $(\{1, 2, \ldots, n\}, \prec_n)$ equals $\prod_{i=1}^{n-1} \frac{2i-1}{2i}$ for $n \in \mathbb{N}$.

The conjecture would imply

Consequence Applied to the reverse natural enumeration, the probabilistic construction of the linear order $(\mathbb{N}, \prec)$ yields with probability 1 a dense chain without endpoints.

Proof. Let c_n be the probability that n is minimal in $(\{1, 2, \ldots, n\}, \prec_n)$. Then, by the conjecture, for any $n \in \mathbb{N}$, n would be minimal in $(\mathbb{N}, \prec)$ with probability $c_n \cdot \prod_{i=1}^{\infty} \frac{2i-1}{2i} = 0$. By symmetry, with probability 1, $(\mathbb{N}, \prec)$ has no endpoints. Now let $p, q \in \mathbb{N}$ with $p \prec q$. With probability 1, there are infinitely many $r \in \mathbb{N}$ with $p, q \leq r$ and $r \prec q$. For these elements $r \prec q$, the probability that $r \prec p$ is strictly smaller than 1 and depends only on $(\{1, 2, \ldots, \max(p, q)\}, \prec)$. Hence with probability 1, there is some $r \in \mathbb{N}$ with $p \prec r \prec q$. $\square$

If, indeed, this consequence is true, the following problem arises.

Open Question: Does for any enumeration s of the 2-subsets of $\mathbb{N}$, the corresponding probabilistic construction of the linear order $(\mathbb{N}, \prec)$ yield with probability 1 a dense chain without endpoints?

However, let us show that a slightly different procedure produces very different results: Consider the following *dynamic* probabilistic construction. In order to define a linear order on $\mathbb{N}$, we construct $\prec$ again successively on $\{1, 2, \ldots, j\}$ for $j = 2, 3, \ldots$. Suppose we have obtained a linear order $(\{1, 2, \ldots, j\}, \prec)$. Now enumerate $\{1, 2, \ldots, j\} = \{a_1, a_2, \ldots, a_j\}$ such that $a_1 \prec a_2 \prec a_3 \prec \cdots \prec a_j$. In order to extend $\prec$ to a linear order on $\{1, 2, \ldots, j, j+1\}$, we decide whether $j+1 \prec a_i$ or $a_i \prec j+1$ randomly with probability $\frac{1}{2}$, successively for $i = j, j-1, j-2, \ldots, 1$. Observe that here the sequence of the 2-subsets of $\mathbb{N}$ dealt with is not given from the beginning, but constructed dynamically during the construction process depending on the earlier choices.

Proposition 3 *With the above dynamic procedure with positive probability $p \approx 0.288$, 1 is the smallest element of $(\mathbb{N}, \prec)$.*

Proof. Assume we have constructed $(\{1, 2, \ldots, j\}, \prec)$. The probability that in $(\{1, 2, \ldots, j+1\}, \prec)$ the element $j+1$ is minimal is 2^{-j}. Hence the probability that 1 is minimal in $(\mathbb{N}, \prec)$ is $p = \prod_{j=1}^{\infty}(1 - 2^{-j})$. Since this product is absolutely convergent, we may reorder and calculate

$$\begin{aligned} p &= 1 - \sum_{j=1}^{\infty} 2^{-j} + \sum_{i<j} 2^{-(i+j)} - \sum_{i<j<k} 2^{-(i+j+k)} + - \ldots \\ &\approx 0.288 \end{aligned}$$

□

3 Random $\mathcal{C}$-structures

In this section, we wish to deal with more general structures of first order predicate calculus. First, let L be a relational language with relation symbols P_j of arity n_j ($j \in \mathbb{N}$). Let $\mathcal{C}$ be a class of countable L-structures closed under isomorphisms, substructures and unions of ascending chains of finite structures from $\mathcal{C}$. Our aim is to construct a random $\mathcal{C}$-structure on $\mathbb{N}$. Here, we decide successively for each possible relation R of arity n, say, and all $x_1, x_2, \ldots, x_n \in \mathbb{N}$ whether $(x_1, x_2, \ldots, x_n)$ belongs to R or not. Again, these choices may depend on earlier choices, and we have to ensure that our 'partially defined' relations can be 'completed' so that we obtain a $\mathcal{C}$-structure. More precisely, we proceed as follows.

We assume that there is an infinite universal structure $\mathbb{U} \in \mathcal{C}$, and that for each finite number n, $\mathcal{C}$ contains only finitely many pairwise non-isomorphic structures of size n. The latter assumption ensures that on a given finite set A, the finitely many isomorphism types of $\mathcal{C}$-structures on A are determined by finitely many relations, i.e., by the relations corresponding to the predicates $P_1, P_2, \ldots, P_p$ for some $p \in \mathbb{N}$. Note that by the Engeler/Ryll-Nardzewski/Svenonius theorem, a structure $\mathbb{U}$ is ω-categorical iff its automorphism group has only finitely many orbits on the set of n-tuples of $\mathbb{U}$, for each $n \in \mathbb{N}$. Hence, if the structure $\mathbb{U}$ is universal and homogeneous, our assumption on $\mathcal{C}$ holds iff $\mathbb{U}$ is ω-categorical, cf. [Cam90, Hod93].

We define an extended relational language L' which contains for each $j \in \mathbb{N}$ two relation symbols P_j and $\overline{P}_j$ of arity n_j. Let $\mathcal{C}'$ be the class of all L'-structures $\langle A, (R_j, \overline{R}_j)_{j\in\mathbb{N}} \rangle$ such that $\langle A, (R_j)_{j\in\mathbb{N}} \rangle$ is a $\mathcal{C}$-structure and $\overline{R}_j = A^{n_j} \setminus R_j$ for any $j \in \mathbb{N}$. Then we call a finite L'-structure $\langle A, (Q_j, \overline{Q}_j)_{j\in\mathbb{N}} \rangle$ *admissible* if there exists a $\mathcal{C}'$-structure $\langle A, (R_j, \overline{R}_j)_{j\in\mathbb{N}} \rangle$ such that $Q_j \subseteq R_j$ and $\overline{Q}_j \subseteq \overline{R}_j$ for each $j \in \mathbb{N}$. In particular then $Q_j \cap \overline{Q}_j = \emptyset$. Now let $\langle A, (Q_j, \overline{Q}_j)_{j\in\mathbb{N}} \rangle$ be an admissible L'-structure, and let B be a finite set with $A \subseteq B$. We claim that then the L'-structure $\langle B, (Q_j, \overline{Q}_j)_{j\in\mathbb{N}} \rangle$ is also admissible.

Indeed, there is a $\mathcal{C}$-structure $\mathbb{A} = \langle A, (R_j)_{j\in\mathbb{N}} \rangle$ such that $Q_j \subseteq R_j$ and $\overline{Q}_j \subseteq A^{n_j} \setminus R_j$ for $j \in \mathbb{N}$, and we may assume that $\mathbb{A} \subseteq \mathbb{U}$. Choose some $\mathbb{B}$ with $|B|$ elements and $\mathbb{A} \subseteq \mathbb{B} \subseteq \mathbb{U}$. Since $\mathcal{C}$ is closed under substructures and isomorphisms, this implies our claim.

Probabilistic construction of a $\mathcal{C}$-structure As underlying set, we take $\mathbb{N}$. Let $s = (t_i)_{i\in\omega}$ be an enumeration of all tuples $t_i = (P, x_1, x_2, \ldots, x_n)$ where P is a relational symbol from L of arity n and $x_1, x_2, \ldots, x_n \in \mathbb{N}$. Now we construct a sequence of admissible L'-structures $\mathbb{A}_k = \langle A_k, (R_{kj}, \overline{R}_{kj})_{j\in\mathbb{N}} \rangle$ for $k \in \omega$ with the following properties for all $k \in \mathbb{N}$:

1. $A_k = \{x \in \mathbb{N} \mid x \text{ occurs in } t_i \text{ for some } i \leq k\}$.
2. Let $t_{k+1} = (P_m, x_1, x_2, \ldots, x_{n_m})$. Then $R_{kj} = R_{k+1,j}$ and $\overline{R}_{kj} = \overline{R}_{k+1,j}$ for each $j \neq m$, and either $R_{k+1,m} = R_{k,m} \cup \{(x_1, x_2, \ldots, x_{n_m})\}$, $\overline{R}_{k+1,m} = \overline{R}_{k,m}$, or else $R_{k+1,m} = R_{k,m}$, $\overline{R}_{k+1,m} = \overline{R}_{k,m} \cup \{(x_1, x_2, \ldots, x_{n_m})\}$.

Indeed, assume $\mathbb{A}_k$ is constructed. By what we showed before, the structure $\langle A_{k+1}, (R_{k,j}, \overline{R}_{k,j})_{j\in\mathbb{N}} \rangle$ is admissible. Let $t_{k+1} = (P_m, x_1, x_2, \ldots, x_{n_m})$. Put $R_{k+1,j} = R_{kj}$ and $\overline{R}_{k+1,j} = \overline{R}_{kj}$ for each $j \neq m$. It follows that $\langle A_{k+1}, (R_{k+1,j}, \overline{R}_{k+1,j})_{j\in\mathbb{N}} \rangle$ with either

(i) $R_{k+1,m} = R_{k,m} \cup \{(x_1, x_2, \ldots, x_{n_m})\}$ and $\overline{R}_{k+1,m} = \overline{R}_{k,m}$ or

(ii) $R_{k+1,m} = R_{k,m}$, $\overline{R}_{k+1,m} = \overline{R}_{k,m} \cup \{(x_1, x_2, \ldots, x_{n_m})\}$

is admissible since $\langle A_{k+1}, (R_{k,j}, \overline{R}_{k,j})_{j\in\mathbb{N}} \rangle$ is admissible. If this is true with only one of (i) or (ii), we are forced to choose $\mathbb{A}_{k+1}$ correspondingly. If both (i) and (ii) can be realized, we decide between them randomly with probability $\frac{1}{2}$.

Finally, we put $R_j = \bigcup_{k\in\omega} R_{k,j}$, an n_j-ary relation on $\mathbb{N}$. It follows that $\mathbb{N}^{n_j} \setminus R_j = \bigcup_{k\in\omega} \overline{R}_{k,j}$. Let $k \in \mathbb{N}$. We show that $\mathbb{A}'_k = \left\langle A_k, (A_k^{n_j} \cap R_j)_{j\in\mathbb{N}} \right\rangle$ is a $\mathcal{C}$-structure.

Indeed, as noted above, there is $p \in \mathbb{N}$ such that the finitely many isomorphism types of $\mathcal{C}$-structures on A_k are determined by the relations corresponding to the relation symbols $P_1, P_2, \ldots, P_p$. Choose any $p < q \in \mathbb{N}$. By construction, there is $k \leq \ell \in \mathbb{N}$ such that $A_k^{n_j} \cap R_j \subseteq R_{\ell,j}$ and $A_k^{n_j} \setminus R_j \subseteq \overline{R}_{\ell,j}$ for each $j \in \{1, 2, \ldots, q\}$. But $\mathbb{A}_\ell$ is admissible, so there is a $\mathcal{C}'$-structure $\mathbb{B} = \left\langle A_\ell, (Q_j, \overline{Q}_j)_{j\in\mathbb{N}} \right\rangle$ with $R_{\ell,j} \subseteq Q_j$ and $\overline{R}_{\ell,j} \subseteq \overline{Q}_j$ for each $j \in \mathbb{N}$. Hence the restriction $\mathbb{B} \restriction A_k$ of $\mathbb{B}$ to the set A_k is a $\mathcal{C}'$-structure, and $A_k^{n_j} \cap Q_j = A_k^{n_j} \cap R_j$ for each $j \in \{1, 2, \ldots, q\}$. Since $q > p$ was arbitrary and $\mathcal{C}$-structures on A_k are determined by the relations $A_k^{n_j} \cap Q_j$ ($j = 1, 2, \ldots, p$), it follows that $A_k^{n_j} \cap Q_j = A_k^{n_j} \cap R_j$ for each $j \in \mathbb{N}$, showing that $\mathbb{A}'_k$ is a $\mathcal{C}$-structure.

Since $\mathcal{C}$ is closed under unions of ascending chains, also $\mathbb{R} = \langle \mathbb{N}, (R_j)_{j\in\mathbb{N}} \rangle = \bigcup_{k\in\mathbb{N}} \mathbb{A}'_k$ is a $\mathcal{C}$-structure, the *random $\mathcal{C}$-structure determined by the enumeration* s and this process.

The result Let us assume that if $j \in \mathbb{N}$, the $\mathcal{C}$-structures of size j are determined by the relations corresponding to the predicates $P_1, P_2, \ldots, P_{k_j}$ and that $k_j \leq k_{j+1}$ for all $j \in \mathbb{N}$. Now we define a *natural enumeration* s_{nat} inductively as follows. Suppose that $m \in \mathbb{N}$ and for $\{1, 2, \ldots, m\}$, s' constitutes a finite listing of tuples of the form $(P, x_1, x_2, \ldots, x_n)$. Then we list tuples for $\{1, 2, \ldots, m+1\}$ as follows: First, list for $\{1, m+1\}$ all tuples $(P_i, x_1, x_2, \ldots, x_{n_i})$ where $1 \leq i \leq k_2$ and $x_1, x_2, \ldots, x_{n_i} \in \{1, m+1\}$. Then list the remaining tuples $(P_j, x_1, x_2, \ldots, x_{n_j})$ with $j > k_2$ and $x_1, x_2, \ldots, x_{n_j} \in \{1, m+1\}$ arbitrarily interspersed later on. Next, we list all tuples $(P_i, x_1, x_2, \ldots, x_{n_i})$ where $1 \leq i \leq k_3$ and $x_1, x_2, \ldots, x_{n_i} \in \{1, 2, m+1\}$, provided these have not been listed already earlier. Again, we intersperse the other tuples, and we continue until we list the 'new' tuples for $\{1, 2, \ldots, m+1\}$.

Theorem 4 *Let $\mathcal{C}$ contain for each $n \in \mathbb{N}$ only finitely many non-isomorphic L-structures of size n and assume there is an infinite universal homogeneous structure $\mathbb{U}$ in $\mathcal{C}$. Then, with probability 1, the random structure $\mathbb{R}$ constructed from a natural enumeration s_{nat} and the above procedure is isomorphic to $\mathbb{U}$.*

Proof. (Similar to the proof of Theorem 1) Let $\mathbb{A} \subseteq \mathbb{B}$ be a one-point extension with $B = A \dot\cup \{x\}$, say, and $\mathbb{A} \subseteq \mathbb{R}$. We show that $\mathbb{R}$ realizes this extension with probability 1. Let m be the maximal element of A, and consider any $j > m$. So $\mathbb{A} \subseteq \mathbb{R} \restriction \{1, 2, \ldots, j-1\}$. Since $\mathcal{C}$ contains a universal homogeneous structure, by Proposition 2(b) the finite $\mathcal{C}$-structures satisfy the amalgamation property. In particular, there is an amalgam in $\mathcal{C}$ of $\mathbb{B}$ and $\mathbb{R} \restriction \{1, 2, \ldots, j-1\}$ over $\mathbb{A}$. This amalgam ensures that when extending $\mathbb{R} \restriction \{1, 2, \ldots, j-1\}$ to $\mathbb{R} \restriction \{1, 2, \ldots, j\}$, there is a positive probability $p > 0$ that j realizes the same R_i-relations with elements of A as does x in $\mathbb{B}$, for any $1 \le i \le k_j$. By our assumption on $\mathcal{C}$, it then follows that $\mathbb{B} \cong \mathbb{R} \restriction (A \cup \{j\})$ over $\mathbb{A}$. Again, p depends only on the structure $\mathbb{R} \restriction \{1, 2, \ldots, m\}$ and on $\mathbb{B}$, but not on $m+1, m+2, \ldots, j$. Consequently, the probability that there is *no* such $j > m$ is zero. $\square$

Next, we want to derive from Theorem 4 a similar result for classes of structures which may contain functional symbols. Let L be a language of first order predicate calculus and $\mathcal{C}$ a class of countable locally finite L-structures which is closed under isomorphisms, substructures and unions of ascending chains of finite structures from $\mathcal{C}$, and which contains an infinite universal homogeneous object $\mathbb{U}$. A first attempt would be to view functions as relations and to let $\mathcal{C}^\star$ comprise all such relational $\mathcal{C}$-structures and their substructures. However, then $\mathbb{U}$ need not be homogeneous as a relational structure in $\mathcal{C}^\star$, since two isomorphic subsets might generate non-isomorphic L-substructures of $\mathbb{U}$, and we cannot apply Theorem 4. Therefore, we enrich the relational language by adding relation symbols describing the isomorphism type of any structure generated by a finite tuple. More precisely, for each isomorphism class $\tilde{\mathbb{A}}$ of a finite structure $\mathbb{A} \in \mathcal{C}$ generated by the n-tuple $\underline{a} = (a_1, a_2, \ldots, a_n)$, say, we introduce an n-ary relational symbol $P_{\tilde{\mathbb{A}},\underline{a}}$ and let $L^\star$ be the language containing all relation symbols of L, for each function symbol of L a corresponding relation symbol, and all symbols $P_{\tilde{\mathbb{A}},\underline{a}}$. If $\mathbb{B} \in \mathcal{C}$, we view the functions as relations and add relations $R^{\mathbb{B}}_{\tilde{\mathbb{A}},\underline{a}}$ on B by putting $(x_1, x_2, \ldots, x_n) \in R^{\mathbb{B}}_{\tilde{\mathbb{A}},\underline{a}}$ iff the bijection $x_i \mapsto a_i$ extends to an L-isomorphism from the $\mathcal{C}$-substructure $\langle x_1, x_2, \ldots, x_n \rangle$ of $\mathbb{B}$ onto $\mathbb{A}$, i.e., the L-substructure generated by $\underline{x} = (x_1, x_2, \ldots, x_n)$ over $\underline{x}$ is canonically isomorphic to $\mathbb{A}$ over $\underline{a}$. We let $\mathcal{C}^\star$ comprise all such extended relational $\mathcal{C}$-structures and all their $L^\star$-substructures. It is clear that now $\mathbb{U}$ is a universal and homogeneous $\mathcal{C}^\star$-structure. Now we make the following assumption on $\mathcal{C}$:

($*$) For each $n \in \mathbb{N}$, up to isomorphism, $\mathcal{C}$ contains only finitely many L-structures generated by n elements.

This assumption means again that $\mathbb{U}$ is ω-categorical, and it ensures that for each $n \in \mathbb{N}$, $\mathcal{C}^\star$ contains only finitely many pairwise non-isomorphic structures of size n. Clearly, the isomorphism type of a finite $\mathcal{C}^\star$-structure X is determined by the single relation $R^X_{\tilde{\mathbb{A}},\underline{a}}$, if for some enumeration $\underline{x} = (x_1, x_2, \ldots, x_n)$ of X, the L-structure $\langle X \rangle$ over $\underline{x}$ is canonically isomorphic to $\mathbb{A}$ over $\underline{a}$. Hence each finite $\mathcal{C}^\star$-structure is determined by the relations corresponding to the predicates $R_{\tilde{\mathbb{A}},\underline{a}}$, and we define our natural enumeration s_{nat} as above with respect to these predicates; if we wish, for each arity we might add a few further predicates from $L^\star$. Now we perform the probabilistic construction of a random $\mathcal{C}^\star$-structure $\mathbb{R}$ as before. It follows from the construction that the $L^\star$-relations of $\mathbb{R}$ corresponding to function symbols of L are partial functions on $\mathbb{R}$. Note that although the relations $R^{\mathbb{B}}_{\tilde{\mathbb{A}},\underline{a}}$ determine the whole L-substructure $\langle \underline{x} \rangle$ of $\underline{x} \in R^{\mathbb{R}}_{\tilde{\mathbb{A}},\underline{a}}$, they are determined randomly and, moreover, whether a particular number j realizes a function value $f(x_1, x_2, \ldots, x_n)$ or not might also be determined randomly. Now we have:

Corollary 5 *Let $\mathcal{C}$ satisfy assumption $(*)$, and let $\mathbb{U}$ be an infinite universal homogeneous $\mathcal{C}$-structure. Then the random structure $\mathbb{R} \in \mathcal{C}^\star$ constructed from the natural enumeration s_{nat} and the procedure before is, with probability 1, an L-structure and as such isomorphic to $\mathbb{U}$.*

Here the first statement means that the $L^\star$-relations of $\mathbb{R}$ corresponding to function symbols of L are actually functions.

Proof. By Theorem 4, as a $\mathcal{C}^\star$-structure, with probability 1, $\mathbb{R}$ is universal and homogeneous. Now let f be an n-ary function symbol from L and $x_1, x_2, \ldots, x_n \in \mathbb{R}$. Then $\mathbb{R} \upharpoonright \{x_1, x_2, \ldots, x_n\} \in \mathcal{C}^\star$ is a substructure of a finite structure $\mathbb{B} \in \mathcal{C}^\star$ which occurs as an extended relational $\mathcal{C}$-structure. In particular, in $\mathbb{B}$, $y = f(x_1, x_2, \ldots, x_n)$ is defined. Since $\mathbb{R}$ realizes all one-point extensions, there is $z \in \mathbb{R}$ with $z = f(x_1, x_2, \ldots, x_n)$. Hence $\mathbb{R}$ can be viewed as an extended relational $\mathcal{C}$-structure, and its L-reduct is clearly isomorphic to $\mathbb{U}$. $\square$

As application we note that for instance the following classes of functional structures all contain universal homogeneous objects and satisfy the assumptions of Corollary 5:

1. countable $\vee$-semilattices
2. countable distributive lattices
3. countable Boolean algebras
4. the basis of ω-Scott-domains, with embedding-projection pairs as morphisms, as well as several other categories of domains, event structures, and information systems occurring in the mathematical foundations of denotational semantics of programming languages (cf. [DG92, DG93]).

Hence, in all these cases, by Corollary 5, the universal homogeneous structure can be constructed probabilistically.

4 A number-theoretic representation

As is well-known [Rad64], there is also an explicit number-theoretic representation of the universal homogeneous graph: As underlying set, take the natural numbers $\mathbb{N}$. For $i, j \in \mathbb{N}$ with $i < j$, let i and j be connected by an edge iff 2^i occurs in the unique expansion of j as a sum of distinct powers of 2. Then the graph obtained realizes all one-point extensions, hence is universal and homogeneous.

Now we wish to derive a similar representation for the universal homogeneous poset. Again, the situation is more complicated than for graphs, since we have to stay inside the class of posets. We proceed inductively and similarly to the probabilistic construction.

Let $s = s_2 s_3 s_4 \cdots = (S_i)_{i\in\mathbb{N}}$ be the natural enumeration of the 2-subsets of $\mathbb{N}$. Let $j \in \omega$ and suppose we have already constructed $(P_j, \prec_j, \|_j)$ with $P_j = \bigcup_{i<j} S_i$ such that the conditions (a)-(d) from page 3 hold. As there, we wish to define $\prec_{j+1}$ and $\|_{j+1}$. Let $S_j = \{a_j, b_j\}$ with $a_j < b_j$ in the natural order of $\mathbb{N}$. If already $a_j \prec_j b_j$ or $b_j \prec_j a_j$ or $a_j \|_j b_j$, we put $\prec_{j+1}=\prec_j$ and $\|_{j+1}=\|_j$.

Now assume we have at least two possibilities of putting either $a_j \prec_{j+1} b_j$ or $b_j \prec_{j+1} a_j$ or $a_j \|_{j+1} b_j$. We put $a_j \prec_{j+1} b_j$ ($b_j \prec_{j+1} a_j$, resp.) provided this is not prevented and 3^{a_j} occurs in the ternary expansion of b_j with coefficient 1 (2, resp.), and $a_j \|_{j+1} b_j$ in all other cases. Finally, let $\prec = \bigcup_{i\in\omega} \prec_j$.

Theorem 6 *The poset* $(\mathbb{N}, \prec)$ *constructed above is universal and homogeneous.*

Proof. We have to show that $(\mathbb{N}, \prec)$ realizes all one-point extensions. So let $(A, \prec) \subseteq (B, \prec)$ be a finite one-point extension with $(A, \prec) \subseteq (\mathbb{N}, \prec)$ and $B = A \cup \{x\}$, say. Let m be the maximal element of A in $\mathbb{N}$. Let $(C, \prec)$ be an amalgam of $(B, \prec)$ and $(\{1, 2, \ldots, m\}, \prec)$ over $(A, \prec)$ with $C = \{1, 2, \ldots, m, x\}$. Choose $j > m$ such that for each $1 \leq c \leq m$, the ternary expansion of j contains 3^c with coefficient 1 (resp., 2) iff $c \prec x$ (resp., $x \prec c$) in $(C, \prec)$. It follows easily that then the substructure $(A \cup \{j\}, \prec)$ of $(\mathbb{N}, \prec)$ is isomorphic to $(B, \prec)$. □

In a similar way, we may give an arithmetic definition of the universal homogeneous structure $\mathbb{U}$ of Theorem 4 or Corollary 5. In fact, an effective representation could also be obtained by following the construction of $\mathbb{U}$ in the proof of Proposition 2(2), (b) implies (a). However, that construction uses an enumeration of all possible amalgamations of finite $\mathcal{C}$-structures, so the present arithmetic representation is much simpler.

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