# Radial basis interpolation on homogeneous manifolds: Convergence rates*

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**Abstract**

Pointwise error estimates for approximation on compact homogeneous manifolds using dadial kernels are presented. Tangent space techniques are used to lift the problem from the manifold to Euclidean space, where methods for proving such error estimates are well established.

## 1 Introduction

There is currently significant interest in approximation on spheres, related to many interesting geophysical problems. There are a number of different approximation methods currently available on spheres, including wavelets [5], piecewise polynomial splines [1], and the subject of this paper, radial functions (sometimes called zonal splines or variational splines) [5, 8]. Error

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estimates and convergence rates for radial approximation on spheres, of an optimal nature, are recent in vintage [7, 6], and rely on some technically demanding mathematics. In this paper we build on an idea of Bos and de Marchi [3] in order to provide convergence rates for radial interpolation on a much wider class of manifolds: the reflexive, compact homogeneous spaces.

Let $M^d \subset \mathbb{R}^{d+k}$ be a $d$-dimensional embedded compact homogeneous manifold in the sense of Definition 1.1 in [10]. In particular there is a compact group $G$ of isometries of $\mathbb{R}^{d+k}$ such that for some $\eta \in M^d$ (often referred to as the pole) $M = \{g\eta : g \in G\}$. The *reflexive* condition means that for all $x, y \in M$ there is a $g \in G$ with $gx = y$ and $gy = x$. A kernel $k : M \times M \to \mathbb{R}$ is termed *zonal* (or $G$-invariant) if $k(x, y) = k(gx, gy)$ for all $g \in G$ and $x, y \in M$. Since the maps in $G$ are isometries of Euclidean space, they preserve both Euclidean distance and the (arc-length) metric $d(\cdot, \cdot)$ induced on the components of $M$ by the Euclidean metric. Thus the distance kernel $d(x, y)$ is zonal, as are all the radial functions, $\phi(d(x, y))$, which are kernels which depend only on the distance between $x, y$.

The manifold carries a unique normalised $G$-invariant measure which we call $\mu$ ($\mu$ can be viewed either as the component of Lebesgue measure tangential to $M$ or as the measure induced by Haar measure on $G$). Then, we can define the inner product of real functions

$$(f, g) \;=\; \int_M fg d\mu.$$

An important part of analysing the interpolation process (and many other processes) is the construction of polynomials on the manifold. If $\Pi_j$ is the space of all polynomials of total degree $j$ in the ambient space $\mathbb{R}^{d+k}$ then $P_j := \Pi_j|_M$ is the space of degree $j$ polynomials on $M$. We can also construct sets of *harmonic polynomials* $H_j := P_j \bigcap P_{j-1}^{\perp}$, where the orthogonality is with respect to the inner product $(\cdot, \cdot)$.

It is straightforward to show that $H_j$ is $G$-invariant, in the sense that $g \cdot p_j(\cdot) := p_j(g^{-1} \cdot) \in H_j$ for all $p_j \in H_j$, and has an orthogonal decomposition into irreducible $G$-invariant subspaces (i.e., subspaces with no proper $G$-invariant subspace)

$$H_j \;=\; \oplus_{l=1}^{h_j} \Xi_{jl}.$$

Now, if $\{Y_{jl}^1, \ldots, Y_{jl}^{d_{jl}}\}$ is an orthonormal basis for $\Xi_{jl}$, then

$$P_{jl}(x, y) \;=\; \sum_{m=1}^{d_{jl}} Y_{jl}^m(x) Y_{jl}^m(y), \quad x, y \in M,$$

is the unique $G$-invariant reproducing kernel for $\Xi_{jl}$. In other words, if $T_{jl}$ is the orthogonal projection onto $\Xi_{jl}$,

$$\begin{aligned} T_{jl}f(x) &= \int_M P_{jl}(x,y)f(y)d\mu(y), \quad x \in M, \\ \text{or } T_{jl}f(x) &= (f, P_{jl}(\cdot,x)), \quad x \in M, \end{aligned} \tag{1}$$

for all $f \in L_2(M)$. In particular,

$$T_{jl}f(x) = (T_{jl}f, P_{jl}(\cdot,x)), \quad x \in M. \tag{2}$$

It is clear that $P_{jl}$ is symmetric for all $j = 0, 1, \ldots$, and $l = 1, \ldots, h_j$, i.e., $P_{jl}(x,y) = P_{jl}(y,x)$. The self-adjoint projection operators $T_{jl}$ associated with the $G$-invariant kernels $P_{jl}$ are special cases of the general *kernel operator* $T_k$ associated with a continuous zonal kernel $k(x,y)$ by

$$T_k f(x) = \int_M k(x,y)f(y)d\mu(y), \quad x \in M. \tag{3}$$

All these operators are $G$-equivariant in the sense that $g \cdot T_k(f) = T_k(g \cdot f)$ and, at least on reflexive spaces, commute with each of the projectors $T_{jl}$ (Proposition 3.7 in [10]). Since $\Xi_{jl}$ is an irreducible $G$-invariant subspace, the commuting properties of the self-adjoint $T_k$ imply $T_k T_{jl} = a_{jl} T_{jl}$ for some scalar $a_{jl}$. Thus on reflexive spaces, by the density of polynomials, each zonal kernel $k$ has an expansion (convergent as operators on $L_2(M)$)

$$k(x,y) = \sum_{j=0}^{\infty} \sum_{m=1}^{h_j} a_{jl} P_{jl}(x,y).$$

In [10] the fundamentality of the set $\{k(\cdot,y) \; : \; y \in M\}$, of translates of a zonal kernel $k$ in the continuous functions on $M$ is explored. The crucial result is that a zonal kernel has dense translates if and only if $a_{jl} \neq 0$ for all $j, l$, i.e. the associated integral operator, $T_k$ does not behave like the null operator on any of the $G$-invariant polynomial subspaces. As is shown in [11] the positive definiteness of a kernel $k$ is equivalent to $k$ being positive definite on each of these $G$-invariant polynomial subspaces. Here, $k$ is positive definite if the quadratic form

$$\sum_{i,j=1}^{N} c_i c_j k(x_i, x_j)$$

is non-negative for any choice of points $x_1, \ldots, x_N \in M$ and coefficients $c_1, \ldots, c_N \in \mathbb{R}$.

A fixed positive definite kernel $k$ gives rise to a reproducing kernel Hilbert space $\mathcal{W}$, often called the *native space*, whose elements have smoothness of the same order as $k$. Given the values of $f$ at distinct $x_1, \ldots, x_N \in M$, we approximate $f \in \mathcal{W}$ by the *k-spline interpolant*

$$s_k(x) \;=\; \sum_{j=1}^{N} c_j k(x, x_j),$$

where the coefficients $c_1, \ldots, c_N$ are determined by satisfying the interpolation equations $s_k(x_i) = f(x_i)$, $i = 1, \ldots, x_N$. This interpolation problem has a unique solution if $k$ is strictly positive definite, i.e., the above quadratic form is always positive when $x_i$ are distinct.. It is such kernels that we shall be interested in.

The reader should view the following work as tracking closely the corresponding analysis on the sphere, but we depart from the rationale of Light and v. Golitschek [6] as we try to convert error bounds to convergence rates. They describe a scaling operation on the sphere under which the Lagrange polynomials remain bounded, this being a key element of the analysis. The novelty of our approach is that we transplant the scaling, and in fact the entire error estimation onto the tangent space, and use well known properties of the scaling of Lagrange polynomials in Euclidean space. This is possible because distances on the manifold and its tangent space are comparable.

In Section 2 we develop the Hilbert space theory and show how to produce an error estimate for interpolation. In Section 3 we use this error estimate to produce convergence rates for positive definite kernels which are also radial (functions of geodesic distance on the manifold) or zonal.

# 2 Hilbert Space, Reproducing Kernel, and Error Estimates

In this section we describe the construction of a Hilbert space from a positive definite zonal kernel $k$, and see that $k$ is the reproducing kernel for this Hilbert space. This construction is very standard (see e.g. Atteia [2]), and is only included for completeness.

As indicated above, when $M$ is reflexive, every positive definite zonal kernel $k$ whose translates are dense in $C(M)$ has an expansion in terms of the reproducing kernels for the minimal $G$-invariant polynomial subspaces of $L_2$ (see Theorems 12 and 16 in [11]):

$$k(x,y) \;=\; \sum_{j=0}^{\infty}\sum_{l=1}^{h_j} a_{kj} P_{jl}(x,y), \quad a_{jl} > 0.$$

For the rest of this paper we assume

$$\sum_{j=0}^{\infty}\sum_{l=1}^{h_j} d_{jl} a_{jl} \;<\; \infty. \tag{4}$$

Since $d_{jl} = P_{jl}(x,x)^{1/2} P_{jl}(y,y)^{1/2} > |P_{jl}(x,y)|$, this condition ensures that the positive definite zonal kernel defined by the sum above is jointly continuous, so that point evaluation makes sense.

Define the inner product

$$[f,g] \;:=\; \sum_{j=0}^{\infty}\sum_{l=1}^{h_j} a_{jl}^{-1}(T_{jl}f, T_{jl}g),$$

where we recall that $(,)$ is the usual $L_2$ inner product. Let $\|f\| = [f,f]^{1/2}$. Then, the Hilbert space $\mathcal{W}$ is the subspace of $L_2$ satisfying

$$\mathcal{W} \;:=\; \{f \in L_2 : \|f\| < \infty\}.$$

Condition (4) also ensures that $k(\cdot,x) \in \mathcal{W}$ for every $x \in M$.

It is easy to show that point evaluation at $x \in M$ in this space is realised by inner product with $k(\cdot,x)$, since

$$\begin{aligned}
[f,k(\cdot,x)] &= \sum_{j=0}^{\infty}\sum_{l=1}^{h_j} a_{jl}^{-1}(T_{jl}f, T_{jl}k(\cdot,x)) \\
&= \sum_{j=0}^{\infty}\sum_{l=1}^{h_j} a_{jl}^{-1}(T_{jl}f, a_{jl}P_{jl}(\cdot,x)) \\
&= \sum_{j=0}^{\infty}\sum_{l=1}^{h_j} T_{jl}f(x) \\
&= f(x),
\end{aligned} \tag{5}$$

where, in the penultimate line we have used Equation (2).

We can now immediately show

**Proposition 1** *Let $s_k$ be the $k$-spline interpolant to $f \in \mathcal{W}$. Then $s_k$ is the interpolant to $f$ from $\mathcal{W}$ with minimum norm.*

**Proof**
The following holds for any $g \in H$:

$$\begin{aligned}[g,g] &= [g - s_k + s_k, g - s_k + s_k] \\ &= [g - s_k, g - s_k] + [s_k, s_k] + 2[s_k, g - s_k].\end{aligned}$$

Since $k$ is the reproducing kernel for $\mathcal{W}$,

$$\begin{aligned}[s_k, g - s_k] &= [\sum_{n=1}^{N} \alpha_n k(\cdot, x_n), g - s_k] \\ &= \sum_{n=1}^{N} \alpha_n g(x_n) - s_k(x_n) \\ &= 0,\end{aligned}$$

if $g$ is also an interpolant to $f$. Thus, if $g$ is an interpolant to $f$,

$$[g,g] = [g - s_k, g - s_k] + [s_k, s_k] \geq [s_k, s_k]. \quad \blacksquare \tag{6}$$

**Remarks 1** In fact, one way of characterising the $k$-spline interpolant is via a minimum norm property (see e.g. Light and Wayne [9] for details of such an approach).

We can use (6) in a different way to give an error estimate for $k$-spline interpolation in $\mathcal{W}$.

**Theorem 1** *Let $f \in \mathcal{W}$ and $s_k$ be the $k$-spline interpolant to $f$ at $x_1, \ldots, x_N$. Then, for $x \in M$,*

$$|f(x) - s_k(x)| \leq \|f\| P(x, x_1, \ldots, x_N),$$

*where*

$$\begin{aligned}&P(x, x_1, \ldots, x_N) \\ &= \min_{\alpha_1, \ldots, \alpha_N \in \mathbb{R}} \left\{ k(x,x) - 2\sum_{n=1}^{N} \alpha_n k(x, x_n) + \sum_{n,m=1}^{N} \alpha_n \alpha_m k(x_n, x_m) \right\}^{1/2}.\end{aligned}$$

**Proof**
Using the reproducing kernel property we have

$$\begin{aligned}
|f(x) - s_k(x)| &= [f(x) - s_k(x), k(\cdot, x)] \\
&= [f(x) - s_k(x), k(\cdot, x) - \sum_{n=1}^{N} \alpha_n k(\cdot, x_n)],
\end{aligned}$$

since $f$ and $s_k$ are equal at $x_n$, $n = 1, \ldots, N$. The coefficients $\alpha_1, \ldots, \alpha_N$ are arbitrary real numbers. Using the Cauchy-Schwarz inequality we get

$$\begin{aligned}
|f(x) - s_k(x)| &\leq \|f - s_k\| \, \|k(\cdot, x) - \sum_{n=1}^{N} \alpha_n k(\cdot, x_n)\| \\
&\leq \|f\| \, \|k(\cdot, x) - \sum_{n=1}^{N} \alpha_n k(\cdot, x_n)\|,
\end{aligned}$$

using (6). Now,

$$\begin{aligned}
&\|k(\cdot, x) - \sum_{n=1}^{N} \alpha_n k(\cdot, x_n)\| \\
&= [k(\cdot, x) - \sum_{n=1}^{N} \alpha_n k(\cdot, x_n), k(\cdot, x) - \sum_{m=1}^{N} \alpha_m k(\cdot, x_m)] \\
&= \left\{ k(x, x) - 2 \sum_{n=1}^{N} \alpha_n k(x, x_n) + \sum_{n,m=1}^{N} \alpha_n \alpha_m k(x_n, x_m) \right\}^{1/2}
\end{aligned}$$

using the reproducing kernel property and the symmetry of $k$. The required result follows by minimisation of the final expression over all values of $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$. ■

# 3 Pointwise Convergence Rates for Zonal Kernels

In this section we consider zonal kernels and in particular radial kernels of the form

$$k(x, y) = \phi(d(x, y)),$$

where $d(x,y)$ is the geodesic distance between $x$ and $y$, and $\phi : \mathbb{R} \to \mathbb{R}$ is a univariate, even, real-valued function. As we have already noted, the $G$-invariance of $d(x,y)$ shows radial kernels are clearly zonal. For the analysis in this section we will assume that $k$ is positive definite and $2r$-times continuously differentiable in each variable near the diagonal in $M \times M$.

We are interested in establishing (pointwise) convergence rates for the interpolants as the as the interpolation points $x_1, \ldots, x_N$ become dense about the point of interest $x$. Since we will not be concerned with global constants, we are able to exploit the compactness of $M$ and simply give local estimates. By passing to the tangent space $T_x = \mathbb{R}^d$ at $x$ it becomes possible to reduce all the critical work to the Euclidean case.

Here are two simple results which lie at the heart of the Euclidean case. The proof of the first is elementary.

**Lemma 1** *Let $Z := \{z_1, \ldots, z_t\}$ be a $\Pi_{2r}$ unisolvent set in $\mathbb{R}^d$ with $t = \dim(\Pi_{2r})$. Let $p_1, \ldots, p_t$ be the Lagrange polynomials for $Z$, i.e. $p_i(z_j) = \delta_{ij}$. Then:*

*i. $\lambda Z := \{\lambda z_1, \ldots, \lambda z_t\}$ is unisolvent.*

*ii. The Lagrange polynomials for $\lambda Z$ are $p_i^\lambda = p_i(\cdot/\lambda)$, $i = 1, \ldots, t$. In particular $p_i^\lambda(0) = p_i(0)$.*

**Lemma 2** *Fix an integer $r \geq 0$ and a $\Pi_{2r}$ unisolvent set $\{z_1, \ldots, z_t\}$ in $\mathbb{R}^d$. Then there exist $d_0 > 0$ and $C_0$ such that:*

*i. Any set $\{v_1, \ldots, v_t\}$ with $\|z_i - v_i\| \leq d_0$ is $\Pi_{2r}$ unisolvent,*

*ii. If $\{v_1, \ldots, v_t\}$ are as in (i) and $\{p_i, i = 1, \ldots, t\}$ is a Lagrange basis for $\Pi_{2r}$ centered on $v_i$ then $|p_i(0)| \leq C_0$.*

**Proof**

If $\{q_1, \ldots, q_t\}$ is any basis for $\Pi_{2r}$, then $\{v_1, \ldots, v_t\}$ is unisolvent if and only if $\det(q_j(v_i)) \neq 0$. Hence by the continuity of $(v_1, \ldots, v_t) \mapsto \det(q_j(v_i))$, and the unisolvency of $\{z_1, \ldots, z_t\}$ there is a $d_0$ for which (i) holds. Since the map from unisolvent sets of distinct centers to Lagrange polynomials is continuous, so is $(v_1, \ldots, v_t) \mapsto (p_1(0), \ldots, p_t(0))$. But $\{(v_1, \ldots, v_t) \ : \ \|v_i - z_i\| \leq d_0\}$ is compact, so (ii) holds with

$$C_0 = \sup_{i, v_j} \{|p_i(0)|\} < \infty \quad \blacksquare$$

These two Lemmas provide the basis for us to estimate the size of the analog of $P(x, x_1, \ldots, x_N)$ for the Euclidean space case.

**Proposition 2** *Suppose $K(x,y)$ is a strictly positive definite symmetric function on some compact neighborhood $\overline{U} \times \overline{U}$ of $(0,0)$ in $\mathbb{R}^d \times \mathbb{R}^d$. Further suppose $K$ is in $C^{2r}(\overline{U} \times \overline{U})$. Then there exist constants $C > 0$ and $\rho_0$ such that for any finite set $u_1, \ldots, u_N$ in $U$ with*

$$\sup_{v \in \overline{U}} \min_j \|v - u_j\| \leq \rho < \rho_0, \tag{7}$$

*the following estimate holds:*

$$\min_{\alpha_1,\ldots,\alpha_N \in \mathbb{R}} \left\{ K(0,0) - 2\sum_{n=1}^{N} \alpha_n K(0,u_n) + \sum_{n,m=1}^{N} \alpha_n \alpha_m K(u_n,u_m) \right\} \leq C\rho^{2r}. \tag{8}$$

**Proof**
Let $Z = \{z_1, \ldots, z_t\}$ be the fixed unisolvent set from from Lemma 2, such that keeping within $d_0$ of the $z_i$ preserves unisolvency. Choose $r > 0$ such that the ball $\overline{B_r(0)} \subset \overline{U}$. Then set $\rho_0 = rd_0/\max_i \|z_i\|$. If $\rho < \rho_0$, then $\frac{\rho}{d_0} Z \subset B_r(0) \subset \overline{U}$. Hence for each $\frac{\rho}{d_0} z_j \in \overline{U}$, the density assumption (7) yields a $u_{i_j}$ with $\|\frac{\rho}{d_0} z_j - u_{i_j}\| \leq \rho$. Hence $v_j = \frac{d_0}{\rho} u_{i_j}$ is $\Pi_{2r}$ unisolvent by the same Lemma and the corresponding Lagrange polynomials satisfy $|p_j(0)| \leq C_0$. If the indexing on the $u_i$ is changed so that $i_j = j$, then scaling the $v_j$ by $\frac{\rho}{d_0}$ produces, by Lemma 1, a $\Pi_{2r}$–unisolvent subset $\{u_1, \ldots, u_t\}$. Furthermore, if $\{p_1, \ldots, p_t\}$ are the Lagrange polynomials for $u_1, \ldots, u_t$, then

$$\max\{|p_1(0)|, \ldots, |p_t(0)|\} \leq C_0. \tag{9}$$

Moreover, for $j = 1, \ldots, t$,

$$\|u_j\| \leq \rho(\|z_j\|/d_0 + 1) \leq \rho C_u, \quad \text{where } C_u = 1 + \max_j \|z_j\|/d_0 \tag{10}$$

Now we proceed to prove the bound (8) by the coefficients $\alpha_1, \ldots, \alpha_N$ as follows. Let $\alpha_j = p_j(0), j = 1, \ldots, t$ and $\alpha_j = 0$ otherwise. This choice of coefficients is made since

$$\sum_{j=1}^{t} p_j(0) q(u_j) \;\; = \;\; q(0), \tag{11}$$

for all $q \in \Pi_{2r}$. Then the left hand side (LHS) of (8) may be bounded as follows:

$$LHS \leq \left\{ K(0,0) - 2\sum_{j=1}^{t} p_j(0)K(0,u_j) + \sum_{i,j=1}^{t} p_i(0)p_j(0)K(u_i,u_j) \right\} \tag{12}$$

Since $K(\cdot,\cdot)$ is $2r$ times continuously differentiable in each variable, for fixed $w \in U$ we may expand

$$K(w,z) = \sum_{|\alpha|<2r} c_\alpha^w (z-w)^\alpha + R_{2r}^w(z), \quad z \in U, \tag{13}$$

where $R_{2r}^w$ is a Taylor series remainder satisfying

$$R_{2r}^w \leq C_R \|z-w\|_2^{2r}, \tag{14}$$

for some constant $C_R$ independent of $z$ and $w$. Putting $z = w$ in the above expansion we see that

$$c_0^w = K(w,w). \tag{15}$$

Plugging (13) into (12) gives

$$\begin{aligned} LHS \leq & \left\{ K(0,0) - 2\sum_{j=1}^{t} p_j(0) \left\{ \sum_{|\alpha|<2r} c_\alpha^0 (u_j)^\alpha + R_{2r}^0(u_j) \right\} \right. \\ & \left. + \sum_{i,j=1}^{t} p_i(0)p_j(0) \left\{ \sum_{|\alpha|<2r} c_\alpha^{u_i}(u_j-u_i)^\alpha + R_{2r}^{u_i}(u_j) \right\} \right\}. \end{aligned}$$

Using the polynomial reproduction (11) we get

$$\begin{aligned} LHS \leq & \left\{ K(0,0) - 2K(0,0) - 2\sum_{j=1}^{t} p_j(0)R_{2r}^0(u_j) \right. \\ & \left. + \sum_{i=1}^{t} p_i(0) \left\{ \sum_{|\alpha|<2r} c_\alpha^{u_i}(-u_i)^\alpha \right\} + \sum_{i,j=1}^{t} p_i(0)p_j(0)R_{2r}^{u_i}(u_j) \right\} \\ = & \left\{ -K(0,0) - 2\sum_{j=1}^{t} p_j(0)R_{2r}^0(u_j) \right. \\ & \left. + \sum_{i=1}^{t} p_i(0) \left\{ K(u_i,0)) - R_{2r}^{u_i}(0) \right\} + \sum_{i,j=1}^{t} p_i(0)p_j(0)R_{2r}^{u_i}(u_j) \right\}, \end{aligned}$$

where we have used (12) and (15) in the above argument.

Now, we can use the symmetry of $K$ and re-expand $K(u_i, 0) = K(0, u_i)$ to give

$$
\begin{aligned}
LHS &\leq \Bigg\{ -K(0,0) - 2\sum_{i=1}^{t} p_i(0) R_{2r}^{0}(u_i) \\
&\qquad + \sum_{i=1}^{t} p_i(0) \Bigg\{ \sum_{|\alpha|<2r} c_\alpha^0 (u_i)^\alpha + R_{2r}^{0}(u_i) - R_{2r}^{u_i}(0) \Bigg\} \\
&\qquad + \sum_{i,j=1}^{t} p_i(0) p_j(0) R_{2r}^{u_i}(u_j) \Bigg\} \\
&= \Bigg\{ -K(0,0) - 2\sum_{j=1}^{t} p_j(0) R_{2r}^{0}(u_j) + K(0,0) \\
&\qquad + \sum_{i=1}^{t} p_i(0)(R_{2r}^{0}(u_i) - R_{2r}^{u_i}(0)) + \sum_{i,j=1}^{t} p_i(0) p_j(0) R_{2r}^{u_i}(u_j) \Bigg\} \\
&= \qquad - \sum_{i=1}^{t} p_i(0)(R_{2r}^{0}(u_i) + R_{2r}^{u_i}(0)) + \sum_{i,j=1}^{t} p_i(0) p_j(0) R_{2r}^{u_i}(u_j) \\
&\leq C_R \max_{i,j=1,t} \{ \|u_i - u_j\|_2^{2r}, \|u_i\|_2^{2r} \} \Bigg\{ \sum_{i,j=1}^{t} |p_i(0)||2 + |p_j(0)|| \Bigg\} \\
&\leq C_R (2C_u)^{2r} \rho^{2r} t^2 C_L (2 + C_L),
\end{aligned}
$$

using (14) and (10). Combining all the constants yields (8). ■

To lift this up to a local error bound on the manifold requires only the Lipschitz equivalence of the Euclidean metric on $\mathbb{R}^{d+k}$ restricted to $M$ and the (arc-length) metric $d(\cdot,\cdot)$. Thus, we fix $x \in M$ and, for $\rho > 0$, let $D_\rho(x) = \{y \in M : d(x,y) \leq \rho\}$. Let $T_x = \mathbb{R}^d$ be the tangent space to $M$ at $x$ considered as a $d$-dimensional hyperplane in $\mathbb{R}^{d+k}$. Let $N_\rho = \{y \in \mathbb{R}^d : \|y\| \leq \rho\}$. Since $M$ is an embedded $C^\infty$ manifold the orthogonal projection, $\pi_x$, from $\mathbb{R}^{d+k}$ to $T_x$ has a derivative at $x$ which is the identity map when restricted to $T_x$. Hence by the inverse function theorem there exists $\rho_0 > 0$ and a $C^\infty$ map $\phi : N_{\rho_0} \to M$ which is the inverse of the orthogonal projection restricted to a closed neighborhood of $x$ in $M$. Since $\phi : N_{\rho_0} \subset \mathbb{R}^d \to M$ is an invertible $C^\infty$ map on a compact neighborhood, whose inverse is orthogonal projection restricted to $M$, there exists a constant $C_1$ such that

$$\begin{aligned} C_1 d(\phi(y),\phi(z)) \le \|y-z\| &\le ||\phi(y)-\phi(z)|| \\ &\le d(\phi(y),\phi(z)), \quad y,z \in N_{\rho_0}. \end{aligned} \tag{16}$$

When we combine the purely local Proposition 2 with the metric equivalence the principal local error estimate for $s_k$ spline interpolation is obtained.

**Theorem 2** *Let $k(x,y)$ be a strictly positive definite zonal kernel on $M$. Assume $k$ is $2r$-times continuously differentiable in each variable. Let $x \in S \subset M$. Suppose that the interpolation points $x_1,\dots,x_N$ satisfy*

$$\max_{y\in S} \min_{i=1,\dots,N} d(y,x_i) \le \rho,$$

*for some sufficiently small $\rho$. Then, for $f \in H$,*

$$|f(x)-s_k(x)| \le C\|f\|\rho^r,$$

*where $C$ is independent of $\rho$ and $f$.*

**Proof**
Let $\pi_x$ be the orthogonal projection onto the tangent space $T_x$ to $M$ at $x$, as in the preceding discussion. So we assume the $x \in T_x$ is the origin. Also if $\phi$ is a local inverse to $\pi_x$, then $K(w,z) = k(\phi(w),\phi(z))$ defines a positive definite function on a compact neighborhood of 0, satisfying the hypotheses of Proposition 2. Now taking account of the metric stretching in (16), the result in (8) carries over to $M$. Then combining with Theorem 1 yields the result. ■

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