Evaluation of some Integrals arising from Approximation on Spheres*

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January 16, 2001

Abstract

Integrals arising in approximation on the sphere using radial basis functions in the ambient Euclidean space are computed.

1 Introduction

The ability to approximate both accurately and efficiently on the unit sphere $S^{d-1} \subset \mathbb{R}^d$ is of interest to practitioners in a variety of scientific disciplines, astronomy, meteorology, and geology to name a few. The integrals we compute below arise from the desire to approximate on the sphere using functions defined on the ambient Euclidean space. We consider two families of radial basis functions on $\mathbb{R}^d$: for fixed $y \in \mathbb{R}^d$, $\phi(x) = \|x-y\|^{2\alpha}$ and $\phi(x) = \|x-y\|^{2\alpha} \log \|x-y\|$, where $\|\cdot\|$ is the Euclidean norm in d–dimensional space. If we consider the restriction of these functions to the unit sphere we become interested in the following functions on S^{d-1}: for fixed $y \in S^{d-1}$, $\phi(x) = (1-xy)^\alpha$ and $\phi(x) = (1-xy)^\alpha \log(1-xy)$, where xy denotes the inner product of vectors $x, y \in S^{d-1}$.

The purpose of this paper is to compute the Fourier–Gegenbauer series for the functions $\phi(t) = (1-t)^\alpha$ and $\phi(t) = (1-t)^\alpha \log(1-t)$;

*Technical Report No. 2001/02, Department of Mathematics & Computer Science, University of Leicester, University Road, Leicester LE1 7RH, England.

see Theorems 2 and 3. These expansions may be used to classify the space of functions which can be approximated using the above Euclidean radial basis functions for approximation on S^{d-1}. These integrals and others have been computed, using different techniques, by Baxter and Hubbert [2].

Let $P_n^\lambda(x)$ be the Gegenbauer polynomial of degree n, orthogonal on $[-1,1]$ with respect to the weight $(1-x^2)^{\lambda-1/2}$. These polynomials are normalised so that

$$\begin{aligned} h_n^\lambda &:= \int_{-1}^{1} P_n^\lambda(x)(1-x^2)^{\lambda-1/2}\,dx \\ &= \frac{\pi 2^{1-2\lambda}\Gamma(n+2\lambda)}{n!(n+\lambda)(\Gamma(\lambda))^2} \end{aligned} \qquad (1)$$

2 Results

Theorem 1 *For $\alpha > 0$ and $\lambda > -1/2$, let*

$$I_n^{\alpha,\lambda} = \int_{-1}^{1} (1-x^2)^\alpha P_n^{(\lambda)}(x)\,dx.$$

If n is even then

$$I_n^{\alpha,\lambda} = \frac{\Gamma(\lambda+n/2)2^{-2\alpha}\pi\Gamma(2\alpha+1)\Gamma(\lambda-\alpha-1/2+n/2)}{(n/2)!\Gamma(\lambda)\Gamma(\lambda-\alpha-1/2)\Gamma(\alpha+n/2+3/2)\Gamma(\alpha+1/2)}.q$$

However, if n is odd then $I_n^{\alpha,\lambda} = 0$.

Proof

The case of n odd is obvious.

From Szegö [3, p. 82], we have the following three term recurrence relation

$$nP_n^{(\lambda)}(x) = 2(n+\lambda-1)xP_{n-1}^{(\lambda)}(x) - (n+2\lambda-2)P_{n-2}^{(\lambda)}(x).$$

Substituting this recurrence relation into the above integral gives:

$$\begin{aligned} I_n^{\alpha,\lambda} &= \int_{-1}^{1} (1-x^2)^\alpha \frac{2(n+\lambda-1)xP_{n-1}^{(\lambda)}(x)}{n}\,dx \\ &\qquad - \int_{-1}^{1} (1-x^2)^\alpha \frac{(n+2\lambda-2)}{n} P_{n-2}^{(\lambda)}(x)\,dx \\ &= A - B, \end{aligned}$$

with the obvious definitions of A and B. Then, using integration by parts,

$$A = \int_{-1}^{1} \frac{(1-x^2)^{\alpha+1}(n+\lambda-1)2\lambda P_{n-2}^{(\lambda+1)}\,dx}{(\alpha+1)n},$$

where we have used the fact that

$$\frac{d}{dx}P_{n-1}^{(\lambda)}(x) = 2\lambda P_{n-2}^{(\lambda+1)}(x);$$

see [3, Page 82]. Thus

$$I_n^{\alpha,\lambda} = \frac{2(n+\lambda-1)\lambda}{n(\alpha+1)} I_{n-2}^{\lambda+1} - \frac{(n+2\lambda-2)}{n} I_{n-2}^{\alpha,\lambda}$$

We now proceed by induction. Assume the truth of the induction hypothesis for the case $n-2$ for all λ and for all α. Using the recurrence relation just derived one obtains:

$$\begin{aligned}
I_n^{\alpha,\lambda} &= \frac{2(n+\lambda-1)\lambda\Gamma(\lambda+n/2)2^{-2\alpha}\pi}{4n(\alpha+1)\Gamma(\lambda+1)\Gamma(\alpha+1/2)} \\
&\qquad \times \frac{(2\alpha+2)(2\alpha+1)\Gamma(2\alpha+1)\Gamma(\lambda-\alpha-3/2+n/2)}{\Gamma(\lambda-\alpha-1/2)\Gamma(\alpha+n/2+3/2)(n/2-1)!} \\
&\quad - \frac{(n+2\lambda-2)\Gamma(\lambda+n/2-1)2^{-2\alpha}\pi\Gamma(2\alpha+1)\Gamma(\lambda-\alpha-3/2+n/2)}{(n/2-1)!n\Gamma(\lambda-\alpha-1/2)\Gamma(\alpha+n/2+1/2)\Gamma(\lambda)\Gamma(\alpha+1)\Gamma(\alpha+1/2)} \\
&= \frac{\Gamma(\lambda+n/2-1)2^{-2\alpha}\pi\Gamma(2\alpha+1)\Gamma(\lambda-\alpha-3/2+n/2)}{(n/2-1)!\Gamma(\lambda)\Gamma(\alpha+1/2)\Gamma(\lambda-\alpha-1/2)\Gamma(\alpha+n/2+1/2)} \\
&\qquad \times \left[\frac{(n+\lambda-1)2\lambda(\lambda+n/2-1)(2\alpha+2)(2\alpha+1)}{4n(\alpha+1)\lambda(\alpha+1/2)(\alpha+n/2+1/2)} - \frac{(n+2\lambda-2)}{n}\right].
\end{aligned}$$

However,

$$\frac{(n+2\lambda-2)}{n}\left(\frac{2(n+\lambda-1)}{2\alpha+n+1}-1\right) = \frac{n+2\lambda-2}{n/2(2\alpha+n+1)}(\lambda-\alpha+n/2-3/2).$$

Therefore

$$\begin{aligned}
I_n^{\alpha,\lambda} &= \frac{\Gamma(\lambda+n/2-1)2^{-2\alpha}\pi\Gamma(2\alpha+1)\Gamma(\lambda-\alpha-3/2+n/2)}{(n/2-1)!\Gamma(\lambda)\Gamma(\alpha+1/2)\Gamma(\lambda-\alpha-1/2)\Gamma(\alpha+n/2+1/2)} \\
&\qquad \times \frac{(\lambda-\alpha+n/2-3/2)(\lambda+n/2-1)}{n/2(\alpha+n/2-1)} \\
&= \frac{\Gamma(\lambda+n/2)2^{-2\alpha}\pi\Gamma(2\alpha+1)\Gamma(\lambda-\alpha-1/2+n/2)}{(n/2)!\Gamma(\lambda)\Gamma(\alpha+1/2)\Gamma(\lambda-\alpha-1/2)\Gamma(\alpha+n/2+3/2)},
\end{aligned}$$

as required.

We complete the proof by noting that, from (1) with $\lambda = \alpha + 1/2$ and $n = 0$,

$$I_0^{\alpha,\lambda} = \frac{2^{-2\alpha}\pi\Gamma(2\alpha+1)}{(\Gamma(\alpha+1/2))^2(\alpha+1/2)}. \quad \blacksquare$$

Theorem 2 *Let $\alpha > 0$ and $\lambda > -1/2$. Then*

$$\begin{aligned} J_n^{\alpha,\lambda} &:= \int_{-1}^{1} (1-x)^\alpha (1-x^2)^{\lambda-1/2} P_n^{(\lambda)}(x) \\ &= \frac{(-1)^n 2^{\alpha+2\lambda+2n}\Gamma(\alpha+\lambda+1/2)\Gamma(\lambda+n+1/2)\Gamma(2\lambda+n)}{n!\Gamma(2\lambda+2n)\Gamma(\alpha+2\lambda+n+1)\Gamma(\alpha-n+1)\Gamma(\lambda)} \\ &\qquad \times\Gamma(\alpha+1)\Gamma(\lambda+n). \end{aligned}$$

Proof

Using integration by parts

$$J_n^{\alpha,\lambda} = \int_{-1}^{1} \frac{(1-x)^{\alpha+1}}{\alpha+1}\frac{d}{dx}\left((1-x^2)^{\lambda-1/2}P_n^{(\lambda)}(x)\right)dx.$$

Now, using Rodriguez' formula (see e.g. [3, Page 82]),

$$(1-x^2)^{\lambda-1/2}P_n^{(\lambda)}(x) = \gamma_n^\lambda \frac{d^n}{dx^n}(1-x^2)^{n+\lambda-1/2}, \tag{2}$$

where

$$\gamma_n^\lambda = \frac{(-2)^n\Gamma(n+\lambda)\Gamma(n+2\lambda)}{n!\Gamma(\lambda)\Gamma(2n+2\lambda)}. \tag{3}$$

Thus

$$\begin{aligned} \frac{d}{dx}(1-x^2)^{\lambda-1/2}P_n^{(\lambda)}(x) &= \gamma_n^\lambda \frac{d^{n+1}}{dx^{n+1}}(1-x^2)^{n+\lambda-1/2} \\ &= \gamma_n^\lambda \frac{d^{n+1}}{dx^{n+1}}(1-x^2)^{n+1+(\lambda-1)-1/2}. \end{aligned}$$

Using (2) again we have

$$\frac{d^{n+1}}{dx^{n+1}}(1-x^2)^{n+\lambda-1/2} = \frac{(1-x^2)^{\lambda-3/2}P_{n+1}^{(\lambda-1)}}{\gamma_{n-1}^{\lambda+1}}.$$

Thus

$$\frac{d}{dx}(1-x^2)^{\lambda-1/2}P_n^{(\lambda)}(x) = \frac{\gamma_n^\lambda}{\gamma_{n+1}^{\lambda-1}}(1-x^2)^{\lambda-3/2}P_{n+1}^{(\lambda-1)}(x).$$

Putting this back into the integral gives

$$J_n^{\alpha,\lambda} = \frac{\gamma_n^\lambda}{\gamma_{n+1}^{\lambda-1}} \int_{-1}^1 \frac{(1-x)^{\alpha+1}}{\alpha+1}(1-x^2)^{\lambda-3/2} P_{n+1}^{(\lambda-1)}(x)dx.$$

Therefore

$$J_n^{\alpha,\lambda} = \frac{\gamma_n^\lambda J_{n+1}^{\alpha+1,\lambda-1}}{\gamma_{n-1}^{\lambda+1}\alpha+1},$$

so that

$$J_n^{\alpha,\lambda} = \frac{\alpha\gamma_n^\lambda J_{n-1}^{\alpha-1,\lambda+1}}{\gamma_{n-1}^{\lambda+1}}.$$

Now, from (3),

$$\frac{\gamma_n^\lambda}{\gamma_{n-1}^{\lambda+1}} = \frac{-2\lambda\alpha}{n(n+2\lambda)},$$

with the result that

$$J_n^{\alpha,\lambda} = \frac{\alpha(-2\lambda)}{n(n+2\lambda)} J_{n-1}^{\alpha-1,\lambda+1}$$

We can now proceed by induction. Assume the result is true for $n-1$, and all $\alpha > 0$ and $\lambda > -1/2$:

$$J_{n-1}^{\alpha-1,\lambda+1} = \frac{(-1)^{n-1}2^{\alpha-1+2\lambda+2n}\Gamma(\alpha+\lambda+1/2)\Gamma(\lambda+n+1/2)}{(n-1)!\Gamma(2\lambda+2n)\Gamma(\alpha+2\lambda+n+1)} \times\frac{\Gamma(2\lambda+n+1)\Gamma(\alpha)\Gamma(\lambda+n-1)}{\Gamma(\alpha-n+1)\Gamma(\lambda-1)}.$$

Substituting this into the recurrence relation gives

$$\begin{aligned} J_n^{\alpha,\lambda} &= \frac{\alpha(-2\lambda)(-1)^{n-1}2^{\alpha+2\lambda+2n-1}}{n(n+2\lambda)(n-1)!\Gamma(2\lambda+2n)} \\ &\quad \times\frac{\Gamma(\alpha+\lambda+1/2)\Gamma(\lambda+n+1/2)\Gamma(2\lambda+n+1)\Gamma(\alpha)\Gamma(\lambda+n)}{\Gamma(\alpha+2\lambda+n+1)\Gamma(\alpha-n+1)\Gamma(\lambda+1)} \\ &= \frac{(-1)^n 2^{\alpha+2\lambda+2n}\Gamma(\alpha+\lambda+1/2)\Gamma(\lambda+n+1/2)\Gamma(2\lambda+n)}{n!\Gamma(2\lambda+2n)\Gamma(\alpha+2\lambda+n+1)\Gamma(\alpha-n+1)\Gamma(\lambda)} \\ &\quad \times\Gamma(\alpha+1)\Gamma(\lambda+n) \end{aligned}$$

We conclude the induction by noting from [3, Page 68], that for all $\alpha > 0$ and $\lambda > -1/2$:

$$J_0^{\alpha,\lambda} = \frac{2^{\alpha+2\lambda}\Gamma(\alpha+\lambda+1/2)\Gamma(\lambda+1/2)}{\Gamma(\alpha+2\lambda+1)}. \quad \blacksquare$$

Let

$$\Psi(x) = \frac{d}{dx}\log\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

be the digamma function (see [1, Chapter 6]).

Theorem 3 *Let $\alpha > 0$, $\lambda > -1/2$ and*

$$K_n^{\alpha,\lambda} \;=\; \int_{-1}^{1}(1-x)^\alpha \log(1-x)(1-x^2)^{\lambda-1/2}P_n^{(\lambda)}(x)\,dx.$$

Then, if $\alpha \in I\!N$ and $n > \alpha$,

$$K_n^{\alpha,\lambda} = \frac{(-1)^\alpha 2^{\alpha+2\lambda+2n}\Gamma(\alpha+\lambda+1/2)\Gamma(\lambda+n+1/2)\Gamma(2\lambda+n)\Gamma(\alpha+1)\Gamma(\lambda+n)}{n!\Gamma(2\lambda+2n)\Gamma(\alpha+2\lambda+n+1)(n-\alpha-1)!\Gamma(\lambda)}.$$

Otherwise,

$$K_n^{\alpha,\lambda} = J_n^{\alpha,\lambda}(\log 2 + \Psi(\alpha+\lambda+1/2) + \Psi(\alpha+1) - \Psi(\alpha+2\lambda+n) - \Psi(\alpha-n+1)).$$

Proof
The key observation in this proof is that

$$\frac{d}{d\alpha}(1-x)^\alpha = (1-x)^\alpha\log(1-x).$$

Thus,

$$K_n^{\alpha,\lambda} = \frac{d}{d\alpha}J_n^{\alpha,\lambda}.$$

If we write

$$A_n^\lambda = \frac{(-1)^n 2^{2\lambda+2}\Gamma(\lambda+n+1/2)\Gamma(2\lambda+n)\Gamma(\lambda+n)}{n!\Gamma(2\lambda+2n)\Gamma(\lambda)}$$

then

$$J_n^{\alpha,\lambda} = A_n^\lambda\frac{2^\alpha\Gamma(\alpha+\lambda+1/2)\Gamma(\alpha+1)}{\Gamma(\alpha+2\lambda+n)\Gamma(\alpha-n+1)}.$$

Thus, using the facts that $\Gamma'(x) = \Gamma(x)\Psi(x)$ and that

$$\frac{d}{dx}\frac{1}{\Gamma(x)} = -\frac{\Psi(x)}{\Gamma(x)},$$

we see that

$$
\begin{aligned}
K_n^{\alpha,\lambda} & \\
&= A_n^\lambda \frac{\partial}{\partial\alpha} \frac{2^\alpha \Gamma(\alpha+\lambda+1/2)\Gamma(\alpha+1)}{\Gamma(\alpha+2\lambda+n)\Gamma(\alpha-n+1)} \\
&= A_n^\lambda \frac{2^\alpha \Gamma(\alpha+\lambda+1/2)\Gamma(\alpha+1)}{\Gamma(\alpha+2\lambda+n)\Gamma(\alpha-n+1)} \\
&\quad (\log 2 + \Psi(\alpha+\lambda+1/2) + \Psi(\alpha+1) - \Psi(\alpha+2\lambda+n) - \Psi(\alpha-n+1))),
\end{aligned}
$$

and the result follows.

Now, when $\alpha \in \mathbb{N}$ and $n > \alpha$, $\Gamma(\alpha - n + 1)$ is undefined, as is $\Psi(\alpha - n + 1)$. Thus in this case we need to compute the limit

$$\lim_{x \to -k} \frac{\Psi(x)}{\Gamma(x)}$$

for $k \in \mathbb{N}$. Since Γ possesses a simple pole at $-k$ with residue $(-1)^k/k!$, (see [1, Page 255]), Ψ also possesses a simple pole with residue -1. Thus, for $k \in \mathbb{N}$,

$$\lim_{x \to -k} \frac{\Psi(x)}{\Gamma(x)} = (-1)^{k+1}/k!.$$

Therefore, for $\alpha \in \mathbb{N}$ and $n \geq \alpha$,

$$K_n^{\alpha,\lambda} = (-1)^{n-\alpha} A_n^\lambda \frac{2^\alpha \Gamma(\alpha+\lambda+1/2)\Gamma(\alpha+1)}{\Gamma(\alpha+2\lambda+n)(n-\alpha-1)!} \quad \blacksquare$$

Acknowledgements: We would like to thank Leon Greenberg for directing us towards the complex analytic approach to the calculation of the limit in Theorem 3.

References

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