Error Estimates For Multilevel Approximation Using Polyharmonic Splines*

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Abstract

Polyharmonic splines are used to interpolate data in a stationary multilevel iterative refinement scheme. By using such functions the necessary tools are provided to obtain simple pointwise error bounds on the approximation. Linear convergence between levels is shown for regular data on a scaled multiinteger grid, and a multilevel domain decomposition method.

Keywords: multilevel approximation, polyharmonic splines, convergence.
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1 Introduction

Consider data $f_i \in \mathbb{R}$ located at distinct points $x_i \in \mathbb{R}^d$ for $i = 1, \ldots, N$. Let the interpolant be a linear combination of translated basis functions, appended by some polynomial thus

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$$s(x)=\sum_{i=1}^{N}\lambda_i\Phi(x-x_i)+p_{m-1}(x), \tag{1}$$

where $p_{m-1}\in\mathcal{P}_{m-1}^d$, and $s(x_i)=f_i$ for $i=1,\ldots,N$, with conditions

$$\sum_{i=1}^{N}\lambda_i q(x_i)=0, \quad \text{for all } q\in\mathcal{P}_{m-1}^d. \tag{2}$$

Here $\mathcal{P}_{m-1}^d$ is the space of all $d-$variate polynomials of degree $(m-1)$ or less, and is of dimension M, where

$$M=\begin{pmatrix} m-1+d \\ d \end{pmatrix}.$$

Then

$$p_{m-1}=\sum_{j=1}^{M}\alpha_j t_j \tag{3}$$

where $t_1,\ldots,t_M$ is a basis for $\mathcal{P}_{m-1}^d$. The function $\Phi(x)=\phi(\|x\|)$ is conditionally positive definite of order m, written CPD_m. Common choices for $\phi(r)$ in this setting are given in [10] and [13],

$$\begin{array}{llll} r^\beta, & \beta\notin 2\mathbb{N}, & m\geq\lceil\beta/2\rceil, & \text{polyharmonic spline,} \\ r^\beta\log r, & \beta\in 2\mathbb{N}, & m>\beta/2, & \text{polyharmonic spline,} \\ (c^2+r^2)^{\beta/2}, & \beta\notin 2\mathbb{N}, & m\geq\lceil\beta/2\rceil, & \text{generalised multiquadric.} \end{array}$$

The weights in (1), (2) and (3) form the solution vector to the following system

$$M\begin{pmatrix}\lambda\\ \alpha\end{pmatrix}=\begin{bmatrix} A & P \\ P^T & 0\end{bmatrix}\begin{pmatrix}\lambda\\ \alpha\end{pmatrix}=\begin{pmatrix} f\\ 0\end{pmatrix}.$$

Interpolating data in this way has been used in a variety of applications; see [1] and [3].

For large values of N the interpolation matrix M becomes ill–conditioned, causing computational difficulties. A method of circumventing this problem is to use a multilevel process of iterative refinement, as will be described in Section 2. A theoretical foundation for such an approach is provided by [9], and numerical viability has been demonstrated in [2] and [4]. As will be seen in

Section 3, polyharmonic splines have properties which allow simple pointwise error bounds to be obtained for multilevel schemes. Such estimates are valid for regular data on a scaled multiinteger grid, and a domain decomposition method as described in [4].

2 Iterative Refinement

Although attention will later be restricted to these specialised cases, it is beneficial to explain iterative refinement in the general setting. Consider a sequence of subsets $X_k = \{x_1^{(k)}, x_2^{(k)}, \ldots, x_{N_k}^{(k)}\}$ of X, where $X = \{x_1, x_2, \ldots, x_N\}$ and $.^{(k)}$ denotes a quantity pertaining to level k. At the k^{th} level, a refinement to the approximation is made by interpolating the error of the previous level $f - (s_1 + s_2 + \cdots + s_{k-1})$ at the points in X_k with

$$s_k(x) = \sum_{i=1}^{N_k} \lambda_i^{(k)} \Phi\left(\Delta^{(k)}(x - x_i^{(k)})\right) + p_{m-1}^{(k)}(x), \tag{4}$$

for some scale factor $\Delta^{(k)}$. Formally

$$\begin{aligned} s_1|_{X_1} &= f|_{X_1} \\ s_2|_{X_2} &= (f - s_1)|_{X_2} \\ &\vdots \\ s_k|_{X_k} &= (f - \sum_{i=1}^{k-1} s_i)|_{X_k}. \end{aligned}$$

For clarity, let the error at each level be

$$e_k = f - \sum_{j=1}^{k} s_j.$$

Then s_k interpolates e_{k-1} on X_k. Therefore the errors at different levels are related via

$$\begin{aligned} e_1(x) &= f(x) - s_1(x), \\ e_k(x) &= e_{k-1}(x) - s_k(x) \text{ for } k \geq 2. \end{aligned} \tag{5}$$

The philosophy of this approach is to use tighter basis functions as the levels increase, so $\Delta^{(k+1)} > \Delta^{(k)}$. The main features of the data f are captured at the outset, and finer details are resolved as the levels progress.

The use of different basis functions at each level causes complications in the error analysis. By scaling the points in X_k it is possible to obtain an equivalent interpolation problem using a fixed basis. By decomposing the domain we can also keep the number of interpolation points inside each subdomain fixed. The benefit of scaling the points and not the basis function is that this allows the analysis to remain in a fixed function space.

For ease consider the case when $\Delta^{(k)} = \delta^{k-1}$ for $\delta > 1$, and define

$$y_j^{(k)} = u + \delta^{k-1} x_j^{(k)}, \text{ for } j = 1, 2, \ldots, N_k, \quad u \in \mathbb{R}^d.$$

Further, let $y = u + x\delta^{k-1}$ and $Y_k = u + \delta^{k-1} X_k = \{y_1^{(k)}, y_2^{(k)}, \ldots, y_{N_k}^{(k)}\}$. We include a translation here because of the application we have in mind in Section 3.

At the k^{th} level, the previous error e_{k-1} is scaled by a factor of δ^{k-1} and interpolated using $\Phi(y)$ at the points in Y_k. The result is a scaled version of the original interpolant. By introducing the affine scaling $y = u + x\delta^{k-1}$, the interpolant in (4) becomes

$$\begin{aligned} s_k(x) &= \sum_{i=1}^{N_k} \lambda_i^{(k)} \Phi\left(\delta^{k-1}(x - x_i^{(k)})\right) + p_{m-1}^{(k)}(x) \\ &= \sum_{i=1}^{N_k} \lambda_i^{(k)} \Phi\left(y - y_i^{(k)}\right) + p_{m-1}^{(k)}\left((y-u)/\delta^{k-1}\right) \\ &= \sum_{i=1}^{N_k} \lambda_i^{(k)} \Phi\left(y - y_i^{(k)}\right) + q_{m-1}^{(k)}(y) =: t_k(y), \end{aligned}$$

where $q_{m-1}^{(k)} \in \mathcal{P}_{m-1}^d$. For clarity let the scaled error at each level be given by

$$g_k(y) = e_k(x) = e_k\left((y-u)/\delta^{k-1}\right). \tag{6}$$

From (5) the errors at different levels are related via

$$e_k(x) = e_{k-1}(x) - s_k(x).$$

With scaling this becomes

$$e_k\left((y-u)/\delta^{k-1}\right) = e_{k-1}\left((y-u)/\delta^{k-1}\right) - s_k(x),$$

and from the definition of g_k and t_k this yields

$$\begin{aligned} g_1 &= f - t_1, \\ g_k(y) &= g_{k-1}(y/\delta) - t_k(y), \qquad k \geq 2. \end{aligned} \tag{7}$$

So as $s_k(x)$ interpolates $e_{k-1}(x)$ on X_k, then $t_k(y)$ interpolates the scaled error $g_{k-1}(y/\delta)$ on Y_k, and is of the same form as (1).

The key aspect is that the interpolant obtained on the scaled set of points Y_k using $\Phi(y)$ is simply a scaled version of the solution to the problem on X_k which uses tighter basis functions $\Phi(x\delta^{k-1})$. From (6) the maximum pointwise error of the original and scaled problem is the same, therefore proof of convergence for $\|g_k\|_\infty$ is also valid for $\|e_k\|_\infty$. It will be shown in Section 3 that the error at each level decreases linearly.

Simplification of the error analysis is possible when the points used in Y_k are the same as those in Y_{k-1}. Examples of data configurations for which this is satisfied are given.

2.1 Data Configuration

Let X be a scaled multiinteger grid where $X = \mathbb{Z}^d$. Consider subsets of X given by $X_k = \delta^{1-k}\mathbb{Z}^d$, then the scaling $y = \delta^{k-1}x$ gives $Y_k = \mathbb{Z}^d$. Therefore the interpolation problem on Y_k uses the same points for all k.

A more practical application o this phenomenon occurs in the domain decomposition method presented in [4]. For some fixed $K \in \mathbb{N}$ let $\Omega = \{x \in \mathbb{R}^d : \|x\|_\infty \leq K\}$, and the associated grid $X = \{z \in \mathbb{Z}^d : z \in \Omega\}$. Let $S = \{-1, 1\}^d$. Then we can construct a refinement of the grid X using the sets

$$\Omega_{\sigma_1} = \frac{\sqrt{d}}{2}\sigma_1 + \frac{1}{2}\Omega, \qquad \sigma_1 \in S :$$

$$X_1 = \bigcup_{\sigma_1 \in S} X_{\sigma_1},$$

where $X_{\sigma_1} = \{z/2 \in \Omega_{\sigma_1} : z \in \mathbb{Z}^d\}$. The cardinality of X_{σ_1} is exactly that of X.

At the kth level we have the decomposition

$$\Omega_{\sigma_1,\ldots,\sigma_{k-1}} = \bigcup_{\sigma_k \in S} \Omega_{\sigma_1,\ldots,\sigma_k},$$

where

$$\Omega_{\sigma_1,\ldots,\sigma_k} = \frac{\sqrt{d}}{2^m}\sigma_k + \frac{1}{2}\Omega_{\sigma_1,\ldots,\sigma_{k-1}}, \qquad \sigma_k \in S.$$

The associated local grid refinement is $X_{\sigma_1,\ldots,\sigma_k} = \{z/2^k \in \Omega_{\sigma_1,\ldots,\sigma_k} : z \in \mathbb{Z}^d\}$, with global grid

$$X_k = \bigcup_{\sigma_1,\ldots,\sigma_k \in S^k} X_{\sigma_1,\ldots,\sigma_k}.$$

Then, each sub-grid $X_{\sigma_1,\ldots,\sigma_k}$ is just an affine scaling of the original grid:

$$X_{\sigma_1,\ldots,\sigma_k} = u_{\sigma_1,\ldots,\sigma_k} + 2^{-k}X, \tag{8}$$

where

$$u_{\sigma_1,\ldots,\sigma_k} = \sqrt{d}\sum_{i=1}^{k} 2^{-i}\sigma_i.$$

The residual interpolation problem on the grid $X_{\sigma_1,\ldots,\sigma_k}$ is solved using the basis function $\Phi(2^k\cdot)$ with support limited to the given sub-domain. This means that the algorithm is highly parallelizable because there is no communication between sub-domains. After the affine scaling prescribed by (8) described above we have $Y_k = X$ for all k and the basis function used in the scaled problem is Φ.

3 Error Analysis

In this section we shall concentrate on the error analysis for the multilevel domain decomposition algorithm, using the refinement described in the previous section. It is straightforward to adapt this analysis to the infinite grid case.

Before addressing the multilevel scheme, let s_f be an interpolant of the form (1) to f at points in X. It is necessary to bound the size of the error $(f - s_f)$ measured in some norm. Let $\hat{\Phi}$ be the Fourier transform of Φ given by

$$\hat{\Phi}(w) = \int_{\mathbb{R}^d} \Phi(x)e^{-ix^Tw}\mathrm{d}x.$$

The native space of Φ, denoted by $\mathcal{G}_\Phi$, contains all functions f such that

$$\hat{f}/\sqrt{\hat{\Phi}} \in L_2(\mathbb{R}^d).$$

If $\Phi \in$CPD$_m$, then

$$\|f\|_\Phi = \|\hat{f}/\sqrt{\hat{\Phi}}\|_2 \tag{9}$$

is a semi-norm on $\mathcal{G}_\Phi$ with kernel $\mathcal{P}_{m-1}$. Then there exists a pointwise error bound of the form

$$|f(x) - s_f(x)| \leq P(x)\|f\|_\Phi, \tag{10}$$

where P is called the power function; see [13]. The following proposition is a direct consequence of Theorem 4.3 of Light and Wayne [7], and gives the form of the power function which we shall use in the sequel.

Proposition 1 *Let $x_1, \ldots, x_N$ be ordered so that $x_1, \ldots, x_M$ are unisolvent with respect to $\mathcal{P}_{m-1}$. Let $p_1, \ldots, p_M$ satisfy $p_i(x_j) = \delta_{ij}$, $i, j = 1, \ldots, M$. Then,*

$$P(x) = \left\{ \Phi(0) - 2\sum_{j=1}^{M} p_j(x)\Phi(x - x_j) + \sum_{i,j=1}^{M} p_i(x)p_j(x)\Phi(x_i - x_j) \right\}^{1/2}. \tag{11}$$

Let P_δ be the power function obtained using $\Phi(\cdot/\delta)$ as a basis function. What we see immediately is that P_δ is invariant under affine scalings, where δ is the scaling parameter. In other words, if $y_j = u + \delta x_j$, $j = 1, \ldots, N$, $u \in \mathbb{R}^d$, $\delta > 0$, and $y = u + \delta x$, then

$$\begin{aligned} P_\delta(y) &= \left\{ \Phi(0/\delta) - 2\sum_{j=1}^{M} p_j^\delta(y)\Phi((y - y_j)/\delta) + \sum_{i,j=1}^{M} p_i^\delta(y)p_j^\delta(y)\Phi((y_i - y_j)/\delta) \right\}^{1/2} \\ &= P(x) \end{aligned} \tag{12}$$

where $p_i^\delta(y_j) = \delta_{ij}$, $i, j = 1, \ldots, M$. Above, we have used the (straightforward to establish) fact that $p_i^\delta(y) = p_i(x)$. Therefore, unlike the case for the usual interpolation problem (see [7]), we will not analyse the power function to infer convergence in the multilevel algorithm. In order to prove pointwise convergence it will be then necessary to show that the error in the $\|\cdot\|_\Phi$ seminorm decreases with successive levels.

For simplicity of notation we will consider the decrease in error at the first level of refinement, when we decompose Ω with the associated grid X and consider one of the 2^d 'quadrants' Ω_σ with grid X_σ for some $\sigma \in S$. The basis function for interpolation on X is Φ and on X_σ is $\Phi(2\cdot)$. The residual on Ω is e and on Ω_σ is e_σ. Thus, if s_σ is the interpolant to e on X_σ, then $e_\sigma = e - s_\sigma$. For $x \in X_\sigma$, if we put $y = 2x - \sqrt{d}\sigma$, $g(y) = e(x)$, $g_\sigma(y) = e_\sigma(x)$, and $t_\sigma(y) = s_\sigma(x)$, then we can rewrite (7) in the form

$$g_\sigma(y) = g(y/2) - t_\sigma(y). \tag{13}$$

Since t_σ is the best approximation to $g(\cdot/2)$ using Φ in $\mathcal{G}_\Phi$ the following holds

$$\begin{aligned} \|g(\cdot/2)\|_\Phi^2 &= \|g(\cdot/2) - t_\sigma\|_\Phi^2 + \|t_\sigma\|_\Phi^2 \\ &= \|g_\sigma\|_\Phi^2 + \|t_\sigma\|_\Phi^2; \end{aligned} \tag{14}$$

see [8]. Thus,

$$\|g_\sigma\|_\Phi \leq \|g(\cdot/2)\|_\Phi.$$

What we wish to show is that

$$\|g_\sigma\|_\Phi \leq C \cdot \|g\|_\Phi, \tag{15}$$

for $C < 1$. This inequality can then be applied recursively, so that if $g_{\sigma_1,\ldots,\sigma_k}$ is the scaled residual from $\Omega_{\sigma_1,\ldots,\sigma_k}$, then

$$\|g_{\sigma_1,\ldots,\sigma_k}\|_\Phi \leq C^k \|g\|_\Phi.$$

Since g is the residual when interpolating f using ϕ we can use (14) again to see that $\|g\|_\Phi \leq \|f\|_\Phi$, so that

$$\|g_{\sigma_1,\ldots,\sigma_k}\|_\Phi \leq C^k \|f\|_\Phi, \tag{16}$$

as long as (15) is satisfied. We can now use (10) to see that, for $x \in \Omega_{\sigma_1,\ldots,\sigma_{k+1}}$,

$$\begin{aligned} |g_{\sigma_1,\ldots,\sigma_{k+1}}(x)| &\leq P_{2^{-k-1}}(x)\, C^k \|f\|_\Phi \\ &= P(y) C^k \|f\|_\Phi, \end{aligned}$$

where

$$y = 2^{k+1}(u_{\sigma_1,\ldots,\sigma_{k+1}} + x);$$

see (8). Since $y \in \Omega$ which is compact, and P is a continuous function, if (15) holds then

$$|g_{\sigma_1,\ldots,\sigma_{k+1}}(x)| \leq AC^k \|f\|_\Phi, \tag{17}$$

for some constant A which is independent of k and x.

It now remains to show that $C < 1$ in (15).

3.1 Polyharmonic Splines and the Constant C

Semi-norms generated by some basis functions are able to be written explicitly in terms of the function itself rather than its Fourier transform; see [6]. Let Φ be a polyharmonic spline which is conditionally positive definite of order m. In order that $\mathcal{G}_\Phi \subset C(\mathbb{R}^d)$, so that interpolation makes sense, it is necessary that $m > d/2$; see [11]. Then $\mathcal{G}_\Phi$ coincides with the Beppo–Levi space of order m on $\mathbb{R}^d$. In these cases (see for instance [13])

$$\|f\|_\Phi = \left(\sum_{|\alpha|=m} c_\alpha \|D^\alpha f\|^2 \right)^{\frac{1}{2}} \tag{18}$$

and the coefficients c_α are chosen such that

$$\sum_{|\alpha|=m} c_\alpha x^\alpha = |x|^{2m}.$$

Now, if $g = f(\cdot/\delta)$ then

$$\begin{aligned} \|D^\alpha g\|_2^2 &= \int_{\mathbb{R}^d} |D^\alpha g|^2 \\ &= \delta^{-2m} \int_{\mathbb{R}^d} |D^\alpha f(\cdot/\delta)|^2 \\ &= \delta^{d-2m} \int_{\mathbb{R}^d} |D^\alpha f|^2 \\ &= \delta^{d-2m} \|D^\alpha f\|_2^2. \end{aligned}$$

Hence

$$\|f(\cdot/\delta)\|_\Phi = \delta^{\frac{d}{2}-m}\|f\|_\Phi. \tag{19}$$

Since $m > d/2$, then by setting $C = \delta^{\frac{d}{2}-m}$ this proves (15). It is interesting to note that condition on m to obtain (18) is exactly required to guarantee that $C < 1$.

Thus, with (17), we have

Theorem 1 *Let e_k be the error at level k approximating f by the multilevel domain decomposition method, using polyharmonic splines. Then*

$$\|e_k\|_{\infty,\Omega} \leq AC^{k-1}\|f\|_\Phi,$$

where A is independent of k and f, and $C = 2^{\frac{d}{2}-m}$.

4 Numerical Results

As a consequence of the error estimate in Theorem 1, one would expect to see the pointwise error decay linearly at a rate of $C = \delta^{\frac{d}{2}-m}$. A number of examples are given where a test point has been chosen to examine the behaviour of $\|g\|_\infty$.

The multilevel domain decomposition scheme as described in Section 2.1 is implemented to interpolate a variety of datasets, where $d = 1$, given by the functions

$$\begin{aligned} F_1(x) &= \sin(10x) + 3x^2, \\ F_2(x) &= \frac{\sin(16x)}{16(x+1)}, \\ F_3(x) &= x\exp(-x^2). \end{aligned}$$

The initial grid X consists of 20 regularly spaced points on $[0,1]$, and basis functions of the form $\|x\|^\beta \log(\|x\|)$ for $\beta = 2,4$ are used. The scale factor is chosen such that $\delta = 2$. Figures 1 & 2 show the error at an arbitrary test

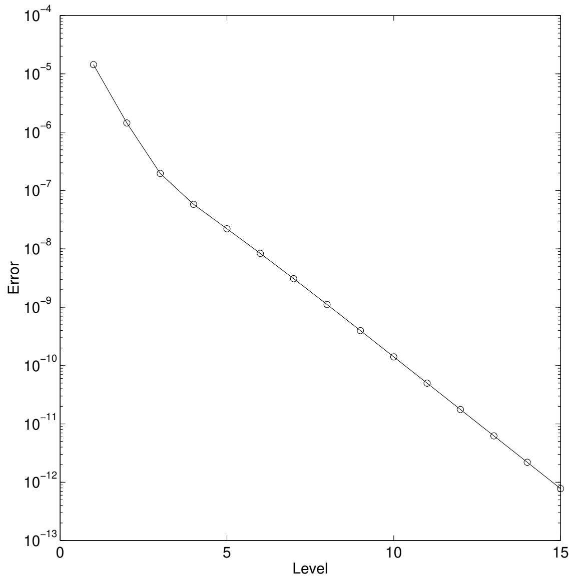

Figure 1: Convergence of Multilevel Method for $r^2 \log(r)$ in $\mathbb{R}$

point $y = \frac{1}{2}$, for initial data F_3 in both cases. Table 1 contains approximate convergence rates, and the appropriate values of $\sqrt{C} = 2^{-\frac{1}{2}(\beta+1)}$ for comparison.

In $\mathbb{R}^2$ the initial grid X consists of a 10×10 regular grid on $[0,1]^2$. Again the scale factor is chosen such that $\delta = 2$, and the test point is located at $y = (\frac{1}{2}, \frac{1}{2})$. The initial data F_4 is given by Franke's function

$$\begin{aligned} F_4(x,y) &= 0.75e^{-0.25(9x-2)^2-0.25(9y-2)^2} + 0.75e^{-(9x-2)^2/49-(9y-2)^2/10} \\ &\quad +0.5e^{-0.25(9x-7)^2-0.25(9y-3)^2} - 0.2e^{-(9x-4)^2-(9y-7)^2}. \end{aligned}$$

In $\mathbb{R}^3$ then initial grid X consists of a $6 \times 6 \times 6$ regular grid on $[0,1]^3$. The test point is located at $y = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, and the data F_5 is given by

$$F_5(x,y,z) = x^2 y^3 \cos(10z).$$

Tables 2 and 3 contain the approximate convergence rate, and the predicted values of C for the two and three dimensional cases.

Theorem 1 predicts the convergence rate well in the one dimensional case. In the two and three dimensional cases however, it becomes clear that the actual pointwise error converges faster than the error bound predicted by the theory. Interestingly, the error appears to decrease at the same rate for the thin plate spline in both $\mathbb{R}$, $\mathbb{R}^2$, and $\mathbb{R}^3$.

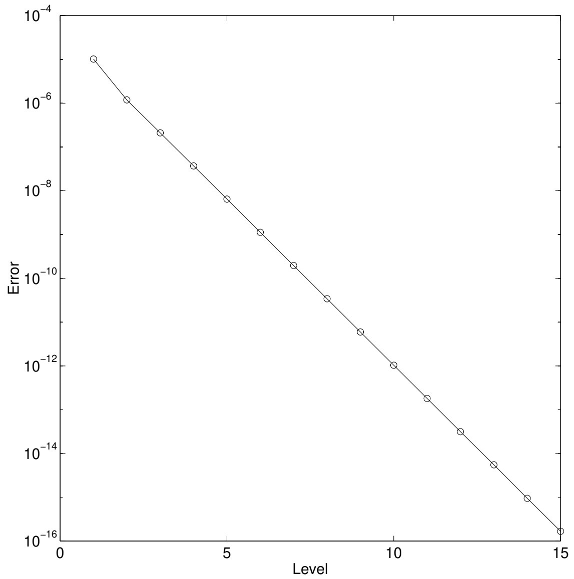

Figure 2: Convergence of Multilevel Method for $r^4 \log(r)$ in $\mathbb{R}$

Dataset	Basis Function	
	$\beta = 2$	$\beta = 4$
F_1	0.353	0.177
F_2	0.353	0.175
F_3	0.353	0.174
C	0.354	0.177

Table 1: Convergence Rates of Multilevel Method in $\mathbb{R}$

Dataset	Basis Function	
	$\beta = 2$	$\beta = 4$
F_4	0.344	0.15
C	0.5	0.25

Table 2: Convergence Rates of Multilevel Method in $\mathbb{R}^2$

Dataset	Basis Function	
	$\beta = 2$	$\beta = 4$
F_5	0.3281	0.086
C	0.707	0.354

Table 3: Convergence Rates of Multilevel Method in $\mathbb{R}^3$

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