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# Towards a language theory for infinite N-free pomsets*

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**Abstract**

N-free or series-parallel pomsets are a model for the behavior of modularly constructed concurrent systems. The investigation of languages of finite N-free pomsets was initiated by Lodaya & Weil who extended the theorems by Kleene and by Myhill & Nerode on recognizable word languages to this setting. In this paper, we extend Lodaya & Weil's results to infinite N-free pomsets and generalize Büchi's theorem. Furthermore, we investigate first-order definable, starfree, and aperiodic sets of infinite N-free pomsets and prove results in the spirit of McNaughton & Papert's and Schützenberger's theorems for finite words.

## 1 Introduction

In theoretical computer science, the notion of a recognizable subset of a monoid and more generally of an algebra is of outstanding importance. Here, recognizability means to be recognized by a homomorphism into a finite algebra. Seen as subset of an algebra, recognizable sets can often be described by certain rational expressions.

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Frequently, this algebraic notion is equivalent to the more combinatorial notion of regularity, i.e. acceptance by a finite automaton. Furthermore, the elements of the algebra in consideration can carry an internal structure. For instance, words can be seen as labeled linear orders, terms as labeled ordered trees, Mazurkiewicz traces as dependence graphs etc. If such an internal structure is present, it is natural to consider sets of such structures that share a typical property. Classical results state that properties expressed in monadic second order logic give rise to recognizable sets. This holds for words, terms, Mazurkiewicz traces, computations of stably concurrent automata, local traces, graphs of bounded tree width, and many others. If one restricts the expressibility of the logic, e.g. to first order logic, corresponding restrictions of the set of recognizable sets can be described.

In this paper, we carry out this program for N-free (or series-parallel) pomsets. They were introduced into theoretical computer science to model the behavior of modularly constructed concurrent systems [Sta81, Gra81, Pra86]. Gischer [Gis88] showed that the equational theory of the set of N-free pomsets with sequential and parallel product is finitely based. In his paper, one also finds a proof that a finite pomset is N-free iff it is constructed from singletons by sequential and parallel product. Bloom & Ésik [BÉ98] gave an order-theoretic characterization of those pomsets that can be constructed by sequential and parallel product and sequential $\omega$-power. A similar result was shown by Ésik & Okawa [ÉO99] if one in addition allows the parallel $\omega$-power. We will consider the set of pomsets that can be constructed from the singletons by sequential and parallel iteration and by the infinite product. It will be shown that these are precisely those at most countable pomsets that do not embed the N-shaped poset, where any principal ideal is finite, and that contain only finite antichains.

The investigation of languages of finite N-free pomsets was initiated by Lodaya & Weil [LW98, LW00b, LW00a]. In particular, they defined branching automata, an extension of finite automata that can accept finite N-free pomsets. Lodaya & Weil showed that a set of finite N-free pomsets (1) can be accepted by a finite branching automaton of bounded depth iff (2) it can be constructed from the singletons by the sequential and parallel product, the union, and the sequential iteration iff (3) it is width-bounded and recognizable by a homomorphism into a finite algebra [LW98, LW00b].

Inspired by the corresponding notion for $\omega$-languages of words, we define when a set of infinite N-free pomsets is $\omega$-recognizable or saturated by a homomorphism into a finite algebra. Furthermore, we extend branching automata to branching Büchi-automata that can accept infinite N-free pomsets. Relating these notions, we show that the following are equivalent for a set of infinite N-free pomsets $X$ of bounded width:

1.1. $X$ is $\omega$-recognizable,

1.2. $X$ is definable in monadic second order logic,

1.3. $X$ is $\omega$-series-rational, i.e. constructed from the singletons by sequential and parallel product, sequential iteration and $\omega$-iteration, and union, and

1.4. $X$ is accepted by a finite branching Büchi-automaton.

These equivalences extend Büchi's and Kleene's results on infinite words to the realm of infinite N-free pomsets. For finite N-free pomsets, the equivalence of 1.1, 1.3 and 1.4 was already shown by Lodaya & Weil in [LW00b]. But the restriction of the first equivalence to finite N-free pomsets as well as the other equivalences for inifinite N-free pomsets are new. These results show that one can generalize the classical results on recognizable $\omega$-languages of words to infinite N-free pomsets.

The restriction of monadic second order logic to first order (or elementary) logic has been considered for finite and infinite words as well. It was shown that a set of finite words is starfree iff it is aperiodic iff it can be defined by a sentence of elementary logic [Sch65, MP71]. These equivalences were extended to $\omega$-words by Ladner [Lad77], Thomas [Tho79, Tho81] and Perrin [Per84]. Here, we also consider sets of N-free pomsets defined by a sentence from elementary logic. We show that the following statements are equivalent for a set of finite N-free pomsets $X$ of bounded width:

2.1. $X$ is aperiodic (i.e. recognized by a homomorphism into a finite sp-algebra with aperiodic sequential product),

2.2. $X$ is definable in elementary logic, and

2.3. $X$ is starfree, i.e. constructed from the singletons by sequential and parallel product, and Boolean operations.

These equivalences extend the results by Schützenberger [Sch65] and McNaughton & Papert [MP71] from finite words to finite N-free pomsets. Finally, we also get these equivalence for sets of infinite N-free pomsets.

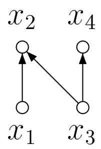
In general, the equivalences mentioned above do not hold for sets of N-free pomsets of unbounded width. In particular, the implications 1.1→1.2, 1.1→1.3, 2.1→2.2, and 2.1→2.3 become false in general (cf. Example 5.4). But we show that the implications 1.2→1.1 and 2.3→2.2→2.1 hold for unbounded sets as well.

# 2 Basic definitions

## 2.1 Order theory

In this paper, any partially ordered set is assumed to be nonempty. Let $(V, \leq)$ be a partially ordered set. We write $x \parallel y$ for elements $x, y \in V$ if they are incomparable. A set $A \subseteq V$ is an *antichain* provided the elements of $A$ are mutually incomparable. The *width* of the partially ordered set $(V, \leq)$ is the least cardinal $w(V, \leq)$ such that $|A| \leq w(V, \leq)$ for any antichain $A$. If $w(V, \leq)$ is a natural number, we say $(V, \leq)$ *has finite width*. Note that there exist partially ordered sets that contain only finite antichains, but have infinite width. $\downarrow y$ denotes the principal ideal $\{x \in V \mid x \leq y\}$ generated by $y \in V$.

An *N-free poset* $(V, \leq)$ is a nonempty, at most countably infinite partially ordered set such that the partially ordered set $(N, \leq_N)$ cannot be embedded into $(V, \leq)$ (cf. picture below), any antichain in $(V, \leq)$ is finite, and $\downarrow x$ is finite for any $x \in V$. We fix an alphabet $\Gamma$, i.e. a nonempty finite set. Then $\mathrm{NF}^\infty$ denotes the set of all $\Gamma$-labeled N-free posets $(V, \leq, \lambda)$. These labeled posets are called *N-free pomsets*. Let $\mathrm{NF}$ denote the set of finite N-free pomsets over $\Gamma$.




The poset $(N, \leq_N)$

Next, we define the sequential and the parallel product of $\Gamma$-labeled posets: Let $p_1 = (V_1, \leq_1, \lambda_1)$ and $p_2 = (V_2, \leq_2, \lambda_2)$ be $\Gamma$-labeled posets with $V_1 \cap V_2 = \emptyset$. The *sequential product* $p_1 \cdot p_2$ of $p_1$ and $p_2$ is the $\Gamma$-labeled partial order

$$(V_1 \cup V_2, \leq_1 \cup V_1 \times V_2 \cup \leq_2, \lambda_1 \cup \lambda_2).$$

Thus, in $p_1 \cdot p_2$, the labeled poset $p_2$ is put on top of the labeled poset $p_1$. On the contrary, the *parallel product* $p_1 \parallel p_2$ is defined to be

$$(V_1 \cup V_2, \leq_1 \cup \leq_2, \lambda_1 \cup \lambda_2),$$

i.e. here the two partial orders are set side by side. By $\mathrm{SP}$, we denote the least class of $\Gamma$-labeled posets containing the singletons that is closed under the application of the sequential product $\cdot$ and the parallel product $\parallel$.

To construct infinite labeled posets, we extend the sequential product $\cdot$ naturally to an infinite one as follows: For $i \in \omega$, let $p_i = (V_i, \leq_i, \lambda_i)$ be mutually disjoint $\Gamma$-labeled posets. Then the infinite sequential product is defined by

$$\prod_{i \in \omega} p_i = (\bigcup_{i \in \omega} V_i, \bigcup_{i \in \omega} \leq_i \cup \bigcup_{\substack{i,j \in \omega \\ i<j}} V_i \times V_j, \bigcup_{i \in \omega} \lambda_i).$$

By $\mathrm{SP}^\infty$, we denote the least class $\mathcal{C}$ of $\Gamma$-labeled posets such that

- $\mathrm{SP} \subseteq \mathcal{C}$,
- $p_1, p_2 \in \mathcal{C}$ implies $p_1 \parallel p_2 \in \mathcal{C}$,
- $p_1, p_2 \in \mathcal{C}$ and $p_1$ finite imply $p_1 \cdot p_2 \in C$, and
- $p_i \in \mathcal{C}$ finite for $i \in \omega$ implies $\prod_{i \in \omega} p_i \in \mathcal{C}$.

Thus, a $\Gamma$-labeled poset belongs to $\mathrm{SP}^\infty$ if it can be constructed from the finite $\Gamma$-labeled N-free posets applying the sequential product, the parallel product or the infinite product.

Bloom & Ésik characterize the pomsets that can be constructed by the sequential and parallel product and by the sequential $\omega$-power [BÉ98, Thm. 4.5]: A pomset can be constructed in this way iff $(N, \leq_N)$ cannot be embedded into it, any principal ideal is finite and there are only finitely many non-isomorphic filter (i.e. if it satisfies the *generalized N-condition*). Note that any such pomset contains only finite antichains: If $\{a_1, a_2, \dots\}$ is an infinite antichain, then the filters generated by $\{a_1, a_2, \dots, a_n\}$ are mutually non-isomorphic. Actually, in the proofs of [BÉ98, Lemma 4.6-4.10], only this weaker condition is used. Therefore, the proof of the following lemma can be taken verbatim from [BÉ98].

**Lemma 2.1 ([BÉ98, Lemma 4.10])** *Let $p \in \mathrm{NF}^\infty$ be connected and infinite. Then precisely one of the following statements holds:*

1. *$p$ is an infinite product of finite N-free pomsets, or*
2. *there exists a finite N-free pomset $p_0$ and finitely many connected N-free pomsets $p_1, p_2, \dots, p_n$ for some $n > 1$ such that $p = p_0 \cdot (p_1 \parallel p_2 \parallel \cdots p_n)$.*

Based on this result, we now extend the known equality $\mathrm{SP} = \mathrm{NF}$ [Gis88] to infinite $\Gamma$-labeled posets.

**Lemma 2.2** *Let $\Gamma$ be an alphabet. Then $\mathrm{SP}^\infty = \mathrm{NF}^\infty$.*

**Proof.** Let $p_1$ and $p_2$ be N-free pomsets. Recall that the left factor in a sequential product is always finite. Hence any principal ideal in $p_1 \cdot p_2$ is finite. Furthermore, any antichain in $p_1 \cdot p_2$ lies entirely in $p_1$ or in $p_2$. Hence one gets similarly to [Gis88] that the sequential product of N-free pomsets is N-free, again. Similarly, as any factor in an infinite sequential product is finite, an infinite sequential product is N-free. As the parallel product of two N-free pomsets is their disjoint union, any of its principal ideals is finite. Furthermore, any antichain in $p_1 \parallel p_2$ is the union of two antichains in $p_1$ and in $p_2$. As these two antichains are finite, so is their union. Thus, the parallel product is N-free, again. This proves the inclusion $\mathrm{SP}^\infty \subseteq \mathrm{NF}^\infty$.

For the converse inclusion, let $p \in \mathrm{NF}^\infty$. If $p$ is not connected, it is a finite parallel product since $p$ does not contain infinite antichains. So, from now on we may assume that $p$ is connected. By Lemma 2.1, either $p$ is an infinite sequential product of finite N-free pomsets, or there exist $p_0 \in \mathrm{NF}$ and connected pomsets $p_1, p_2, \ldots, p_n \in \mathrm{NF}^\infty \setminus \{\emptyset\}$ with $p = p_0 \cdot (p_1 \parallel p_2 \parallel \cdots p_n)$. In the first case, we are done since any finite N-free pomset is an element of $\mathrm{SP}$ [Gis88]. In the second case, we proceed inductively with $p_1, p_2, \ldots, p_n$. It remains to prove that this inductive decomposition eventually terminates: By contradiction, assume $p = p_0, p_1, p_2, \ldots$ is an infinite sequence of connected N-free pomsets such that for any $i > 0$, there exist $q_i \in \mathrm{NF}$ and $p'_i \in \mathrm{NF}^\infty$ with $p_i = q_{i+1} \cdot (p_{i+1} \parallel p'_{i+1})$. Let $v_i$ be some node from $p'_i$. Note that $v_i$ for $i > 0$ is a node from $p'_i$ and $v_j$ for $j > i$ belongs to $p_i$. Hence the nodes $v_i$ are mutually incomparable nodes of $p = p_0$ which contradicts our assumption that $p$ contains only finite antichains. Hence, indeed, $\mathrm{NF}^\infty \subseteq \mathrm{SP}^\infty$. $\square$

## 2.2 Languages of N-free pomsets

In this section, we define several classes of languages of N-free pomsets. First, the generalizations of rational, starfree, $\omega$-rational, and $\omega$-starfree languages are introduced:

The sequential, the parallel and the infinite sequential products can easily be extended to sets of (finite) N-free pomsets as follows: Let $L \subseteq \mathrm{NF}$ and $L', M' \subseteq$

$\mathrm{NF}^\infty$. Then we define

$$\begin{aligned} L \cdot M' &:= \{p \cdot q \mid p \in L, q \in M'\}, \\ L' \parallel M' &:= \{p \parallel q \mid p \in L', q \in M'\}, \\ L^+ &:= \{p_1 \cdot p_2 \cdots p_n \mid n > 0, p_i \in L\}, \text{ and} \\ L^\omega &:= \{\prod_{i \in \omega} p_i \mid p_i \in L\}. \end{aligned}$$

The class of *series-rational languages* [LW98, LW00b] is the least class $\mathcal{C}$ of subsets of $\mathrm{NF}$ such that

- $\{p\} \in \mathcal{C}$ for $p \in \mathrm{NF}$, and
- $L \cup M, L \cdot M, L \parallel M, L^+ \in \mathcal{C}$ for $L, M \in \mathcal{C}$.

Note that we do not allow the iteration of the parallel product in the construction of series-rational languages. Therefore, for any series-rational language $L$ there exists an $n \in \omega$ with $w(p) \leq n$ for any $p \in L$, i.e. any series-rational language is *width-bounded*.

The class of *starfree languages* is the least class $\mathcal{C}$ of subsets of $\mathrm{NF}$ such that

- $\{p\} \in \mathcal{C}$ for $p \in \mathrm{NF}$, and
- $L \cup M, L \cdot M, L \parallel M, \mathrm{NF} \setminus L \in \mathcal{C}$ for $L, M \in \mathcal{C}$.

Differently from series-rational languages, a starfree language is in general not width-bounded since, e.g., $\mathrm{NF}$ is starfree.

The class of *$\omega$-series-rational languages* is the least class $\mathcal{C}$ of subsets of $\mathrm{NF}^\infty$ such that

- $\{p\} \in \mathcal{C}$ for $p \in \mathrm{NF}$,
- $L \cup M, L \parallel M \in \mathcal{C}$ for $L, M \in \mathcal{C}$, and
- $L^+, L^\omega, L \cdot M \in \mathcal{C}$ for $L, M \in \mathcal{C}$ and $L \subseteq \mathrm{NF}$.

For the same reason as for series-rational languages, any $\omega$-series-rational language is width-bounded. It is easily seen that the series-rational languages are precisely those $\omega$-series-rational languages that contain only finite labeled posets.

The class of *$\omega$-starfree languages* is the least class $\mathcal{C}$ of subsets of $\mathrm{NF}^\infty$ such that

- $L \in \mathcal{C}$ if $L \subseteq \mathrm{NF}$ is starfree,
- $L \cup M, L \parallel M \in \mathcal{C}$ for $L, M \in \mathcal{C}$, and
- $L \cdot M, \mathrm{NF}^\infty \setminus M \in \mathcal{C}$ for $L, M \in \mathcal{C}$ and $L \subseteq \mathrm{NF}$.

Next, we want to define when a language of N-free pomsets is recognizable. Let $S$ be a set that is equipped with two binary operations $\cdot$ and $\|$. We assume these two operations to be associative and, in addition, $\|$ to be commutative. Then $(S, \cdot, \|)$ is an *sp-algebra* [LW00b]. Note that the set of finite N-free pomsets is an sp-algebra. An sp-algebra $(S, \cdot, \|)$ is *aperiodic* if there exists $n \in \mathbb{N}$ such that $s^n = s^{n+1}$ for any $s \in S$ where $s^n$ is the $n$-fold sequential product of $s$. Mappings between sp-algebras that commute with the two products will be called *sp-homomorphisms*.

A set $X$ of finite N-free pomsets is *recognizable* (*aperiodic*, resp.) if there exists a finite (and aperiodic, resp.) sp-algebra $(S, \cdot, \|)$ and an sp-homomorphism $\eta : \mathrm{NF} \to S$ such that $X = \eta^{-1}\eta(X)$. In [LW98, LW00b] Lodaya & Weil showed, e.g.,. that a set of finite N-free pomsets is series-rational iff it is recognizable and width-bounded.

Recall that a set of infinite words $L \subseteq \Gamma^\omega$ is Büchi-recognizable if there exists a finite semigroup $(S, \star)$ and a semigroup homomorphism $\eta : \Gamma^+ \to (S, \star)$ such that for any $u_i, v_i \in \Gamma^+$ with $\eta(u_i) = \eta(v_i)$ for $i \in \omega$, we have $u_0 u_1 u_2 \cdots \in L$ if and only if $v_0 v_1 v_2 \cdots \in L$ (cf. [PP99]). Here, we use this characterization as a definition and transfer it into the realm of N-free pomsets:

Let $\mathcal{X}$ be a set of variables that will range over elements of $\mathrm{NF}$. We call the terms over $\cdot$ and $\|$ that contain variables in $\mathcal{X}$ *finite terms*. Now let $t_i$ be finite terms for $i \in \omega$. Then $\prod_{i\in\omega} t_i$ is a *term* and any finite term is a term. Furthermore, if $t$ is a finite term and $t_i$ are terms for $1 \leq i \leq n$, then $t \cdot t_1$ and $t_1 \parallel t_2 \parallel \cdots t_n$ are terms, too. Now let $f : \mathcal{X} \to \mathrm{NF}$. Then $f(t)$ is defined naturally for any term $t$. Let $L \subseteq \mathrm{NF}^\infty$ and let $\eta : (\mathrm{NF}, \cdot, \|) \to (S, \cdot, \|)$ be an sp-homomorphism into a finite sp-algebra. Then $\eta$ *recognizes* $L$ iff for any term $t$ and any mappings $f, g : \mathcal{X} \to \mathrm{NF}$ with $\eta \circ f = \eta \circ g$, we have $f(t) \in L$ if and only if $g(t) \in L$. In this case, we will say that $L$ is *$\omega$-recognizable*. Furthermore, $L$ is *$\omega$-aperiodic* if it is recognized by an sp-homomorphism into a finite aperiodic sp-algebra.

One can easily check that $L \subseteq \mathrm{NF}$ is recognizable if and only if it is $\omega$-recognizable (and that the same holds for the notions aperiodic and and $\omega$-aperiodic): First, let $L = \eta^{-1}\eta(L)$. Furthermore, let $f, g : \mathcal{X} \to \mathrm{NF}$ with $\eta \circ f = \eta \circ g$ and let $t$ be a term with $f(t) \in L$. Since $L$ consists of finite pomsets, only, $t$ is a finite term (recall that we excluded the empty pomset from our consideration). Since $\eta$ is an sp-homomorphism, we get $\eta(f(t)) = \eta(g(t))$ and therefore $g(t) \in \eta^{-1}\eta(L) = L$. Hence $L$ is $\omega$-recognizable. Conversely, let $L$ be $\omega$-recognizable and let $\eta$ be an sp-homomorphism that recognizes $L$. Suppose $p \in L$ and $q \in \mathrm{NF}$ with $\eta(p) = \eta(q)$. Define two functions $f, g : \mathcal{X} \to \mathrm{NF}$ by $f(x) = p$ and $g(x) = q$ for any $x \in \mathcal{X}$. Then, of course, $\eta \circ f = \eta \circ g$. Since $x$ is a term and $\eta$ recognizes $L$, this implies $q = g(x) \in L$. Thus, $L$ is recognizable.

## 2.3 Monadic second order logic

In this section, we will define monadic second order formulas and their interpretations over $\Gamma$-labeled pomsets. *Monadic formulas* involve first order variables $x, y, z \dots$ for vertices and monadic second order variables $X, Y, Z, \dots$ for sets of vertices. They are built up from the atomic formulas $\lambda(x) = a$ for $a \in \Gamma$, $x \leq y$, and $x \in X$ by means of the boolean connectives $\neg$, $\vee$, $\wedge$, $\rightarrow$, $\leftrightarrow$ and quantifiers $\exists, \forall$ (both for first order and for second order variables). Formulas without free variables are called sentences. The satisfaction relation $\models$ between $\Gamma$-labeled posets $p = (V, \leq, \lambda)$ and monadic sentences $\varphi$ is defined canonically with the understanding that first order variables range over the vertices of $V$ and second order variables over subsets of $V$.

Let $\mathcal{C}$ be a set of $\Gamma$-labeled posets and $\varphi$ a monadic sentence. Furthermore, let $L = \{p \in \mathcal{C} \mid p \models \varphi\}$ denote the set of posets from $\mathcal{C}$ that satisfy $\varphi$. Then we say that the sentence $\varphi$ *axiomatizes the set* $L$ *relative to* $\mathcal{C}$ or that $L$ is *monadically axiomatizable relative to* $\mathcal{C}$.

An *elementary formula* is a monadic formula that does not contain any set variable. A set $X \subseteq \mathcal{C}$ is *elementarily axiomatizable relative to* $\mathcal{C}$ if there exists an elementary sentence that axiomatizes $X$ relative to $\mathcal{C}$.

**Lemma 2.3** *An N-free pomset from* $\mathrm{NF}^\infty$ *is finite iff it satisfies*

$$\varphi = \forall x \exists y (x \leq y \wedge \forall z (y \leq z \to y = z)),$$

*i.e.* $\mathrm{NF}$ *is elementary axiomatizable relative to* $\mathrm{NF}^\infty$.

**Proof.** If $p = (V, \leq, \lambda)$ is finite, then any node $x$ is dominated by a maximal node, so $p \models \varphi$. Converselly let $p \models \varphi$ for some $p = (V, \leq, \lambda) \in \mathrm{NF}^\infty$. Since antichains in $p$ are finite, there are only finitely many maximal nodes in $(V, \leq)$. Each of these maximal nodes dominates only finitely many nodes. Since $p$ satisfies $\varphi$, any node $x$ is dominated by some maximal node. Thus, indeed, $V$ is finite. $\square$

The *quantifier depth* of a monadic formula is defined canonically. For a $\Gamma$-labeled partial order $s$ and a positive integer $k$, let $\mathrm{MTh}_k(s)$ denote the set of all monadic sentences $\varphi$ of quantifier depth at most $k$ that are satisfied by $s$. The set $\mathrm{MTh}_k(s)$ is the *$k$-bounded monadic theory of $s$*. Analogously, the *$k$-bounded elementary theory* $\mathrm{Th}_k(s)$ comprises all elementary sentences in $\mathrm{MTh}_k(s)$. By $\mathrm{MTH}_k$ and $\mathrm{TH}_k$, we denote the set of all $k$-bounded monadic and elementary theories of finite N-free pomsets. Up to logical equivalence, there are only finitely many monadic sentences of quantifier depth at most $k$; hence the set of $k$-bounded monadic theories is finite (and the same holds for the elementary theories).

# 3 From rational to axiomatizable sets

In this section, we will show that ($\omega$-)series-rational sets are monadically axiomatizable relative to $\mathrm{NF}^\infty$ (to $\mathrm{NF}$, resp.). Furthermore, we show a similar relation between $\omega$-starfree and elementarily axiomatizable sets.

Let $p = (V, \leq, \lambda)$ be some N-free pomset and $X \subseteq V$. Since any antichain and any principal ideal in $p$ is finite, the set $X$ is finite if and only if it is bounded by an antichain. Hence the formula

$$\exists A \forall a, b, x(((a, b \in A \wedge a \leq b) \rightarrow b \leq a) \wedge (x \in X \rightarrow \exists c : (c \in A \wedge x \leq c)))$$

expresses that the set $X$ is finite. We denote this formula, that will be useful in the following proof, by $\mathrm{finite}(X)$.

**Lemma 3.1** *Let $X \subseteq \mathrm{NF}^\infty$ be ($\omega$-)series-rational. Then $X$ is monadically axiomatizable relative to $\mathrm{NF}$ (to $\mathrm{NF}^\infty$, resp.).*

**Proof.** Any set $\{p\}$ with $p$ finite can be monadically axiomatized. Now let $L$ and $M$ be two sets of N-free pomsets axiomatized by the monadic sentences $\sigma$ and $\tau$, respectively. Then $L \cup M$ is axiomatized by $\sigma \vee \tau$. The set $S \parallel T$ consists of all N-free pomsets satisfying

$$\exists X(X \neq \emptyset \wedge X^{co} \neq \emptyset \wedge \forall x \forall y(x \in X \wedge y \notin X \rightarrow x \parallel y) \wedge \sigma \restriction X \wedge \tau \restriction X^{co})$$

where $\sigma \restriction X$ is the restriction of $\sigma$ to the set $X$ and $\tau \restriction X^{co}$ that of $\tau$ to the complement of $X$. The sequential product can be dealt with similarly.

Next we show that $L^+$ can be described by a monadic sentence: The idea of a sentence axiomatizing $L^+$ is to color the vertices of an N-free pomset $p$ by two colors such that the coloring corresponds to a factorization in factors $p = p_1 \cdot p_2 \cdot p_3 \cdots p_n$ where every factor $p_i$ belongs to $L$. The identification of the $L$-factors will be provided by the property of being a maximal convex one-colored set. More formally, we define the sentence $\varphi$ to be

$$\sigma \vee \exists X \exists Y (\varphi_1 \wedge \varphi_X \wedge \varphi_Y \wedge \mathrm{finite}(X) \wedge \mathrm{finite}(Y))$$

where $\varphi_1$ asserts that $X$ and $Y$ form a partition of the set of vertices such that vertices from $X$ and from $Y$ are mutually comparable. The formula $\varphi_X$ states that the maximal subsets of $X$ that are convex satisfy $\sigma$, i.e.

$$\varphi_X = \forall Z \left[ \left( \begin{array}{c} Z \subseteq X \wedge Z \text{ is convex} \wedge Z \neq \emptyset \wedge \\ \forall Z' (Z \subseteq Z' \subseteq X \wedge Z' \text{ is convex} \rightarrow Z = Z') \end{array} \right) \rightarrow \sigma \restriction Z \right]$$

and the formula $\varphi_Y$ is defined similarly with $Y$ taking the place of $X$. Asserting that the sets $X$ and $Y$ are finite ensures that the sentence $\varphi$ is satisfied by finite N-free pomsets, only. Since the nodes from $X$ and from $Y$ are mutually comparable by $\varphi_1$, a pomset satisfying $\varphi$ is the sequential product of the convex factors determined by the partition $(X, Y)$. Hence we get indeed that $\varphi$ axiomatizes $L^+$. This finishes the proof of the first statement. Since we can axiomatize $L^\omega$ similarly to $L^+$, the second statement follows as well. $\square$

Next we will show that starfree sets of N-free pomsets are elementary axiomatizable. This can be shown by induction on the construction of a starfree set from the singletons $\{p\}$ for $p \in \mathrm{NF}$. The only nontrivial step is the parallel product. To handle it, we first define some more notions: Let $t = (V, \leq, \lambda)$ be an N-free pomset. The *distance* $d(x, y)$ between two nodes $x$ and $y$ of $V$ is the least number $n$ such that there are $x = x_0, x_1, \ldots, x_n = y$ in $V$ with $x_i$, $x_{i+1}$ comparable. Let $r$ be some positive integer and $x \in V$. The $r$-sphere around $x$ is the restriction of $t$ to those elements $y$ whose distance from $x$ is at most $r$. If $x$ and $y$ belong to different connected components of $t$, their distance is $\infty$. If they happen to lie in the same connected component, they are bounded from above or from below (since $t$ is N-free). Hence their distance is at most 2. In other words, the only spheres of finite radius around $x$ are (1) the set $\{x\}$, (2) the elements comparable with $x$, or (3) the connected component containing $x$.

Now let $\varphi$ be some elementary sentence. By Gaifman's Theorem [Gai82] (cf. also [EF91]), there exists a positive integer $r$ such that $\varphi$ is equivalent to a Boolean combination of sentences of the form

$$\exists x_1 \exists x_2 \ldots \exists x_n (d(x_i, x_j) > 2r \wedge \bigwedge_{i=1,\ldots n} \delta(x_i))$$

where $\delta(x)$ depends on the $r$-sphere around $x$, only. Above, we observed that $r$ can be bounded by $2$. Hence, we can w.l.o.g. assume $r = 2$. Therefore, any elementary sentence $\varphi$ is equivalent to a Boolean combination of statements of the form

"there are $\geq n$ connected components $C \subseteq V$ with $\mathrm{Th}_k(C, \leq, \lambda) = T$"

for some bounded theory $T \in \mathrm{TH}_k$. This will be used in the following proof.

**Lemma 3.2** *Let $\varphi_1$ and $\varphi_2$ be elementary sentences. Then there exists an elementary sentence $\varphi$ such that $p \models \varphi$ iff there exist $p_i \in \mathrm{NF}^\infty$ with $p_i \models \varphi_i$ and $p = p_1 \parallel p_2$ for any $p \in \mathrm{NF}^\infty$.*

**Proof.** Above, we saw that any elementary sentence is equivalent to a Boolean combination of statements of the form
"there are $\leq n$ connected components $C \subseteq V$ with $\mathrm{Th}_k(C, \leq, \lambda) = T$" $\quad (*)$
for some bounded theory $T \in \mathrm{TH}_k$.

Let $t = (V, \leq, \lambda) \in \mathrm{NF}$ and $x, y \in V$. Then $x$ and $y$ lie in the same connected component of $t$ iff they are bounded from above or from below. Hence the statements of the form $(*)$ are expressible by an elementary sentence $\psi(n, k, T)$. Note that $\psi(n, k, T)$ is equivalent to a finite disjunction of sentences $\psi(n, k+1, T')$. Since in our consideration, only finitely many formulas $\psi(n, k, T)$ will occur, we can assume that all of them use the same argument $k$. Hence we can omit it from now on and write $\psi(n, T)$ instead of $\psi(n, k, T)$.

Thus, for any sentence $\varphi$, there exists a Boolean combination $\psi$ of sentences $\psi(n, T)$ such that $p \models \varphi$ iff $p \models \psi$ for any N-free pomset $t \in \mathrm{NF}^\infty$. The sentence

$$\delta = \bigwedge_P \psi(n, T) \wedge \bigwedge_N \neg\psi(n, T) \qquad (**)$$

is a particularly simple such Boolean combination where the two conjunctions run over finite subsets $P$ and $N$ of $\mathbb{N} \times \mathrm{TH}_k$. We say that $\psi(n, T)$ occurs positively (negatively, resp.) in $\delta$ if $(n, T) \in P$ ($(n, T) \in N$, resp.) Let $\delta_1$ and $\delta_2$ be two conjunctions of the form $(**)$. Then we define a new such formula $\delta$ as the conjunction of the following formulas (where $T \in \mathrm{TH}_k$ and $n_1, n_2 \in \mathbb{N}$):

| | |
|---|---|
| $\psi(n_1 + n_2, T)$ | if $\psi(n_1, T)$ occurs positively in $\delta_1$ and $\psi(n_2, T)$ occurs positively in $\delta_2$, |
| $\psi(n_i, T)$ | if $\psi(n_i, T)$ occurs positively in $\delta_i$ (for $i = 1, 2$), and |
| $\neg\psi(n_1 + n_2 - 1, T)$ | if $\psi(n_1, T)$ occurs negatively in $\delta_1$ and $\psi(n_2, T)$ occurs negatively in $\delta_2$. |

Let $p$ be a N-free pomset satisfying $\delta$. Now we can divide the connected components of $p$ into $p_1$ and $p_2$ in the following way: Suppose $\psi(n_i, T)$ occurs positively in $\delta_i$. Then $p$ satisfies $\psi(n_1+n_2,T)$, i.e. it contains at least $n_1+n_2$ connected components whose $k$-bounded theory is $T$. Then $n_1$ of these components become part of $p_1$ and the remaining ones of $p_2$. If $\psi(n_1,T)$ occurs positively in $\delta_1$ and $\psi(n_2,T)$ negatively in $\delta_2$, the N-free pomset $p$ satisfies $\psi(n_1,T)$. All the connected components whose $k$-bounded theories are $T$ become part of $p_1$. Now suppose $\psi(n_i,T)$ occurs negatively in $\delta_i$. Then $p$ contains at most $n_1+n_2-2$ connected components with theory $T$. In this case, we place $n_1-1$ of them into $p_1$ and the remaining ones into $p_2$. Any connected component of $p$ whose $k$-bounded theory is not mentioned in $\delta$ becomes part of $p_1$. This way we obtain two N-free pomsets $p_1$ and $p_2$ with $p_i \models \delta_i$. Since $p_1$ and $p_2$ are a division of the connected components of $p$, their disjoint union equals $p$, i.e. $p_1 \parallel p_2 = p$.

Converselly let $p_i \in \mathrm{NF}^\infty$ with $p_i \models \delta_i$ and let $p = p_1 \parallel p_2$. Suppose $\psi(n_i,T)$ occurs positively in $\delta_i$. Then $p_i$ contains at least $n_i$ connected components whose $k$-bounded elementary theory is $T$. Hence $p$ contains at least $n_1+n_2$ such connected components, i.e. $p \models \psi(n_1+n_2,T)$. If $\psi(n_1,T)$ occurs positively in $\delta_1$ and $\psi(n_2,T)$ negatively in $\delta_2$, there are at least $n_1$ connected components of $p_1$ whose $k$-bounded elementary theory is $T$. Hence $p$ has at least $n_1$ such connected components, i.e. $p \models \psi(n_1,T)$. Now suppose $\psi(n_i,T)$ occurs negatively in $\delta_i$. Then $p_i$ contains at most $n_i-1$ connected components with theory $T$. Hence there are at most $n_1+n_2-2$ such connected components in $p$, i.e. $p \models \psi(n_1+n_2-2,T)$.

Thus, we showed the statement of the lemma for sentences $\varphi_i$ of the form $(**)$. In general, one can write $\varphi_i$ as disjunction of conjunctions of the above form $(**)$. Combining any pair of disjuncts from $\varphi_1$ and $\varphi_2$ in the above described manner, we obtain the desired sentence $\varphi$. $\square$

**Theorem 3.3** *Let $X \subseteq \mathrm{NF}^\infty$ be ($\omega$-)starfree. Then $X$ is elementary axiomatizable relative to* $\mathrm{NF}$ *(to* $\mathrm{NF}^\infty$*, resp.).*

**Proof.** As usual, this can be shown by induction on the construction of starfree sets.

The base case is trivial since any finite N-free pomset can be described completely by an elementary sentence. The Boolean operations are easily dealt with. Now let $X_i \subseteq \mathrm{NF}$ be axiomatized by $\varphi_i$ for $i=1,2$. Then $X_1 \cdot X_2$ is axiomatized by $\exists y(\psi \wedge \varphi_1' \wedge \varphi_2')$ where $\varphi_1'$ is the relativisation of $\varphi_1$ to the elements below $y$ and $\varphi_2'$ is the relativisation of $\varphi_2$ to the elements not below $y$. Furthermore, $\psi$ states that any element $y$ below $x$ is below any element $z$ not below $x$. Furthermore, $X_1 \parallel X_2$ can be elementarily axiomatized by Lemma 3.2. $\square$

# 4 From axiomatizable to recognizable sets

In this section, we will show that any set of N-free pomsets that is axiomatized in monadic second order logic (elementary logic, resp.) is $\omega$-recognizable ($\omega$-starfree). Since by Lemma 2.3 the finite N-free pomsets can be elementarily axiomatized relative to $\mathrm{NF}^\infty$, the corresponding results hold for finite N-free pomsets as well. The first ingredient in the proof is the fact that the mapping $\mathrm{MTh}_k$ from $\mathrm{NF}$ to the finite set $\mathrm{MTH}_k$ can be seen as an sp-homomorphism. Even more, it is also a homomorphism with respect to the infinite sequential product $\prod$ and the same holds for the mapping $\mathrm{Th}_k$:

**Theorem 4.1** *Let $k \in \omega$ and let $p_\ell$ and $q_\ell$ be N-free pomsets for $\ell \in \omega$.*

1. *Suppose* $\mathrm{MTh}_k(p_\ell) = \mathrm{MTh}_k(q_\ell)$ *for* $\ell \in$
   $omega.$
   *Then* $\mathrm{MTh}_k(p_1 \parallel p_2) = \mathrm{MTh}_k(q_1 \parallel q_2)$.
   *If in addition* $p_1$ *is finite, then* $\mathrm{MTh}_k(p_1 \cdot p_2) = \mathrm{MTh}_k(q_1 \cdot q_2)$.
   *If all* $p_i$ *are finite, then* $\mathrm{MTh}_k(\prod p_i) = \mathrm{MTh}_k(\prod q_1)$.
2. *Suppose* $\mathrm{Th}_k(p_\ell) = \mathrm{Th}_k(q_\ell)$ *for* $\ell \in \mathbb{N}$.
   *Then* $\mathrm{Th}_k(p_1 \parallel p_2) = \mathrm{Th}_k(q_1 \parallel q_2)$.
   *If in addition* $p_1$ *is finite, then* $\mathrm{Th}_k(p_1 \cdot p_2) = \mathrm{Th}_k(q_1 \cdot q_2)$.
   *If all* $p_i$ *are finite, then* $\mathrm{Th}_k(\prod p_i) = \mathrm{Th}_k(\prod q_1)$.

All our operations sequential product $\cdot$, parallel product $\|$, and infinite sequential product $\prod$ are special cases of the generalized sum as considered by Shelah [She75]. The preceding theorem follows from his result [She75, Thm. 2.4]. A condensed proof of Shelah's result can be found in [Gur79], for lexicographic sums of linear orders, [Tho97] contains the full proof. In the terminology of [CM00], the first statements say that the sequential and the parallel product are Hintikka operations. These results can also be derived from [CM00] on quantifier free definable operations. Shelah is also interested in the effective computability of the combined theory from the argument theories. If one is only interested in the result as stated above, an alternative and much simpler proof of both statements can be given using Ehrenfeucht-Fraïssé-games for the respective logics:

**Proof.** We show that $\mathrm{Th}_k(p_i) = \mathrm{Th}_k(q_i)$ for $i \in \omega$ implies $\mathrm{Th}_k(\prod p_i) = \mathrm{Th}_k(\prod q_i)$ using Ehrenfeucht-Fraïssé-games, see [EF91, p. 16 ff.] for an introduction. Following the notation in [EF91], we write $G_k(p, q)$ to denote the game on the structures $p$ and $q$ with $k$ rounds.

By the requirements on $p_\ell$ and $q_\ell$, Duplicator has a winning strategy for the game $G_k(p_\ell, q_\ell)$ for $\ell \in \omega$. A winning strategy for $G_k(\prod p_\ell, \prod q_\ell)$ is described as follows:

The first rule is that, if Spoiler plays a position $x$ in the $\ell$-th factor of $\prod p_\ell$, then Duplicator chooses a position $y$ in the $\ell$-th factor of $\prod q_\ell$.

To explain which $y$ to choose, suppose in the first $i < k$ rounds, Spoiler and Duplicator played $(x_n, y_n)_{1 \leq n \leq i}$. Now, in round $i + 1$, Spoiler chooses a position $x$ in the factor $p_\ell$ of $\prod p_\ell$. Let $1 \leq i_1 < i_2 < \cdots < i_m \leq i$ be those indices for which $x_{i_j}$ belongs to the factor $p_\ell$. Since Duplicator always plays according to the strategy we are describing (first rule), $y_{i_j}$ belongs to $q_\ell$ for all these indices. Hence $(x_{i_j}, y_{i_j})_{1 \leq j \leq m}$ is the result of $m$ rounds of the game $G_k(p_\ell, q_\ell)$ and Duplicator's winning strategy for this game tells him how to answer to Spoiler's move $x$.

Let $(x_\ell, y_\ell)_{1 \leq \ell \leq k}$ be the outcome of this game when played according to the strategy described above. Since $y_i$ is chosen according to the winning strategy for the game $G_k(p_\ell, q_\ell)$, we obtain $\lambda(x_i) = \lambda(y_i)$. Now suppose $x_r \leq x_s$ in $\prod p_\ell$. Suppose $x_r$ and $x_s$ belong to the same factor $p_\ell$. Since $y_r$ and $y_s$ are chosen according to the winning strategy for $G_k(p_\ell, q_\ell)$, we obtain $y_r \leq y_s$. Now suppose that $x_r$ and $x_s$ belong to factors $p_i$ and $p_j$ with $i \neq j$. Since $x_r \leq x_s$, this implies $i < j$. Then $y_r$ and $y_s$ are choosen in $q_i$ and $q_j$. Hence $y_r \leq y_s$.

The statements on the first order theories of the parallel and the sequential product are shown similarly. Allowing in addition moves in which sets are chosen, the first statement on monadic theories can also be shown along the lines of the above proof. □

**Theorem 4.2** *Let $\varphi$ a monadic second order sentence. Then $X = \{p \in \mathrm{NF}^\infty \mid p \models \varphi\}$ is $\omega$-recognizable.*

**Proof.** Let $k$ be the quantifier depth of $\varphi$. Recall that $\mathrm{MTH}_k$ is the set of $k$-bounded monadic theories of finite N-free pomsets. By Theorem 4.1, this set can be endowed with two operations $\cdot, \|$: $(\mathrm{MTH}_k)^2 \to \mathrm{MTH}_k$ such that $\mathrm{MTh}_k : \mathrm{NF} \to \mathrm{MTH}_k$ becomes an sp-homomorphism. We show that this sp-homomorphism $\omega$-recognizes $X$: Let $f, g : \mathcal{X} \to \mathrm{NF}$ be two mappings with $\mathrm{MTh}_k \circ f = \mathrm{MTh}_k \circ g$. Then, using Theorem 4.1, one can easily show by induction on the construction of terms that $\mathrm{MTh}_k(f(t)) = \mathrm{MTh}_k(g(t))$ for any term $t$. Hence $f(t) \models \varphi$ iff $\varphi \in \mathrm{MTh}_k(f(t))$ iff $g(t) \models \varphi$. □

Let $\varphi$ be an elementary sentence. Then by the above theorem the set of N-free pomsets $L$ satisfying $\varphi$ is recognized by the homomorphism $\mathrm{MTh}_k$. Using the second statement from Theorem 4.1 on elementary sentences, one obtains immediately that $L$ can be accepted by the homomorphism $\mathrm{Th}_k$ into the sp-algebra $\mathrm{TH}_k$ of $k$-bounded elementary theories. To show that $L$ is aperiodic, it therefore suffices to show that $\mathrm{TH}_k$ is an aperiodic sp-algebra.

We start with the definition of the lexicographic sum of a finite linear order $(\{1, 2, \ldots n\}, \leq)$ and a finite N-free pomset $p = (V, \sqsubseteq, \lambda)$: Let $V' = \{1, 2, \ldots, n+1\} \times V$ and define $\lambda'(i, v) = \lambda(v)$. Furthermore, $(i, v) \sqsubseteq' (j, w)$ iff $i \leq j$ or if $i = j$ and $v \sqsubseteq w$. Note that the lexicographic sum $(V', \sqsubseteq', \lambda')$ of $(\{1, 2, \ldots, n\}, \leq)$ and $p$ is the $n$-fold sequential product of $p$; therefore, we denote it by $p^n$. Now let $k \in \mathbb{N}$. By Shelah's composition theorem (cf. [She75, Thm. 2.4]), there is $\ell \in \mathbb{N}$ with the following properties: If $m, n \in \mathbb{N}$ are such that the linear orders on $m$ and $n$ vertices have the same $\ell$-bounded theories, then $\mathrm{Th}_k(p^m) = \mathrm{Th}_k(p^n)$. By [Lad77, Lemma 4.4], the elementary theories of the linear orders on $2^\ell - 1$ and on $2^\ell$ points have the same $\ell$-bounded elementary theories. Hence $\mathrm{Th}_k(p^{2^\ell - 1}) = \mathrm{Th}_k(p^{2^\ell})$ for any $p \in \mathrm{NF}$. Since $\mathrm{Th}_k : \mathrm{NF} \to \mathrm{TH}_k$ is a surjective sp-homomorphism, this implies that the sp-algebra $(\mathrm{TH}_k, \cdot, \|)$ is aperiodic. Hence we obtain

**Theorem 4.3** *Let $\varphi$ an elementary sentence. Then $X = \{p \in \mathrm{NF}^\infty \mid p \models \varphi\}$ is $\omega$-aperiodic.*

# 5 From recognizable to rational sets

## 5.1 Simple terms

First, we show that any $\omega$-recognizable set of N-free pomsets of finite width is of a special form (cf. Proposition 5.3) that is defined using the notion of a simple term.

Let $(S, \cdot, \|)$ be an sp-algebra. Then a pair $(\alpha, \beta) \in S^2$ is *linked* if $\alpha \cdot \beta = \alpha$ and $\beta \cdot \beta = \beta$. A *simple term* of order 1 is an element of $S$ or a linked pair $(\alpha, \beta)$. Now let $n > 1$, $\sigma_i$ for $i = 1, 2, \ldots n$ be simple terms of order $n_i$, and $\alpha \in S$. Then $\alpha \cdot (\sigma_1 \parallel \sigma_2 \parallel \cdots \sigma_n)$ is a simple term of order $n_1 + n_2 + \ldots n_n$.

For an sp-homomorphism $\eta : \mathrm{NF} \to S$ and a simple term $\sigma$, we define the language $L_\eta(\sigma)$ inductively: If $\sigma \in S$, we set $L_\eta(\sigma) := \eta^{-1}(\sigma)$. For a linked pair $(\alpha, \beta)$, we define $L_\eta(\alpha, \beta) := \eta^{-1}(\alpha) \cdot (\eta^{-1}(\beta))^\omega$. Now the induction step is described by $L_\eta(\alpha \cdot (\sigma_1 \parallel \sigma_2 \parallel \ldots \sigma_n)) := \eta^{-1}(\alpha) \cdot (L_\eta(\sigma_1) \parallel L_\eta(\sigma_2) \parallel \cdots L_\eta(\sigma_n))$.

**Lemma 5.1** *Let $\Gamma$ be an alphabet, $(S, \cdot, \|)$ an sp-algebra and $\eta : \mathrm{NF} \to (S, \cdot, \|)$ an sp-homomorphism. Let furthermore $p \in \mathrm{NF}^\infty$ be an N-free pomset of finite width. Then there exist simple terms $\tau_1, \tau_2, \ldots, \tau_n$ of order at most $w(p)$ with $n \leq w(p)$ and $p \in L_\eta(\tau_1) \| L_\eta(\tau_2) \| \cdots L_\eta(\tau_n)$.*

**Proof.** If $p$ is finite, the lemma is obvious since the element $\alpha = \eta(t)$ of $S$ is a simple term of order 1. Thus, we may assume $p$ to be infinite. First consider the case that $p = \prod_{i\in\omega} p_i$ is an infinite product of finite N-free pomsets $p_i$. Let $\alpha_i := \eta(p_i)$. A standard application (cf. [PP99, Thm. II.3.2]) of Ramsey's Theorem [Ram30] (cf. also [Cam94]) yields the existence of a strictly increasing sequence of positive integers $n_i$ for $i \in \omega$ and a linked pair $(\alpha, \beta) \in S^2$ such that $\alpha = \alpha_0\alpha_1 \cdots \alpha_{n_0}$ and $\beta = \alpha_{n_i+1} \cdot \alpha_{n_i+2} \cdots \alpha_{n_{i+1}}$ for $i \in \omega$. Hence $p \in L_\eta(\alpha, \beta)$. Since $(\alpha, \beta)$ is a simple term of order $1 \leq w(p)$, we showed the lemma for infinite products of finite N-free pomsets.

Now we prove the lemma for connected N-free pomsets by induction on the width $w(p) \leq |\Gamma|$ (a pomset is *connected* if it is not the parallel product of two pomsets). If $w(p) = 1$, the pomset $p$ is actually linearly ordered and therefore an infinite product of finite pomsets. But this case was already dealt with. Now assume that the statement of the lemma holds for any connected N-free pomset of width at most $n$ and let $p$ be a connected pomsets with $w(p) = n+1$. By what we showed above, we may assume that $p$ is not an infinite product of finite pomsets. Hence, by Lemma 2.1, there exist a finite N-free pomset $p_0$, a natural number $m > 1$ and nonempty connected N-free pomsets $p_i$ for $1 \leq i \leq m \leq w(p)$ with $p = p_0 \cdot (p_1 \| p_2 \| \cdots p_m)$. Since $w(p) \geq w(p_1) + w(p_2) + \cdots w(p_m)$, and $w(p_i) \geq 1$, we have $w(p_i) \leq n$. Hence we can apply the induction hypothesis and obtain simple terms $\sigma_i$ of order at most $w(p_i)$ with $p_i \in L_\eta(\sigma_i)$. With $\alpha := \eta(p_0)$, we therefore have a simple term $\tau = \alpha \cdot (\sigma_1 \| \sigma_2 \| \cdots \sigma_m)$ with $p \in L_\eta(\tau)$. By $w(p_1) + w(p_2) + \cdots w(p_m) \leq w(p)$, the simple term $\tau$ is of order at most $w(p)$.

Finally, assume that $p \in \mathrm{NF}^\infty$ is not connected. Since $p$ is of finite width, it is the parallel product of at most $w(p)$ connected N-free pomsets $p_i$. Note that $w(p_i) \leq w(p)$. Hence $w(p_i)$ belongs to a set $L_\eta(\tau_i)$ for some simple term $\tau_i$ of order at most $w(p)$. $\square$

The following lemma implies that the set $L_\eta(\tau)$ is contained in any $\omega$-recognizable set $L$ that intersects $L_\eta(\tau)$.

**Lemma 5.2** *Let $\Gamma$ be an alphabet, $(S, \cdot, \|)$ a finite sp-algebra, and $\eta : \mathrm{NF} \to (S, \cdot, \|)$ an sp-homomorphism and $\tau_i$ a simple term for $1 \leq i \leq m$. Let further-*

*more* $p, q \in L_\eta(\tau_1) \parallel L_\eta(\tau_2) \parallel \cdots L_\eta(\tau_m)$. *Then there exist a term* $t$ *and mappings* $f, g : \mathcal{X} \to \mathrm{NF}$ *with* $\eta \circ f = \eta \circ g$ *such that* $f(t) = p$ *and* $g(t) = q$.

**Proof.** First, we show the lemma for the case $m = 1$ (for simplicity, we write $\tau$ for $\tau_1$): In this restricted case, the lemma is shown by induction on the construction of the simple term $\tau$. If $\tau = \alpha \in S$, the set $L_\eta(\tau) = \eta^{-1}(\alpha)$ consists of finite N-free pomsets, only. Therefore, we can define functions $f, g : \mathcal{X} \to \mathrm{NF}$ with $f(x) = p$ and $g(x) = q$ for all $x \in \mathcal{X}$. Since $p, q \in L_\eta(\tau) = \eta^{-1}(\alpha)$, we get $\eta(p) = \alpha = \eta(q)$ and therefore $\eta \circ f = \eta \circ g$. With $t = x$, we have in addition $f(t) = p$ and $g(t) = q$ as required.

Now let $\tau = (\alpha, \beta)$ be a linked pair. Then $p, q \in \eta^{-1}(\alpha) \cdot (\eta^{-1}(\beta)^\omega$. Hence there are finite N-free pomsets $p_i$ and $q_i$ with $\eta(p_0) = \eta(q_0) = \alpha$, $\eta(p_i) = \eta(q_i) = \beta$ for $i > 0$, $p = \prod_{i \in \omega} p_i$ and $q = \prod_{i \in \omega} q_i$. Now let $t = \prod_{i \in \omega} x_i$, $f(x_i) = p_i$ and $g(x_i) = q_i$. Then the term $t$ and the functions $f$ and $g$ satisfy the requirements of the lemma.

Now suppose $n > 1$, $\tau_i$ $(1 \le i \le n)$ are simple terms that satisfy the statement of the lemma above. Furthermore, let $\alpha \in S$ and let $\tau = \alpha \cdot (\tau_1 \parallel \tau_2 \parallel \cdots \tau_n)$ and suppose $p, q \in L_\eta(\tau)$. Then there exist finite N-free pomsets $p_0, q_0 \in \eta^{-1}(\alpha)$ and N-free pomsets $p_i$ and $q_i$ in $L_\eta(\tau_i)$ for $1 \le i \le n$ such that $p = p_0 \cdot (p_1 \parallel p_2 \parallel \cdots p_n)$ and $q = q_0 \cdot (q_1 \parallel q_2 \parallel \cdots q_n)$. By the induction hypothesis on the simple terms $\tau_i$, we find terms $t_i$ and functions $f_i, g_i : \mathcal{X} \to \mathrm{NF}$ such that $\eta \circ f_i = \eta \circ g_i$, $p_i = f_i(t_i)$ and $q_i = g_i(t_i)$ for $1 \le i \le n$. We might assume that the variables that occur in the term $t_i$ are disjoint from those that occur in $t_j$ for $1 \le i < j \le n$. Furthermore, let $y \in \mathcal{X}$ be a variable that does not occur in any of the terms $t_i$. Now we define $t = y \cdot (t_1 \parallel t_2 \parallel \cdots t_n)$, $f(x) = f_i(x)$ and $g(x) = g_i(x)$ if $x$ occurs in $t_i$, and $f(x) = p_0$ and $g(x) = q_0$ if $x$ does not occur in any of the terms $t_i$. Then it is easily checked that $\eta \circ f = \eta \circ g$, $f(t) = p$ and $g(t) = q$.

Finally, the general case $m \ge 1$ is handled similarly to the induction step above. $\square$

Let $L$ be an $\omega$-recognizable set of N-free pomsets of finite width. Next, we show that $L$ is the union of languages of the form $L_\eta(\tau_1) \parallel L_\eta(\tau_2) \parallel \cdots L_\eta(\tau_n)$. But this union might be infinite.

**Proposition 5.3** *Let* $\Gamma$ *be an alphabet and* $L \subseteq \mathrm{NF}^\infty$ *be an* $\omega$*-recognizable set of N-free pomsets of finite width. Let* $L$ *be recognized by the sp-homomorphism* $\eta : \mathrm{NF} \to (S, \cdot, \parallel)$, *and let* $\mathcal{T}$ *denote the set of finite tuples of simple terms*

$(\tau_1, \tau_2, \ldots, \tau_k)$ *such that* $\emptyset \neq L \cap (L_\eta(\tau_1) \parallel L_\eta(\tau_2) \parallel \cdots L_\eta(\tau_k))$. *Then*

$$L = \bigcup_{(\tau_1, \tau_2, \ldots, \tau_k) \in \mathfrak{T}} (L_\eta(\tau_1) \parallel L_\eta(\tau_2) \parallel \cdots L_\eta(\tau_k)).$$

**Proof.** For simplicity, let $K = \bigcup_{(\tau_1, \tau_2, \ldots, \tau_k) \in \mathfrak{T}} (L_\eta(\tau_1) \parallel L_\eta(\tau_2) \parallel \cdots L_\eta(\tau_k))$.

Let $p \in L$. By Lemma 5.1, there exist simple terms $\tau_i$ for $1 \leq i \leq k$ such that $p \in L_\eta(\tau_1) \parallel L_\eta(\tau_2) \parallel \cdots L_\eta(\tau_k) \subseteq K$. Hence $L \subseteq K$ and it remains to show $K \subseteq L$. So, let $(\tau_1, \tau_2, \cdots \tau_k) \in \mathfrak{T}$ be a tuple of simple terms and let $p, q \in L_\eta(\tau_1) \parallel L_\eta(\tau_2) \parallel \cdots L_\eta(\tau_k)$ such that $p \in L$. We have to show that $q$ belongs to $L$. By the preceding lemma, there exist mappings $f, g : \mathcal{X} \to \mathrm{NF}$ and a term $t$ with $p = f(t)$ and $q = g(t)$ and $\eta \circ f = \eta \circ g$. Since $L$ is recognized by $\eta$, this implies $q \in L$ as required. $\square$

## 5.2 $\omega$-recognizable and $\omega$-series-rational sets

By [LW98], any recognizable and width-bounded set of finite N-free pomsets is series-rational. In this section, we extend this result to infinite N-free pomsets. We start with an example showing that there are $\omega$-recognizable sets which are not $\omega$-series-rational:

**Example 5.4** Let $(V, \leq)$ be a tree without maximal elements, such that $\downarrow v$ is finite for any $v \in V$, any node has at most 2 upper neighbors, and almost all nodes from $V$ have only one upper neighbor. Let $n$ be the number of branching points of $(V, \leq)$. Then we call $(V, \leq)$ a *tree with $n$ branching points*. Note that $(V, \leq, \lambda)$ is an N-free pomset.

Now let $N$ be a set of natural numbers and let $L_N$ denote the set of all $\Gamma$-labeled trees with $n$ branching points for some $n \in N$. We show that $L_N$ is $\omega$-recognizable:

We consider the sp-algebra $S = \{1, 2\}$ with the mapping $\eta : \mathrm{NF} \to S$ defined by $\eta(p) = \min(w(p), 2)$ for any $p \in \mathrm{NF}$. To turn this into an sp-homomorphism, let $\alpha \parallel \beta = \min(2, \alpha + \beta)$ and $\alpha \cdot \beta = \max(\alpha, \beta)$ for any $\alpha, \beta \in S$. Now let $t$ be a term and $f, g : \mathcal{X} \to \mathrm{NF}$ with $\eta \circ f = \eta \circ g$. Furthermore, assume $f(t) \in L_N$, i.e. that $f(t)$ is a tree with $n \in N$ branching points. As $f(t)$ has no leaves, every parallel product $\parallel$ in $t$ is applied to two infinite terms and similarly the second factor of every sequential product $\cdot$ in $t$ is an infinite term. Hence every variable $x_i$ (that occurs in $t$ at all) occurs in $t$ in a left factor of a sequential product $\cdot$ or within the scope of an infinite product $\prod$. Since $f(t)$ is a tree, this implies that $f(x_i)$ is a (finite) linear order, i.e. $w(f(x_i)) = 1$. Now $\eta \circ f = \eta \circ g$ implies $w(g(x_i)) = 1$. Hence the N-free pomset $g(t)$ differs from the tree with $n$ branching points $f(t)$ only in some non-branching pieces. Thus, $g(t)$ is a tree with $n$ branching points, i.e. $g(t) \in L_N$ as required. Hence we showed that $L_N$ is indeed $\omega$-recognizable.

By the example above, the number of $\omega$-recognizable subsets of $\mathrm{NF}^\infty$ is $2^{\aleph_0}$. Since there are only countably many $\omega$-series-rational languages, not all $\omega$-recognizable sets are $\omega$-series-rational. Therefore, we will restrict to $\omega$-recognizable sets of bounded width and show that any such set is $\omega$-series-rational:

**Proposition 5.5** *Let $\Gamma$ be an alphabet and $L \subseteq \mathrm{NF}^\infty(\Gamma)$ be $\omega$-recognizable and width-bounded. Then $L$ is $\omega$-series-rational.*

**Proof.** Let $\eta : \mathrm{NF}(\Gamma) \to (S, \cdot, \|)$ be an sp-homomorphism that recognizes $L$. Furthermore, let $n \in \omega$ such that $w(p) \leq n$ for any $p \in L$. Now let $\mathfrak{T}_n$ denote the set of tuples of simple terms $(\tau_1, \tau_2, \dots, \tau_k)$ with $k \leq n$ of order at most $n$ such that $\emptyset \neq L \cap (L_\eta(\tau_1) \parallel L_\eta(\tau_2) \parallel \cdots L_\eta(\tau_k))$. Note that $\mathfrak{T}_n$ is finite.

By Proposition 5.3, $L$ contains all languages $L_\eta(\tau_1) \parallel L_\eta(\tau_2) \parallel \cdots L_\eta(\tau_k)$ for $(\tau_1, \tau_2, \dots, \tau_n) \in \mathfrak{T}_n$. Conversely, let $p \in L$. By Lemma 5.1, there exists a tuple $(\tau_1, \tau_2, \dots, \tau_n)$ of simple terms of order at most $w(p) \leq n$ such that $p \in L_\eta(\tau_1) \parallel L_\eta(\tau_2) \parallel \cdots L_\eta(\tau_k)$. Note that $k \leq w(p) \leq n$ implying $(\tau_1, \tau_2, \dots, \tau_k) \in \mathfrak{T}_n$. This finishes the proof that $L$ equals the union of all sets $L_\eta(\tau_1) \parallel L_\eta(\tau_2) \parallel \cdots L_\eta(\tau_k)$ for $(\tau_1, \tau_2, \dots, \tau_k) \in \mathfrak{T}_n$.

Hence it remains to show that $L_\eta(\tau)$ is $\omega$-series-rational for any simple term $\tau$ such that there exists $(\sigma_1, \sigma_2, \dots, \sigma_k) \in \mathfrak{T}_n$ with $\sigma_i = \tau$ for some $1 \leq i \leq k$. The proof proceeds by induction on the subterms of $\tau$: So let $\sigma$ be a subterm of $\tau$ of order 1. First assume $\sigma = \alpha \in S$. Then $L_\eta(\sigma) = \eta^{-1}(\alpha)$ is sp-recognizable. Note that any $p \in \eta^{-1}(\alpha)$ is a subposet of some element of the language $L$. Since $L$ is width-bounded, the set $\eta^{-1}(\alpha)$ is width-bounded. By [LW00b] it follows that $L_\eta(\alpha)$ is series-rational. Now let $\sigma = (\alpha, \beta)$ be a linked pair. Similarly to the first case, any element of $\eta^{-1}(\alpha) \cup \eta^{-1}(\beta)$ is a subposet of some element of $L$. Hence $\eta^{-1}(\alpha)$ and $\eta^{-1}(\beta)$ are width-bounded and sp-recognizable and therefore series-rational. Hence $L_\eta(\alpha, \beta) = \eta^{-1}(\alpha) \cdot (\eta^{-1}(\beta))^\omega$ is $\omega$-series-rational. For the induction step, let $\sigma = \alpha \cdot (\tau_1 \parallel \tau_2 \parallel \cdots \tau_n)$ be a subterm of $\tau$ such that $L_\eta(\tau_i)$ is $\omega$-series-rational. Again, we obtain that $\eta^{-1}(\alpha)$ is width-bounded and sp-recognizable and therefore series-rational. Hence so is $L_\eta(\sigma) = \eta^{-1}(\alpha) \cdot (L_\eta(\tau_1) \parallel$

$L_\eta(\tau_2) \parallel \cdots L_\eta(\tau_n))$. $\square$

## 5.3 Aperiodic and starfree sets

In the proof above, we used the result by Lodaya & Weil on recognizable and series-rational sets of bounded width. In this section, we show a similar result for sets of finite N-free pomsets regarding aperiodic and starfree sets of bounded width.

Recall that we did not allow a poset to be empty. Therefore, in general, $X \cdot \mathrm{NF}$ does not contain $X$. Since, occasionally it will be convenient to have this, we will use the abbreviations $X \cdot \mathrm{NF}_\varepsilon = X \cdot \mathrm{NF} \cup X$, $X \cdot \mathrm{NF}^\infty_\varepsilon = X \cdot \mathrm{NF}^\infty \cup X$, and $\mathrm{NF}_\varepsilon \cdot X = \mathrm{NF} \cdot X \cup X$ for $X \subseteq \mathrm{NF}$.

**Example 5.6** For $k \in \mathbb{N}$, let $\mathrm{NF}_k = \{t \in \mathrm{NF} \mid w(t) \leq k\}$ and. The set $\mathrm{NF}_\varepsilon \cdot (\mathrm{NF} \parallel \mathrm{NF}) \cdot \mathrm{NF}_\varepsilon$ of all N-free pomsets of width at least 2 is starfree. Hence, the set $\mathrm{NF}_1$ of linear pomsets is starfree, too. Words over $\Gamma$ can be identified with linear pomsets, i.e. with the elements of $\mathrm{NF}_1$. Then, of course, $\Gamma^+ \setminus L$ corresponds to $\mathrm{NF}_1 \setminus L'$ where $L' \subseteq \mathrm{NF}_1$ corresponds to $L$. Hence, starfree word languages correspond to starfree subsets of $\mathrm{NF}$.

We saw above that $\mathrm{NF}_1$ is starfree. Note that, for any $k \in \mathbb{N}$, the set $\mathrm{NF}_\varepsilon \cdot ((\mathrm{NF} \setminus \mathrm{NF}_{k-1}) \parallel \mathrm{NF}) \cdot \mathrm{NF}_\varepsilon$ is the set of N-free pomsets of width at least $k+1$. Hence, its complement $\mathrm{NF}_k$ is starfree.

Similarly, one can show that $\mathrm{NF}^\infty_k = \{t \in \mathrm{NF}^\infty \mid w(t) \leq k\}$ is $\omega$-starfree.

Let $S$ be a set and let $X_s \subseteq \mathrm{NF}$ for $s \in S$. Then the *set of sets starfree over* $\{X_s \mid s \in S\}$ is the least class $\mathcal{C} \subseteq \mathcal{P}(\mathrm{NF})$ containing $X_s$ for $s \in S$, $\{t\}$ for $t \in \mathrm{NF}$, that is closed under the Boolean operations and the sequential and parallel product. A set of N-free pomsets is $\omega$-starfree over $\{X_s \mid s \in S\}$ if it can be constructed from the sets starfree over $\{X_s \mid s \in S\}$ by Boolean combinations, parallel product, and sequential product. Note that $X \subseteq \mathrm{NF}$ is ($\omega$)-starfree if it is ($\omega$)-starfree over the empty family.

**Example 5.7** We show that the sequential iteration of a starfree subset of $\mathrm{NF} \parallel \mathrm{NF}$ is starfree: More generally, let $L \subseteq (\mathrm{NF} \parallel \mathrm{NF}) \cup \Gamma$. Then $L^+$ is starfree over $\{L\}$ which, in case $L$ is starfree, implies the starfreeness of $L^+$: A *minimal sequential factor* of a poset $t$ is a poset $t' \in (\mathrm{NF} \parallel \mathrm{NF}) \cup \Gamma$ such that $t \in \mathrm{NF}_\varepsilon \cdot t' \cdot \mathrm{NF}_\varepsilon$. Now one easily observes that $L^+$ is the set of all pomsets from $\mathrm{NF}$ all of whose minimal sequential factors belong to $L$. Hence

$$L^+ = \mathrm{NF} \setminus (\mathrm{NF}_\varepsilon \cdot (\mathrm{NF} \setminus L \cap ((\mathrm{NF} \parallel \mathrm{NF}) \cup \Gamma)) \cdot \mathrm{NF}_\varepsilon),$$

a set starfree over $L$.

Let $S$ be a set. Then $S^+$ denotes the set of nonempty words over $S$. We denote the concatenation operation of words over $S$ by $\odot$. Now let $f : (\mathrm{NF} \parallel \mathrm{NF}) \cup \Gamma \to S$ be a function. By [Gis88], the semigroup $(\mathrm{NF}, \cdot)$ is freely generated by $(\mathrm{NF} \parallel \mathrm{NF}) \cup \Gamma$. Hence, we can uniquely extend the function $f$ to a semigroup homomorphism (also denoted $f$) from $(\mathrm{NF}, \cdot)$ to $(S, \odot)$. Even more, the equation $f(\prod p_i) = f(p_1) \odot f(p_2) \odot \dots$ extends $f$ uniquely to a mapping from $\mathrm{NF} \cup \mathrm{NF}^\infty \setminus (\mathrm{NF}^\infty \parallel \mathrm{NF}^\infty)$ to the set of finite and infinite words $S^\infty$. Note that $\mathrm{NF} \cup \mathrm{NF}^\infty \setminus (\mathrm{NF}^\infty \parallel \mathrm{NF}^\infty)$ is the set of finite N-free pomsets and of connected N-free pomsets.

**Lemma 5.8** *Let $f : (\mathrm{NF} \parallel \mathrm{NF}) \cup \Gamma \to S$ be a function.*

1. *Let $K \subseteq S^+$ be a starfree word language. Then $K' = f^{-1}(K) \subseteq \mathrm{NF}$ is starfree over $\{f^{-1}(s) \mid s \in S\}$.*
2. *Let $K \subseteq S^\infty$ be an $\omega$-starfree word language. Then $K' = f^{-1}(K) \subseteq \mathrm{NF}^\infty$ is $\omega$-starfree over $\{f^{-1}(s) \mid s \in S\}$.*

**Proof.** The proof proceeds by induction on the construction of the starfree set $K$: For $K = \{s\}$ with $s \in S$, the statement is trivial. The union is easily dealt with as $f^{-1}(K_1 \cup K_2) = f^{-1}(K_1) \cup f^{-1}(K_2)$. Note that $f^{-1}(S) = \mathrm{NF} \parallel \mathrm{NF} \cup \Gamma$, i.e. all preimages of a free generator of $S^+$ belong to the free generators of $(\mathrm{NF}, \cdot)$. Hence, for $s \in S$ and $w \in S^\infty$, we obtain $f^{-1}(sw) = f^{-1}(s) \cdot f^{-1}(w)$. Inductively, this implies $f^{-1}(vw) = f^{-1}(v) \cdot f^{-1}(w)$ for $v \in S^+$ and $w \in S^\infty$. Hence $f^{-1}(K_1 \odot K_2) = f^{-1}(K_1) \cdot f^{-1}(K_2)$. Finally, we have to handle the complements: $f^{-1}(S^+ \setminus K) = f^{-1}(S^+) \setminus f^{-1}(K) = \mathrm{NF} \setminus f^{-1}(K)$ and $f^{-1}(S^\infty \setminus K) = f^{-1}(S^\infty) \setminus f^{-1}(K) = [\mathrm{NF} \cup \mathrm{NF}^\infty \setminus (\mathrm{NF}^\infty \parallel \mathrm{NF}^\infty)] \setminus f^{-1}(K)$. Since NF is starfree and $\mathrm{NF}^\infty$ is $\omega$-starfree, $f^{-1}$ is indeed $\omega$-starfree over $\{f^{-1}(s) \mid s \in S\}$. $\square$

**Lemma 5.9** *Let $X \subseteq \mathrm{NF}$ be recognized by a homomorphism $\eta : (\mathrm{NF}, \cdot, \parallel) \to (S, \cdot, \parallel)$ into the finite aperiodic sp-algebra $(S, \cdot, \parallel)$. Then $X$ is starfree over $\{\eta^{-1}(s) \cap (\mathrm{NF} \parallel \mathrm{NF}) \mid s \in S\}$.*

**Proof.** Let $f : (\mathrm{NF}, \cdot) \to (S^+, \odot)$ be the uniquely determined semigroup homomorphism with $f(t) = \eta(t)$ for $t \in \mathrm{NF} \parallel \mathrm{NF} \cup \Gamma$. From the free, finitely generated semigroup $(S^+, \odot)$ we have the canonical semigroup homomorphism $\alpha$ onto $(S, \cdot)$.

Let $t \in \mathrm{NF}$. Then there are uniquely determined $t_i \in \mathrm{NF} \parallel \mathrm{NF} \cup \Gamma$ such that $t = t_1 \cdot t_2 \cdots t_n$. Hence

$$\begin{aligned} \eta(t) &= \eta(t_1) \cdot \eta(t_2) \cdots \eta(t_n) \\ &= \alpha(f(t_1) \odot f(t_2) \odot \cdots \odot f(t_n)) \\ &= \alpha \circ f(t), \end{aligned}$$

i.e. $\eta = \alpha \circ f$.

By [Sch65], $K := \alpha^{-1}(\eta(X)) \subseteq S^+$ is a starfree word language since $(S, \cdot)$ is a finite aperiodic semigroup. By Lemma 5.8, $f^{-1}(K) \subseteq \mathrm{NF}$ is starfree over $\{\eta^{-1}(s) \cap (\mathrm{NF} \parallel \mathrm{NF} \cup \Gamma) \mid s \in S\}$ since $\eta^{-1}(s) \cap (\mathrm{NF} \parallel \mathrm{NF} \cup \Gamma) = f^{-1}(s)$ for $s \in S$.

Since $\eta^{-1}(s) \cap (\mathrm{NF} \parallel \mathrm{NF} \cup \Gamma)$ is the union of $\eta^{-1}(s) \cap (\mathrm{NF} \parallel \mathrm{NF})$ and $\eta^{-1}(s) \cap \Gamma$, the set $f^{-1}(K)$ is starfree over $\{\eta^{-1}(s) \cap (\mathrm{NF} \parallel \mathrm{NF}) \mid s \in S\} \cup \{\eta^{-1}(s) \cap \Gamma \mid s \in S\}$. Since any of the latter sets is starfree anyway, we showed that $f^{-1}(K)$ is starfree over $\{\eta^{-1}(s) \cap \mathrm{NF}(\parallel \mathrm{NF}) \mid s \in S\}$. It remains to show that $f^{-1}(K)$ equals $X$: So let $t \in \mathrm{NF}$. Then $t \in f^{-1}(K)$ iff $f(t) \in K = \alpha^{-1}\eta(X)$ iff $\eta(t) = \alpha \circ f(t) \in \eta(X)$ iff $t \in \eta^{-1}\eta(X) = X$. $\square$

**Lemma 5.10** *Let $k \in \mathbb{N}$ and let $X \subseteq \mathrm{NF}_k$ be starfree over $\{Y_s \mid s \in S\} = \mathcal{H}$ where $Y_s \subseteq \mathrm{NF}$ for $s \in S$. Then $X$ is starfree over $\{Y_s \cap \mathrm{NF}_\ell \mid s \in S, 1 \leq \ell \leq k\} = \mathcal{H}_k$*

**Proof.** What we actually show is that $X \cap \mathrm{NF}_k$ is starfree over $\mathcal{H}_k$ whenever $X$ is starfree over $\mathcal{H}$. This is done by induction on the starfree construction of the set $X$ from the sets $Y_s$ and $\{t\}$ for $s \in S$ and $t \in \mathrm{NF}$ (and *not* by induction on $k$ as one might think). Since $Y_s \cap \mathrm{NF}_k$ belongs to $\mathcal{H}_k$ and since $\{t\} \cap \mathrm{NF}_k$ contains at most one element, the base step is obvious. Now let $X_1, X_2$ be starfree over $\mathcal{H}$

such that $X_i \cap \mathrm{NF}_k$ is starfree over $\mathcal{H}_k$. Then

$$\begin{aligned}
(X_1 \cdot X_2) \cap \mathrm{NF}_k &= (X_1 \cap \mathrm{NF}_k) \cdot (X_2 \cap \mathrm{NF}_k), \\
(X_1 \cup X_2) \cap \mathrm{NF}_k &= (X_1 \cap \mathrm{NF}_k) \cup (X_2 \cap \mathrm{NF}_k), \\
(\mathrm{NF} \setminus X_1) \cap \mathrm{NF}_k &= \mathrm{NF}_k \setminus (X_1 \cap \mathrm{NF}_k), \text{ and} \\
(X_1 \parallel X_2) \cap \mathrm{NF}_k &= \bigcup_{\ell_1+\ell_2=k} (X_1 \cap \mathrm{NF}_{\ell_1}) \parallel (X_2 \cap \mathrm{NF}_{\ell_2})
\end{aligned}$$

which proves that the left hand sides of all these equations are starfree over $\mathcal{H}_k$.
□

Mezei's Theorem characterizes the recognizable languages in a direct product of two monoids; it states that these are precisely the finite unions of direct products of recognizable languages in the monoids involved. Using "the same proof" we show a similar statement for nonconnected aperiodic languages in $\mathrm{NF}$:

**Lemma 5.11** *Let $X \subseteq \mathrm{NF} \parallel \mathrm{NF}$ be aperiodic. Then there exist $n \in \mathbb{N}$ and $K_i, L_i \in \mathrm{NF}$ aperiodic for $1 \leq i \leq n$ such that $X = \bigcup_{1 \leq i \leq n} K_i \parallel L_i$.*

**Proof.** There exist a finite aperiodic sp-algebra $(S, \cdot, \|)$ and an sp-homomorphism $\eta : \mathrm{NF} \to S$ such that $X = \eta^{-1}\eta(X)$. Let $G = \{(s_1, s_2) \in S \times S \mid s_1 \parallel s_2 \in \eta(X)\}$. Note that $\eta^{-1}(s) \subseteq \mathrm{NF}$ is aperiodic for any $s \in S$. Hence it suffices to show that $X = \bigcup\{\eta^{-1}(s_1) \parallel \eta^{-1}(s_2) \mid (s_1, s_2) \in G\}$.

First, let $t \in X \subseteq \mathrm{NF} \parallel \mathrm{NF}$. Then there are $t_1, t_2 \in \mathrm{NF}$ with $t = t_1 \parallel t_2$. Hence $\eta(t_1) \parallel \eta(t_2) = \eta(t) \in \eta(X)$, i.e. $(\eta(t_1), \eta(t_2)) \in G$ and, trivially, $t \in \eta^{-1}(\eta(t_1)) \parallel \eta^{-1}(\eta(t_2))$. But this proves the inclusion "$\subseteq$". Conversely let $(s_1, s_2) \in G$, i.e. $s_1 \parallel s_2 \in \eta(X)$, and let $t \in \eta^{-1}(s_1) \parallel \eta^{-1}(s_2)$. Then we find $t_1, t_2 \in \mathrm{NF}$ such that $t = t_1 \parallel t_2$ and $t_i \in \eta^{-1}(s_i)$. Hence $\eta(t) = \eta(t_1) \parallel \eta(t_2) = s_1 \parallel s_2 \in \eta(X)$, i.e. $t \in \eta^{-1}\eta(X) = X$ proving the remaining inclusion "$\supseteq$". □

**Theorem 5.12** *Let $k \in \mathbb{N}$ and let $X \subseteq \mathrm{NF}_k$ be aperiodic. Then $X$ is starfree.*

**Proof.** We show the theorem by induction on $k$. By Example 5.6, Schützenberger's Theorem [Sch65] implies the theorem for $k = 1$. Now suppose the theorem holds for all $\ell < k$ and let $X \subseteq \mathrm{NF}_k$. Then $X = (X \cap (\mathrm{NF} \parallel \mathrm{NF})) \cup (X \setminus (\mathrm{NF} \parallel \mathrm{NF}))$.

Since $\mathrm{NF} \parallel \mathrm{NF}$ is aperiodic and aperiodic sets are closed under intersection, the set $X \cap (\mathrm{NF} \parallel \mathrm{NF})$ is aperiodic. By Lemma 5.11, $X \cap (\mathrm{NF} \parallel \mathrm{NF})$ is a finite union of sets $K \parallel L$ with $K, L \subseteq \mathrm{NF}$ aperiodic. Since $X \subseteq \mathrm{NF}_k$, we obtain $K, L \subseteq \mathrm{NF}_{k-1}$. Thus, by the induction hypothesis, $K$ and $L$ are starfree. Hence $X \cap (\mathrm{NF} \parallel \mathrm{NF})$ is starfree. In particular, we showed the theorem for aperiodic sets contained in $(\mathrm{NF} \parallel \mathrm{NF}) \cap \mathrm{NF}_k$.

Next, we deal with $X \setminus (\mathrm{NF} \parallel \mathrm{NF})$: Since $X$ and $\mathrm{NF} \parallel \mathrm{NF}$ are aperiodic, so is their difference. Hence there exists a finite aperiodic sp-algebra $(S, \cdot, \parallel)$ and an sp-homomorphism $\eta : \mathrm{NF} \to S$ that recognizes $X \setminus (\mathrm{NF} \parallel \mathrm{NF})$. By Lemma 5.9, $X \setminus (\mathrm{NF} \parallel \mathrm{NF})$ is starfree over $\{\eta^{-1}(s) \cap \mathrm{NF} \parallel \mathrm{NF} \mid s \in S\}$. By Lemma 5.10, it is therefore starfree over $\{\eta^{-1}(s) \cap (\mathrm{NF} \parallel \mathrm{NF}) \cap \mathrm{NF}_\ell \mid s \in S, \ell \leq k\}$. Note that any of the sets $\eta^{-1}(s) \cap (\mathrm{NF} \parallel \mathrm{NF}) \cap \mathrm{NF}_\ell$ is aperiodic and contained in $\mathrm{NF} \parallel \mathrm{NF} \cap \mathrm{NF}_k$. Thus, by what we showed above, they are starfree. $\square$

## 5.4 $\omega$-aperiodic and $\omega$-starfree sets

Now we generalize the above result to infinite N-free pomsets. In the proof of Prop. 5.5, we used simple terms that in particular involved idempotent elements of the sequential semigroup $(S, \cdot)$. The following lemma implies immediately that $L_\eta(\alpha, \beta)$ is $\omega$-starfree for any connected pair $(\alpha, \beta)$ in a finite aperiodic sp-algebra:

**Lemma 5.13** *Let $(S, \cdot, \parallel)$ be a finite and aperiodic sp-algebra, let $\beta \in S$ with $\beta \cdot \beta = \beta$, and let $\eta : \mathrm{NF} \to S$ be an sp-homomorphism. Then $(\eta^{-1}(\beta))^\omega$ is $\omega$-starfree.*

**Proof.** Let $X = \eta^{-1}(\beta)$. As before, let $f : (\mathrm{NF}, \cdot) \to (S^+, \odot)$ be the uniquely determined semigroup homomorphism with $f(t) = \eta(t)$ for $t \in \mathrm{NF} \parallel \mathrm{NF} \cup \Gamma$ and let $\alpha$ be the natural homomorphism from $(S, \odot)$ to $(S, \cdot)$. Then, as we saw in the proof of Lemma 5.9, $\eta = \alpha \circ f$ implying $K := f(X) = \alpha^{-1}(\beta) \subseteq S^+$. Since $(S, \cdot)$ is an aperiodic semigroup, $K$ is an aperiodic word language. By $\beta \cdot \beta = \beta$, the set $K^\omega$ is $\omega$-aperiodic and therefore $\omega$-starfree. Hence, by Lemma 5.8, $f^{-1}(K^\omega)$ is $\omega$-starfree over $\{f^{-1}(s) \mid s \in S\}$. Note that $f^{-1}(s) = \eta^{-1}(s) \cap (\mathrm{NF} \parallel \mathrm{NF} \cup \Gamma)$. Since $\mathrm{NF}$ and $\Gamma$ are starfree, this implies that $f^{-1}(s)$ is starfree for $s \in S$, i.e. $f^{-1}(K^\omega)$ is $\omega$-starfree.

It remains to show that $X^\omega = f^{-1}(K^\omega)$: Let $p_i \in X$. Then $f(p_i) \in K$ and therefore $f(\prod p_i) \in K^\omega$. This proves $X^\omega \subseteq f^{-1}(K^\omega)$. For the other inclusion let $w_i \in K$. Then there are $p_i \in X$ with $f(p_i) = w_i$. Hence $\prod w_i = \prod f(p_i) = f(\prod p_i) \in f(X^\omega)$ showing $X^\omega \supseteq f^{-1}(K^\omega)$. $\square$

**Proposition 5.14** *Let $\Gamma$ be an alphabet and $L \subseteq \mathrm{NF}^{\infty}(\Gamma)$ be $\omega$-aperiodic and width-bounded. Then $L$ is $\omega$-starfree.*

**Proof.** Let $\eta : \mathrm{NF}(\Gamma) \to (S, \cdot, \|)$ be an sp-homomorphism that recognizes $L$. Furthermore, let $n \in \omega$ such that $w(p) \leq n$ for any $p \in L$. Now let $\mathcal{T}_n$ denote the set of tuples of simple terms $(\tau_1, \tau_2, \ldots, \tau_k)$ with $k \leq n$ of order at most $n$ such that $\emptyset \neq L \cap (L_\eta(\tau_1) \| L_\eta(\tau_2) \| \cdots L_\eta(\tau_k))$. Note that $\mathcal{T}_n$ is finite.

By Proposition 5.3, $L$ contains all languages $L_\eta(\tau_1) \| L_\eta(\tau_2) \| \cdots L_\eta(\tau_k)$ for $(\tau_1, \tau_2, \ldots, \tau_n) \in \mathcal{T}_n$. Conversely, let $p \in L$. By Lemma 5.1, there exists a tuple $(\tau_1, \tau_2, \ldots, \tau_n)$ of simple terms of order at most $w(p) \leq n$ such that $p \in L_\eta(\tau_1) \| L_\eta(\tau_2) \| \cdots L_\eta(\tau_k)$. Note that $k \leq w(p) \leq n$ implying $(\tau_1, \tau_2, \ldots, \tau_k) \in \mathcal{T}_n$. This finishes the proof that $L$ equals the union of all sets $L_\eta(\tau_1) \| L_\eta(\tau_2) \| \cdots L_\eta(\tau_k)$ for $(\tau_1, \tau_2, \ldots, \tau_k) \in \mathcal{T}_n$.

Hence it remains to show that $L_\eta(\tau)$ is $\omega$-starfree for any simple term $\tau$ such that there exists $(\sigma_1, \sigma_2, \ldots, \sigma_k) \in \mathcal{T}_n$ with $\sigma_i = \tau$ for some $1 \leq i \leq k$. The proof proceeds by induction on the subterms of $\tau$: So let $\sigma$ be a subterm of $\tau$ of order 1. First assume $\sigma = \alpha \in S$. Then $L_\eta(\sigma) = \eta^{-1}(\alpha)$ is aperiodic. Note that any $p \in \eta^{-1}(\alpha)$ is a subposet of some element of the language $L$. Since $L$ is width-bounded, the set $\eta^{-1}(\alpha)$ is width-bounded. By Theorem 5.12 it follows that $L_\eta(\alpha)$ is starfree. Now let $\sigma = (\alpha, \beta)$ be a linked pair. Similarly to the first case, any element of $\eta^{-1}(\alpha) \cup \eta^{-1}(\beta)$ is a subposet of some element of $L$. Hence $\eta^{-1}(\alpha)$ and $\eta^{-1}(\beta)$ are width-bounded and aperiodic and therefore starfree. Since $\beta \cdot \beta = \beta$, by Lemma 5.13 the set $(\eta^{-1}(\beta))^\omega$ is $\omega$-starfree. Hence $\eta^{-1}(\alpha) \cdot (\eta^{-1}(\beta)^\omega = L_\eta(\alpha, \beta)$, is $\omega$-starfree. For the induction step, let $\sigma = \alpha \cdot (\tau_1 \| \tau_2 \| \cdots \tau_n)$ be a subterm of $\tau$ such that $L_\eta(\tau_i)$ is $\omega$-series-rational. Again, we obtain that $\eta^{-1}(\alpha)$ is width-bounded and aperiodic and therefore starfree. Hence $L_\eta(\sigma) = \eta^{-1}(\alpha) \cdot (L_\eta(\tau_1) \| L_\eta(\tau_2) \| \cdots L_\eta(\tau_n))$ is $\omega$-starfree. $\square$

# 6 Branching Büchi-automata

In this section, we extend the concept of a branching automaton that accepts only finite N-free pomsets to a branching Büchi-automaton that is capable of accepting infinite N-free pomsets. Similarly to the finite case, we show that a set of N-free pomsets is $\omega$-series-rational iff it can be accepted by a finite branching Büchi-automaton and is width-bounded.

**Notational conventions** To ease the notation, we will identify an element $s \in S$ with the function $g : S \to \omega$ that sends $s$ to 1 and any other element of $S$ to 0 (i.e. with the characteristic function of the set $\{s\}$). In this sense, we will say that $S$ is a subset of $\omega^S$. The function that sends any element of $S$ to 0 is denoted by 0, too. Let $F(S)$ be the set of pairs of functions from $S$ to $\omega$, i.e. $F(S) = (\omega^S)^2$. Note that $(s, 0)$ and $(0, t)$ for $s, t \in S$ are particular elements of $F(S)$. An element $g = (g_f, g_i) \in F(S)$ is *non-zero* iff $g_f \neq 0 \neq g_i$. The addition of elements of $F(S)$ is defined componentwise: $(g_f, g_i) + (h_f, h_i) = (g_f + h_f, g_i + h_i)$ and $(g_f + h_f)(s) = g_f(s) + h_f(s)$ for $s \in S$.

Now we can introduce our notion of a device accepting N-free pomsets:

**Definition 6.1** A *branching Büchi-automaton* is a tuple $\mathcal{A} = (S, T_s, T_f, T_j, P, I, \mathfrak{F})$ where

1. $S$ is a set of *states*,
2. $T_s \subseteq S \times \Gamma \times S$ is a set of *sequential transitions*,
3. $T_f, T_j, P \subseteq S^3$ are sets of *fork*, *join*, and *pairing transitions*, resp.,
4. $I \subseteq S$ is a set of *initial states*, and
5. $\mathfrak{F} \subseteq F(S)$ is a set of pairs of *accepting functions* $g = (g_f, g_i)$ with $g_f, g_i : S \to \omega$.

The branching Büchi-automaton $\mathcal{A}$ is *finite* iff $S$ and $\mathfrak{F}$ are finite. It is a *branching automaton* if $g_i = 0$ and $g_f \in S$ for any $g \in \mathfrak{F}$.

There are three differences between our branching Büchi-automata and the branching automata defined by Lodaya & Weil: First, our fork and join transitions are simpler than those of Lodaya & Weil. These authors allowed forking into more than two branches. But it is an easy exercise to show that the restriction to binary branching does not influence the expressive power. Second, we have an additional set of pairing transitions. These transitions will work similar to fork transitions but in a run, they need not be paired with a join transition (see below). As a consequence of these pairing transitions, a run has in general more than one final state. Therefore the set of accepting states from [LW00b] is replaced by accepting functions.

Next, we define how a branching Büchi-automaton runs on finite N-free pomsets. These runs are defined inductively on the construction of the pomset: Let

$p_i = (V_i, \leq, \lambda_i)$ for $i = 0, 1$ be finite N-free pomsets and $s, t \in S$ be states. If $V_0 = \{x\}$ with $\lambda(x) = a$, then there is a *finite run of $\mathcal{A}$ on $p_0$ of depth $0$ from $s$ to $t$* (written $s \xrightarrow{p_0}_0 t$) iff $(s, a, t) \in T_s$. Furthermore, $s \xrightarrow{p_0 \cdot p_1}_n t$ iff there exists a state $r \in S$ with $s \xrightarrow{p_0}_k r \xrightarrow{p_1}_\ell t$ and $n = \max(k, \ell)$. Finally, $s \xrightarrow{p_0 \| p_1}_n t$ iff there exist states $s_i$ and $t_i$ such that $(s, s_0, s_1) \in T_f$ is a fork transition, $(t_0, t_1, t) \in T_j$ is a join transition, $s_i \xrightarrow{p_i}_{n_i} t_i$ for $i = 0, 1$, and $n = 1 + \max(n_0, n_1)$.

Thus, finite runs transform states of $\mathcal{A}$ to other states of $\mathcal{A}$. The depth of a finite run measures the maximal number of nested fork transitions in this run. Therefore, if the finite N-free pomset $p$ admits a finite run of depth $n$ in $\mathcal{A}$, then the width of $p$ is at least $n + 1$ and at most $2^n$.

Next, we define runs on possibly infinite N-free pomsets. These runs transform states of $\mathcal{A}$ to pairs of functions from $\omega^S$:

- If $s \xrightarrow{p}_n t$ is a finite run, then $s \overset{p}{\Longrightarrow}_n (t, 0)$ is a run of depth $n$.
- Let $p_i \in \mathrm{NF}$ be finite N-free pomsets, $s_i \in S$ be states such that $s_i \xrightarrow{p_i}_{n_i} s_{i+1}$ for $i \in \omega$. Furthermore suppose that there are infinitely many $i \in \omega$ with $s_i = t$ and let $n = \sup\{n_i \mid i \in \omega\}$. Then $s \overset{\prod_{i \in \omega} p_i}{\Longrightarrow}_n (0, t)$.
- Let $p \in \mathrm{NF}$, $q \in \mathrm{NF}^\infty$, $s, t \in S$ and $g \in F(S)$ with $s \xrightarrow{p}_k t$, $t \overset{q}{\Longrightarrow}_\ell g$ and $n = \max(k, \ell)$. Then $s \overset{p \cdot q}{\Longrightarrow}_n g$.
- Let $q_0, q_1 \in \mathrm{NF}^\infty$, $s, s_0, s_1 \in S$ and $g, g_0, g_1 \in F(S)$, such that $s_j \overset{q_j}{\Longrightarrow}_{n_j} g$ for $j = 0, 1$, $(s, s_0, s_1) \in P$, and $n = 1 + \max(n_0, n_1)$ (where we set $1 + \infty = \infty$). Then $s \overset{q_0 \| q_1}{\Longrightarrow}_n g_0 + g_1$.

Similarly to finite runs, the depth of a run measures the maximal nesting of fork and pairing transitions in this run. Hence, as above, $n < w(p) \leq 2^n$ whenever $p \in \mathrm{NF}^\infty$ admits a run of depth $n$.

A run $s \overset{p}{\Longrightarrow}_n g$ is *successful* if $s \in I$ and $g \in \mathfrak{F}$. For a branching Büchi-automaton $\mathcal{A}$, let $L(\mathcal{A})$ denote the set of all $p \in \mathrm{NF}^\infty$ that admit a successful run. A set $L \subseteq \mathrm{NF}^\infty$ is *$\omega$-regular* if it can be accepted by a finite branching Büchi-automaton.

## 6.1 From $\omega$-regular to $\omega$-series-rational sets

A branching Büchi-automaton is *of bounded depth* if there exists a natural number $n \in \omega$ such that $s \overset{p}{\Longrightarrow}_m g$ implies $m \leq n$ for any $p \in \mathrm{NF}^\infty$, $s \in S$ and $g \in$

$F(S)$. In [LW00b], fork-acyclic branching automata are considered that satisfy this condition for *successful* runs, only. Therefore, any branching automaton of bounded depth is fork-acyclic, but the converse is false in general.

Since $w(p) \leq 2^n$ whenever $s \overset{p}{\Longrightarrow}_n g$, a branching Büchi-automaton $\mathcal{A}$ of bounded depth accepts a set of pomsets $L(\mathcal{A})$ of bounded width. For fork-acyclic branching automata and finite N-free pomsets, this was observed in [LW00b]. There, the converse is shown as well: If $L(\mathcal{A}) \subseteq \mathrm{NF}$ is of bounded width, than $\mathcal{A}$ is fork-acyclic. But there are branching automata $\mathcal{A}$ with $L(\mathcal{A})$ of bounded width that are not of bounded depth. Since we need this stronger notion, next we show that any $\omega$-regular set of bounded width can be accepted by a finite branching Büchi-automaton of bounded depth:

**Lemma 6.2** *Let $\mathcal{A}$ be a finite branching Büchi-automaton and suppose that $L(\mathcal{A})$ has bounded width. Then there exists a finite branching Büchi-automaton $\mathcal{A}'$ of bounded depth with $L(\mathcal{A}) = L(\mathcal{A}')$.*

**Proof.** Let $\mathcal{A} = (S, T_s, T_f, T_j, P, I, \mathfrak{F})$ and $n \in \omega$ with $w(p) \leq n$ for any $p \in \mathrm{NF}^\infty$ that is accepted by $\mathcal{A}$. Let $S' = S \times \{0, 1, 2, 3, \dots, n\}$ be the state space of the automaton $\mathcal{A}'$. This automaton simulates the automaton $\mathcal{A}$ in the first component and counts the nesting of fork and pairing transitions in its second component. Therefore, we define $((s,k), a, (t,\ell)) \in T_s'$ iff $k = \ell$ and $(s, a, t) \in T_s$ is a sequential transition of the old automaton. Furthermore, $((s,m), (s_0, m_0), (s_1, m_1)) \in T_f'$ is a fork transition of the new automaton iff $(s, s_0, s_1) \in T_f$ is one of the old and $m_0 = m_1 = m + 1$. The pairing transitions from $P'$ are defined similarly. Dually, $((s_0, m_0), (s_1, m_1), (s, m)) \in T_j'$ is a join transition of the new automaton iff $(s_0, s_1, s) \in T_j$ is one of the old and $m_0 = m_1 = m + 1$. The initial states of the new automaton are those states whose first component is initial in the old automaton: $I' = I \times \{1, 2, \dots, n\}$. Similarly, the final functions are defined: A pair of functions $g = (g_f, g_i) \in F(S')$ belongs to $\mathfrak{F}'$ iff $(g_f \circ \pi_1, g_i \circ \pi_1) = g \circ \pi_1$ belongs to $\mathfrak{F}$.

As any fork and pairing transition in $\mathcal{A}'$ increments the depth counter and since this counter is bounded by $n$, any run in $\mathcal{A}'$ has depth at most $n$. Hence the automaton $\mathcal{A}'$ is indeed of bounded depth.

Now let $(s,k) \overset{p}{\Longrightarrow}_m g'$ be a successful run of depth $m$ in the automaton $\mathcal{A}'$. Since this automaton simulates $\mathcal{A}$ in its first component, there is a run $s \overset{p}{\Longrightarrow} g' \circ \pi_1$ in $\mathcal{A}$ where $\pi_1 : S' \to S$ is the projection to the first component. Since $g'$ is a final function in $\mathcal{A}'$, the function $g' \circ \pi_1$ is final in $\mathcal{A}$, i.e. $p$ is accepted by $\mathcal{A}$. Thus $L(\mathcal{A}') \subseteq L(\mathcal{A})$.

Conversely suppose $s \xRightarrow{p}_m g$ is a successful run in $\mathcal{A}$. Then $m < w(p)$ since $p$ admits a run of depth $m$. On the other hand $w(p) \leq n$ since $p$ is accepted by $\mathcal{A}$. This implies $m < n$, i.e. in the run $s \xrightarrow{p}_m f$, the nesting of fork and pairing transitions is smaller than $n$. Hence this run can be simulated by $\mathcal{A}'$, i.e. there is a run $(s,0) \xRightarrow{p}_m g'$ in $\mathcal{A}'$ that, in the first component, simulates the run $s \xRightarrow{p}_m g$ of $\mathcal{A}$. In particular $g = g' \circ \pi_1$ implying that the run $(s,0) \xrightarrow{p}_m g'$ is successful in $\mathcal{A}'$. Hence $L(\mathcal{A}) \subseteq L(\mathcal{A}')$. $\square$

Lodaya & Weil showed that a set of finite N-free pomsets is series-rational iff it can be accepted by a finite branching automaton and is width-bounded. This result and the lemma above are the basis for the proof of the following corollary:

**Corollary 6.3** *Let $L \subseteq \mathrm{NF}$ be a set of finite N-free pomsets. Then $L$ is series-rational iff it can be accepted by a finite branching automaton of bounded depth.*

**Proof.** If $L$ is series-rational, it can be accepted by a finite branching automaton [LW00b, Thm. 4.12] and is of bounded width. Hence, by Lemma 6.2, it can be accepted by a finite branching automaton of bounded depth. Conversely, let $L$ be accepted by a finite branching automaton $\mathcal{A}$ whose depth is bounded by $n$. Then, as we saw above, any N-free pomset accepted by $\mathcal{A}$ has width at most $2^n$, i.e. $L(\mathcal{A})$ has bounded width. Hence $L(\mathcal{A})$ is series-rational by [LW00b, Thm. 4.12]. $\square$

Next we show that any $\omega$-regular set of N-free pomsets of bounded width is $\omega$-series-rational:

**Proposition 6.4** *Let $\mathcal{A}$ be a finite branching Büchi-automaton such that $L(\mathcal{A})$ is of bounded depth. Then $L(\mathcal{A})$ is $\omega$-series-rational.*

**Proof.** By Lemma 6.2, we can assume $\mathcal{A} = (S, T_s, T_f, T_j, P, I, \mathfrak{F})$ to be of bounded depth. For $s \in S$ and $g \in F(S)$, let $\mathcal{A}_{s,f} = (S, T_s, T_f, T_j, P, \{s\}, \{g\})$ be the branching Büchi-automaton with one initial state $s$ and one accepting function $g$. Note that in case $g \in S$, the branching Büchi-automaton $\mathcal{A}_{s,g}$ is actually a branching automaton and therefore falls into the realm of Cor. 6.3. Hence, in this case, $L(\mathcal{A}_{s,g})$ is series-rational. Now suppose $g_f = 0$ and $g_i = t \in S$. Then an N-free pomset $p$ is accepted by $\mathcal{A}_{s,g}$ iff it admits a run $s \xrightarrow{p}_n g$, i.e. iff it is an infinite sequential product $p = \prod_{i \in \omega} p_i$ of finite pomsets $p_i$ such that $p_0$ admits a finite run from $s$ to $t$, and $p_i$ leads from $t$ to $t$ for $i > 0$. Thus,

$$L(\mathcal{A}_{s,f}) = L(\mathcal{A}_{s,t}) \cdot (L(\mathcal{A}_{t,t}))^\omega$$

which is $\omega$-series-rational since the sets $L(\mathcal{A}_{s,t})$ and $L(A_{t,t})$ are series-rational subsets of $\mathrm{NF}$.

Now let $g$ be the sum of two non-zero functions. Then a pomset $p \in \mathrm{NF}^\infty$ is accepted by $\mathcal{A}_{s,g}$ iff there are two non-zero functions $g^0$ and $g^1$ with $g = g^0 + g^1$, two N-free pomsets $p_0$ and $p_1$,

1. and a pairing transition $(s, s_0, s_1) \in P$ such that $p = p_0 \parallel p_1$ and $s_i \stackrel{p_i}{\Longrightarrow} g^i$ for $i = 0, 1$, or

2. a pairing transition $(s', s_0, s_1) \in P$, and a finite N-free pomset $q$ such that $p = q \cdot (p_1 \parallel p_2)$, $s \stackrel{q}{\longrightarrow} s'$, and $s_i \stackrel{p_i}{\Longrightarrow} g^i$ for $i = 0, 1$.

Hence $L(\mathcal{A}_{s,g})$ is the union of all sets

$$\bigcup_{(s,s_0,s_1)\in P} L(\mathcal{A}_{s_0,g^0}) \parallel L(\mathcal{A}_{s_1,g^1}) \cup \bigcup_{(s',s_0,s_1)\in P} L(\mathcal{A}_{s,(s',0)}) \cdot (L(\mathcal{A}_{s_0,g^0}) \parallel L(\mathcal{A}_{s_1,g^1}))$$

over all pairs $(g^0, g^1)$ of non-zero functions with $g = g^0 + g^1$. Since this union is finite, we showed inductively that $L(\mathcal{A}_{s,g})$ is $\omega$-series-rational for any $s \in S$ and $g \in F(S)$.

Now $L(\mathcal{A}) = \bigcup_{s\in I, g\in\mathcal{F}} L(\mathcal{A}_{s,g})$. Since $\mathcal{A}$ is finite, $L(\mathcal{A})$ is $\omega$-series-rational. $\square$

## 6.2 From $\omega$-series-rational to $\omega$-regular sets

Similar to [LW00b], we will show that the substitution of a regular set $K$ for a letter $\xi$ in an $\omega$-regular set $L$ yields an $\omega$-regular set $L[\xi, K]$. Since, e.g., $K^\omega = \xi^\omega[\xi, K]$, this will imply that the $\omega$-regular sets satisfy the closure properties required for the $\omega$-series-rational ones.

We start with the formal definition of the substitution: Let $\xi \notin \Gamma$ and $K \subseteq \mathrm{NF}$. Then

$$\begin{aligned}
\xi[\xi, K] &= K, \\
a[\xi, K] &= \{a\} \text{ for } a \neq \xi, \\
(q_0 \parallel q_1)[\xi, K] &= q_0[\xi, K] \parallel q_1[\xi, K], \\
(p_0 \cdot q_0)[\xi, K] &= p_0[\xi, K] \cdot q_0[\xi, K], \text{ and} \\
(\prod_{i\in\omega} p_i)[\xi, K] &= \prod_{i\in\omega}(p_i[\xi, K])
\end{aligned}$$

where $p_i \in \mathrm{NF}(\Gamma \cup \{\xi\})$ and $q_i \in \mathrm{NF}^\infty(\Gamma \cup \{\xi\})$. Furthermore, for $L \subseteq \mathrm{NF}^\infty(\Gamma \cup \{\xi\})$ let

$$L[\xi, K] = \bigcup_{q \in L} q[\xi, K].$$

Now one can show that the set of $\omega$-regular sets is closed under substitutions of regular sets:

**Lemma 6.5** *Let $K \subseteq \mathrm{NF}$ and $L \subseteq \mathrm{NF}^\infty(\Gamma \cup \{\xi\})$ be $\omega$-regular. Then $L[\xi, K]$ is $\omega$-regular, too.*

**Proof.** The proof is a simple adaptation of the proof of [LW00b, Prop. 4.9] which involves the notion of a behaved branching automaton.

□

**Proposition 6.6** *Let $L \subseteq \mathrm{NF}^\infty$ be $\omega$-series-rational. Then there exists a finite branching Büchi-automaton $\mathcal{A}$ that accepts $L$.*

**Proof.** It is easily seen that $\{p\}$ can be accepted by a branching automaton for $p \in \mathrm{NF}$ (cf. [LW00b]). The union $L(\mathcal{A}_0) \cup L(\mathcal{A}_1)$ is accepted by the disjoint union of the two branching Büchi-automata $\mathcal{A}_0$ and $\mathcal{A}_1$.

Now let $\mathcal{A}^i = (S^i, T^i_s, T^i_f, T^i_j, P^i, I^i, \mathfrak{F}^i)$ for $i = 0, 1$ be finite branching Büchi-automata with disjoint state sets. To accept the parallel product $L(\mathcal{A}^0) \parallel L(\mathcal{A}^1)$, take the union of these two automata, add a new initial state $\iota$, pairing transitions $(\iota, s^0, s^1)$ for any initial states $s^i \in I^i$, and require that "both parts" of the new automaton are accepting. More formally, a pair of functions $g_f$ and $g_i$ from $\omega^{S^0 \cup S^1}$ is accepting (i.e. belongs to $\mathfrak{F}$) if $(g_f \restriction S^i, g_i \restriction S^i) \in \mathfrak{F}^i$ for $i = 0, 1$. So let $\mathcal{A}$ denote the finite branching automaton

$$(S^0 \cup S^1 \dot\cup \{\iota\}, T^0_s \cup T^1_s, T^0_f \cup T^1_f, T^0_j \cup T^1_j, P, \{\iota\}, \mathfrak{F})$$

with $P = P^0 \cup P^1 \cup \{\iota\} \times I^0 \times I^1$. Any successful run of this automaton starts with a pairing transition of the form $(\iota, s^0, s^1)$ with $s^i \in I^i$. Hence $L(\mathcal{A}) = L(\mathcal{A}^0) \parallel L(\mathcal{A}^1)$.

Now suppose that $K = L(\mathcal{A}^0)$ consists of finite N-free pomsets, only. Since $\{\xi\}^+$ and $\{\xi\}^\omega$ are $\omega$-regular, so are $K^+ = (\{\xi\}^+)[\xi, K]$ and $K^\omega = (\{\xi\}^\omega)[\xi, K]$ by Lemma 6.5. To show that $K \cdot L(\mathcal{A}^1)$ is $\omega$-regular, we first show that $\{\xi\} \cdot L(\mathcal{A}^1)$ is $\omega$-regular: Replace the set of initial states $I^1$ by a new initial state $\iota$ and add sequential transitions $(\iota, \xi, s)$ for $s \in I^1$. The resulting finite branching Büchi-automaton obviously accepts $\{\xi\} \cdot L(\mathcal{A}^1)$. Since $(\{\xi\} \cdot L(\mathcal{A}^1))[\xi, K] = L(\mathcal{A}^0) \cdot L(\mathcal{A}^1)$, Lemma 6.5 implies that this sequential product is $\omega$-regular, too.

Since the $\omega$-regular sets are closed under all necessary operations, and since the set of $\omega$-series-rational sets is the least such set, we showed that any $\omega$-regular set is $\omega$-series-rational. □

# 7 Summary

Now we can prove our main theorems:

**Theorem 7.1** *Let $\Gamma$ be an alphabet and $L \subseteq \mathrm{NF}^\infty$. Then the following are equivalent:*

1. *$L$ is $\omega$-series-rational.*
2. *$L$ is monadically axiomatizable relative to $\mathrm{NF}^\infty$ and width-bounded.*
3. *$L$ is $\omega$-recognizable and width-bounded.*
4. *$L$ is $\omega$-regular and width-bounded.*
5. *$L$ is accepted by a finite branching Büchi-automaton of bounded depth.*

**Proof.** The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$ are Lemma 3.1, Theorem 4.2 and Proposition 5.5, resp. The equivalence of $(1)$ and $(4)$ is implied by Proposition 6.4 and Proposition 6.6. The implication $(4) \Rightarrow (5)$ is Lemma 6.2 and the remaining implication $(5) \Rightarrow (4)$ is trivial. □

**Theorem 7.2** *Let $\Gamma$ be an alphabet and $L \subseteq \mathrm{NF}^\infty$. Then the following are equivalent:*

1. *$L$ is $\omega$-starfree.*
2. *$L$ is elementary axiomatizable relative to $\mathrm{NF}^\infty$ and width-bounded.*
3. *$L$ is $\omega$-aperiodic and width-bounded.*

**Proof.** The implications $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$ are Theorem 3.3, Theorem 4.3 and Proposition 5.14, resp. □

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