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Distributive lattices with a decidable monadic second order theory

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Abstract

The present paper characterizes classes of distributive lattices whose monadic, monadic chain, or monadic antichain theory are decidable.
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1 Introduction

The elementary theory of a finite distributive lattice can easily be interpreted in the monadic second order theory of its join irreducible elements. Therefore, the decidability of the monadic second order theory of the join irreducible elements implies that of the elementary theory of the finite distributive lattices. This observation was used to show that the set of finite separated distributive lattices has a decidable elementary theory [Lac68, Gur83]; in this case, the join irreducible sets form precisely the finite trees whose monadic second order theory is decidable [Don65].[1] In this paper, we seek a characterization of those classes of finite distributive lattices that have a decidable monadic second order theory.

Let us introduce some notations: An element x of a join semilattice $(L, \leq)$ is *join-irreducible* if it cannot be written as the supremum of finitely many other elements; by $\mathbb{J}(L, \leq)$, we denote the set of all join irreducible elements of $(L, \leq)$. For a partially ordered set $(P, \leq)$, we denote the set of finitely generated ideals by $\mathbb{H}_f(P, \leq)$. Ordered by set inclusion, it is a semilattice. It is well known that $(\mathbb{H}_f(P, \leq), \subseteq)$ is a distributive lattice whenever P is finite, and that $(L, \leq) \cong \mathbb{H}_f(\mathbb{J}(L, \leq))$ for any finite distributive lattice $(L, \leq)$ [Bir40]. For classes of posets

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[1] The finiteness of the lattices is necessary for the set of infinite separated distributive lattices has an undecidable elementary theory [Gur83].

$\mathfrak{P}$ and of join semilattices $\mathfrak{L}$, we write $\mathbb{H}_f(\mathfrak{P})$ for the set of all structures $\mathbb{H}_f(P, \leq)$ with $(P, \leq) \in \mathfrak{P}$ and $\mathbb{J}(\mathfrak{L})$ is understood similarly.

For incomparable elements x and y, we write $x \parallel y$. An *antichain* is a set of mutually incomparable elements. By $w(P, \leq)$, we denote the width of a partially ordered set, i.e., the supremum of the cardinalities of its elements.

We will use the language of partial orders. For a partial order $(P, \leq)$, $\mathrm{Th}(P, \leq)$ denotes its elementary theory. The monadic second order theory is denoted by $\mathrm{MTh}(P, \leq)$; here, we can in addition quantify over subsets of P (but not over relations of larger arity) and state that an element x belongs to a subset X. In monadic antichain logic, monadic chain logic, and weak monadic antichain logic, the quantification over sets is restricted to antichains, chains, and finite antichains, respectively. The corresponding theories of $(P, \leq)$ are denoted by $\mathrm{MATh}(P, \leq)$, $\mathrm{MCTh}(P, \leq)$, and $\mathrm{wMATh}(P, \leq)$. For classes of partial orders $\mathfrak{P}$, $\mathrm{Th}(\mathfrak{P})$ denotes the intersection of the elementary theories of the elements of $\mathfrak{P}$, and $\mathrm{MTh}(\mathfrak{P})$ *etc* are defined similarly.

The following diagram summarizes our results for a set $\mathfrak{L}$ of distributive lattices $(L, \leq)$ satisfying $(L, \leq) \cong \mathbb{H}_f(\mathbb{J}(L, \leq))$ (cf. Corollary 5.2):

$$
\begin{array}{ccl}
\mathrm{MCTh}(\mathfrak{L}) \text{ is decidable} & \iff & \mathrm{MCTh}(\mathbb{J}(\mathfrak{L})) \text{ is decidable and} \\
 & & w(\mathfrak{L}) \text{ is bounded} \\
\Updownarrow & & \Updownarrow \\
\mathrm{MTh}(\mathfrak{L}) \text{ is decidable} & \iff & \mathrm{MTh}(\mathbb{J}(\mathfrak{L})) \text{ is decidable and} \\
 & & w(\mathfrak{L}) \text{ is bounded} \\
\Downarrow & & \Downarrow \\
\mathrm{MATh}(\mathfrak{L}) \text{ is decidable} & \iff & \mathrm{Th}(\mathbb{J}(\mathfrak{L})) \text{ is decidable and} \\
 & & w(\mathfrak{L}) \text{ is bounded} \\
\Downarrow & & \Downarrow \\
\mathrm{Th}(\mathfrak{L}) \text{ is decidable} & \iff & \mathrm{wMATh}(\mathbb{J}(\mathfrak{L})) \text{ is decidable}
\end{array}
$$

No further implications hold: By [Tho78], there exists a set $X \subseteq \omega$ such that the elementary theory of $(\omega, \leq, X)$ is decidable, but its monadic theory is not decidable. Let $(P, \leq)$ be the partial order obtained from $(\omega, \leq, X)$ by duplicating the elements of X. Then the elementary theory of this set can be reduced to the elementary theory of $(\omega, \leq, X)$ and is therefore decidable. Since the width of $(P, \leq)$ is 2, this implies also the decidability of its antichain theory. On the other hand, the monadic theory of $(\omega, \leq, X)$ can be reduced to the monadic theory of $(P, \leq)$. Hence $\mathrm{MTh}(P, \leq)$ is undecidable. Thus, we found an example of a class $\mathfrak{L} = \{(P, \leq)\}$ with decidable antichain and undecidable monadic theories. Next let $\mathfrak{L}$ be the set of all finite Boolean algebras. Then $\mathrm{Th}(\mathfrak{L})$ is decidable, but the width of the elements of $\mathfrak{L}$ is not bounded. Hence the antichain theory of $\mathfrak{L}$ is not decidable.

We will also show that the sets $\mathfrak{L}_n$ of finite distributive lattices of width at most n have a decidable monadic theory. Hence there is no maximal set of finite distributive lattices with a decidable monadic (chain or antichain, resp.) theory.

2 The elementary theory of finitely generated ideals

As a warm up, in this section we characterize those sets of partial orders $\mathfrak{P}$ whose associated set of semilattices $\mathbb{H}_f(\mathfrak{P})$ has a deciable elementary theory:

Theorem 2.1 *Let $\mathfrak{P}$ be a set of partially ordered sets. Then* $\mathrm{Th}(\mathbb{H}_f(\mathfrak{P}))$ *is decidable if and only of the weak monadic antichain theory* $\mathrm{wMATh}(\mathfrak{P})$ *is decidable.*

Proof. We first show that the elementary theory $\mathrm{Th}(\mathbb{H}_f(\mathfrak{P}))$ can be reduced to $\mathrm{wMATh}(\mathfrak{P})$: The idea of the proof is to identify the elements of $\mathbb{H}_f(P, \leq)$, i.e., the finitely generated ideals, with the finite antichain of their maximal elements. Then, quantification over the elements of $\mathbb{H}_f(P, \leq)$ is replaced by quantification over finite antichains in $(P, \leq)$. More formally, the reduction r distributes over the boolean connectives and is defined by

$$\begin{array}{rcl} r(\exists x\alpha) & = & (\exists X r(\alpha)), \text{ and} \\ r(x \leq y) & = & (\forall a \exists b(a \in X \to (b \in Y \wedge a \leq b))). \end{array}$$

Converselly, the weak monadic antichain theory $\mathrm{wMATh}(\mathfrak{P}))$ can be reduced to the elementary theory $\mathrm{Th}(\mathbb{H}_f(PP))$ as follows:

$$\begin{array}{rcl} r(\exists X\alpha) & = & (\exists \overline{x}\ r(\alpha)), \\ r(\exists x\alpha) & = & (\exists x(x \text{ is join irreducible } \wedge r(\alpha))), \\ r(x \in X) & = & (x \leq \overline{x} \wedge \forall y((y \text{ join irreducible } \wedge x \leq y \leq \overline{x}) \to x = y)), \\ r(x \leq y) & = & (x \leq y), \text{ and} \end{array}$$

r distributes over boolean connectives. The elements of $(P, \leq)$ correspond to the join irreducible elements of $\mathbb{H}_f(P, \leq)$, hence the above definition of $r(\exists x\alpha)$. Again, the idea is to replace a finite antichain X in $(P, \leq)$ by its supremum $\overline{x}$ in $\mathbb{H}_f(P, \leq)$. Therefore, an element x of P belongs to the antichain X iff it is a maximal join irreducible element below $\overline{x}$. $\square$

If the partial orders from $\mathfrak{P}$ all have width at most n, we can encode the weak monadic antichain theory of $\mathfrak{P}$ in its elementary theory since the antichains contain at most n elements. Hence we obtain as a simple consequence:

Corollary 2.2 *Let $\mathfrak{P}$ be a set of partial orders of width at most n. Then the elementary theory of $\mathbb{H}_f(\mathfrak{P})$ is decidable iff the elementary theory of $\mathfrak{P}$ is decidable.*

3 From $\mathrm{MTh}(\mathbb{H}_f(\mathfrak{P}))$ to $\mathrm{MTh}(\mathfrak{P})$

Our next aim is to show a similar result for the monadic theory $\mathrm{MTh}(\mathbb{H}_f(\mathfrak{P}))$. Note that the basic idea of the reduction of $\mathrm{Th}(\mathbb{H}_f(\mathfrak{P}))$ to $\mathrm{wMATh}(\mathfrak{P})$ is the replacement of an element of $\mathbb{H}_f(P,\leq)$ by a finite antichain in $(P,\leq)$. If we want to extend this to *sets* of elements in $\mathbb{H}_f(P,\leq)$, it would be natural to consider *sets of finite antichains* in $(P,\leq)$. But this is not possible in monadic second order logic. To make this possible at least in full second order logic, the size of antichains in $(P,\leq)$ should by bounded, i.e., the width of the elements of $\mathfrak{P}$ has to be bounded. Then, a set in $\mathbb{H}_f(P,\leq)$ could be represented by an n-ary relation in $(P,\leq)$. But in monadic second order logic, we cannot quantify over relations, but only over sets. Therefore, it does not suffice to restrict attention to sets $\mathfrak{P}$ of bounded width, but we have in addition to assume that the diabolo width of the elements of $\mathfrak{P}$ is bounded:

Definition 3.1 The partial order $(P,\leq)$ has *diabolo width at most m* if, for any $X,Y\subseteq P$ such that $X\times Y\subseteq\|$, we have $|X|\leq m$ or $|Y|\leq m$.

Figure 1 depicts this notion: Let $(P,\leq)$ be a partially ordered set of diabolo width at most m and let $X\subseteq P$ be a set with more than m elements. Then the set $Y:=P\setminus(X\uparrow\cup X\downarrow)$ is incomparable with X. Hence it contains at most m elements.[2]

Note that the width is at most double the diabolo width of a partially ordered set.

In this section, we will show that for any set $\mathfrak{P}$ of partial orders of bounded diabolo width, the monadic theory of $\mathbb{H}_f(\mathfrak{P})$ can be reduced to the monadic theory of $\mathfrak{P}$. In particular, we have to ensure that any n-ary relation in $\mathfrak{P}$ that contains only antichains can be encoded by a bounded number of sets.

We start with two technical lemmas.

Lemma 3.2 *Let $(C,\leq)$ be a linearly ordered set and let $k\in\mathbb{N}$. Then C splits into $2k$ mutually disjoint subsets $C(j)$ $(1\leq j\leq 2k)$ satisfying*

$(\star)$ *For any $1\leq j\leq 2k$ and for any $x,y\in C(j)$ with $x<y$, the interval $x\uparrow\cap y\downarrow\subseteq C$ contains at least k elements.*

Proof. Let α be an ordinal and let $C=\{x_\beta\mid\beta<\alpha\}$ be an enumeration of C. By transfinite induction, we construct the subsets $C(j)$ as follows: Let $\beta<\alpha$ and assume that we constructed a partition $(C^\beta(j))_{1\leq j\leq 2k}$ of $\{x_\gamma\mid\gamma<\beta\}$ satisfying $(\star)$. Consider the sets

$$\begin{aligned} M &= \{x\in C\mid x<x_\beta, |x\uparrow\cap x_\beta\downarrow|\leq k\}\text{ and}\\ N &= \{x\in C\mid x>x_\beta, |x_\beta\uparrow\cap x\downarrow|\leq k\}. \end{aligned}$$

[2]The name "diabolo width" was chosen since in this picture the set $X\uparrow\cup X\downarrow$ looks like a diabolo – a juggling prop that the author hopes to master eventually.

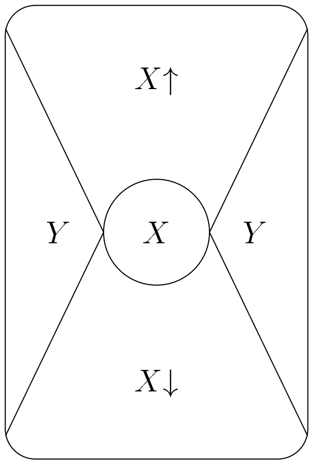

Figure 1: Diabolo width

Since they are linearly ordered, M and N both contain at most $k-1$ elements. Hence there is $1 \leq j \leq 2k$ with $C^{\beta}(j) \cap (M \cup N) = \emptyset$. Now define $C^{\beta+1}(j) := C^{\beta}(j) \cup \{x_{\beta}\}$ and $C^{\beta+1}(i) := C^{\beta}(i)$ for $i \neq j$. Then $(C^{\beta+1}(j))_{1 \leq j \leq 2k}$ is a partition of $\{x_{\gamma} \mid \gamma < \beta+1\}$ satisfying $(\star)$. For a limit ordinal β, we set $C^{\beta}(j) := \bigcup_{\gamma<\beta} C^{\gamma}(j)$ for $1 \leq j \leq 2k$. Now $C(j) := C^{\alpha}(j)$ finishes the construction. $\square$

Let $(P, \leq)$ be a partial order and let $C_1, C_2 \subseteq P$ be chains in $(P, \leq)$. We define an equivalence relation $\sim$ on C_2 by $x \sim y$ iff $C_1 \cap (x{\uparrow} \cup x{\downarrow}) = C_1 \cap (y{\uparrow} \cup y{\downarrow})$, i.e. iff x and y are comparable with the same elements of the first chain C_1.

Note that except $C_1 \cap C_2$ all equiavalence classes of $\sim$ are intervals in C_2. But since $C_1 \cap C_2$ is an equivalence class, in the following lemma we might have $x_i \sim x_j$ for some $i < j$.

Lemma 3.3 *Let $(P, \leq)$ be a partial order of diabolo width at most m. Let $C_1, C_2 \subseteq P$ be chains in $(P, \leq)$ and let $k = (2m+3)^2$. Let $x_i \in C_2$ with $x_i \not\sim x_{i+1}$ and $x_i < x_{i+1}$ for $1 \leq i < k$. Then $C_1 \subseteq x_1{\uparrow} \cup x_k{\downarrow}$, i.e. there is no element in C_1 that is incomparable with both x_1 and x_k.*

Proof. For $1 \leq i \leq k$, let $\Downarrow x_i = \{x \in C_1 \mid x < x_i\}$ and $\Uparrow x_i = \{x \in C_1 \mid x > x_i\}$. Then $\Downarrow x_i \cup \Uparrow x_i$ consists of those elements from $C_1 \setminus \{x_i\}$ that are comparable with x_i. Hence $\Downarrow x_i \subseteq \Downarrow x_{i+1}$ and $\Uparrow x_i \supseteq \Uparrow x_{i+1}$. Since $x_i \not\sim x_{i+1}$ and $x_i < x_{i+1}$, we have $\Downarrow x_i \subsetneq \Downarrow x_{i+1}$ or $\Uparrow x_i \supsetneq \Uparrow x_{i+1}$ for $1 \leq i < k$. First we show $\Downarrow x_i \subsetneq \Downarrow x_{i+2m+3}$

for $1 \le i \le k - 2m - 3$:

By contradiction, assume $1 \le i \le k-2m-3$ with $\Downarrow x_i = \Downarrow x_{i+2m+3}$. Then, for $0 \le j < m+1$, we obtain $\Uparrow x_{i+j} \supsetneq \Uparrow x_{i+j+1}$ and therefore $\Uparrow x_i \setminus \Uparrow x_{i+m+2}$ contains at least $m+2$ elements. Let $z \in \Uparrow x_i \setminus \Uparrow x_{i+m+2}$ and $2 \le j \le m+3$. Then $z \not\ge x_{i+m+j}$ since otherwise $x_{i+m+1} < x_{i+m+j} \le z$. Furthermore, $z \not< x_{i+m+j}$ since otherwise $z < x_{i+2m+3}$ and $z \not< x_i$, contradicting $\Downarrow x_i = \Downarrow x_{i+2m+3}$. Hence, $\Uparrow x_i \setminus \Uparrow x_{i+m+1}$ and $\{x_{i+m+2}, x_{i+m+3}, \ldots, x_{i+2m+3}\}$ are mutually incomparable sets of more than m elements each. But this contradicts our assumption on the diabolo width of $(P, \le)$. Thus, we proved $\Downarrow x_i \subsetneq \Downarrow x_{i+2m+3}$.

Now let $y_i := x_{(2m+3)i}$ for $1 \le i \le \ell := \frac{k}{2m+3} = 2m+3$, i.e. $y_1 < y_2 < \cdots < y_\ell$ is a subsequence of $x_1 < x_2 < \cdots < x_k$ such that $\Downarrow y_i \subsetneq \Downarrow y_{i+1}$ for $1 \le i < \ell$.

To prove the final goal $C_1 \subseteq \uparrow x_1 \cup \downarrow x_k$, assume by contradiction that x is an element of $C_1 \setminus (x_1 \uparrow \cup x_k \downarrow)$ and let $1 \le i \le m+1$ and $z \in Z := \Downarrow y_{2m+3} \setminus \Downarrow y_{m+2}$. Since z and x belong to the chain C_1, they are comparable. Then $z < x$ for otherwise $x \le z \le y_{2m+3} \le x_k$ would contradict our assumption on x. In case $z \ge y_i$ ($z \le y_i$), we had $x > z \ge y_i \ge x_1$ ($z \le y_i \le y_{m+2}$), contradicting our assumption on x (on z, respectively). Hence z and y_i are incomparable, i.e. the sets $\{y_1, y_2, \ldots, y_{m+1}\}$ and Z are incomparable. Since they both contain more than m elements, the diabolo width of $(P, \le)$ is larger than m, a contradiction.$\Box$

Let $(P, \le)$ be a partially ordered set. By a slight abuse of notation, we call an n-tuple $(x_1, x_2, \ldots, x_n) \in P^n$ an *antichain* if the set $\{x_1, x_2, \ldots, x_n\}$ is an antichain. Note that in particular the tuple $(a, a, \ldots, a)$ is an antichain for any $a \in P$. By AC, we denote the set of all *tuples* that are antichains.

Recall that our aim was to encode an n-ary relation whose elements form antichains by subsets of P. With the following lemma, we reach this aim for the restricted case where $(P, \le)$ has width at most 2.

For an element x of direct product $\prod_{j \in J} X_i$ and $I \subseteq J$, we write $\pi_I(x)$ for its projection to the components from I.

Lemma 3.4 *Let $(P, \le)$ be a partial order of diabolo width at most m. Let $C_1, C_2 \subseteq P$ be chains in $(P, \le)$. For any $M \subseteq C_1 \times C_2 \cap \mathrm{AC}$, there exist sets $M_{i,j} \subseteq C_i$ for $i = 1, 2$ and $0 \le j < 4m(2m+3)^2$ such that*

$$M = \left(\bigcup_{j=0}^{4m(2m+3)^2} M_{1,j} \times M_{2,j} \right) \cap \mathrm{AC}.$$

Proof. Similarly to the preceding lemma, let $k = (2m+3)^2$. First, we split the chain C_2 into the set C_2^s of those elements that belong to a small $\sim$-equivalence class and its complement, i.e.

$$\begin{aligned} C_2^s &= \{y \in C_2 : |[y]| \leq m\}, \text{ and} \\ C_2^l &= C_2 \setminus C_2^s = \{y \in C_2 : |[y]| > m\} \end{aligned}$$

where $[y]$ denotes the $\sim$-equivalence class containing y. Using Lemma 3.2, next we split C_1 and C_2^s into $2 \cdot k \cdot m$ disjoint subchains $C_1(j)$ and $C_2^s(j)$ for $1 \leq j \leq 2km$ such that, for any $x, y \in C_1(j)$ ($\in C_2^s(j)$, respectively) with $x < y$ the interval $\uparrow x \cap \downarrow y$ contains at least km elements from $C_1(j)$ (from $C_2^s(j)$, respectively). This ensures that between any two elements of $C_1(j)$, there are more than m elements of C_1. Similarly, we will use that between two elements of $C_2^s(j)$ there are at least $k \cdot m$ elements of C_2^s. To finish the construction, let

$$\begin{aligned} M^l(j) &= M \cap C_1(j) \times C_2^l, \text{ and} \\ M^s(j) &= M \cap C_1 \times C_2^s(j) \end{aligned}$$

for $1 \leq j \leq 2km$. We establish the lemma showing that M equals the set of antichains that occur in

$$H = \bigcup_{1 \leq j \leq 2km} \left[\pi_1(M^l(j)) \times \pi_2(M^l(j)) \cup \pi_1(M^s(j)) \times \pi_2(M^s(j))\right].$$

Let $(x_1, x_2) \in M$. In case $x_2 \in C_2^l$, there is $1 \leq j \leq 2km$ with $x_1 \in C_1(j)$. Hence $(x_1, x_2) \in M \cap C_1(j) \times C_2^l = M^l(j)$. Now $(x_1, x_2) \in \pi_1(M^l(j)) \times \pi_2(M^l(j)) \subseteq H$ follows immediately. In case $x_2 \in C_2^s$, we find $1 \leq j \leq 2km$ with $x_2 \in C_2^s(j)$. Now $(x_1, x_2) \in \pi_1(M^s(j)) \times \pi_2(M^s(j)) \subseteq H$ follows, i.e. we showed $M \subseteq H \cap \mathrm{AC}$.

Conversely, we have to show that antichains from $\pi_1(M^l(j)) \times \pi_2(M^l(j))$ or from $\pi_1(M^s(j)) \times \pi_2(M^s(j))$ belong to M for any $1 \leq j \leq 2km$. So let $1 \leq j \leq 2km$ and $(x_1, x_2), (y_1, y_2) \in M^l(j)$ with $x_1 \parallel y_2$. We want to show $x_1 = y_1$ implying $(x_1, y_2) = (y_1, y_2) \in M$. By contradiction assume $x_1 \neq y_1$. Since $x_1, y_1 \in C_1(j)$, they are comparable. We assume $x_1 < y_1$ (the case $y_1 < x_1$ is dual). As remarked earlier, there are more than m elements of C_1 between x_1 and y_1, in particular $|x_1 \uparrow \cap y_1 \downarrow| > m$. All elements of this interval are incomparable with y_2 since its endpoints x_1 and y_1 are. Thus we found sets $x_1 \uparrow \cap y_1 \downarrow$ and $[y_2]$ both larger than m whose elements are mutually incomparable. Since this contradicts the assumption on the diabolo width of $(P, \leq)$, we obtain $x_1 = y_1$ and therefore

$$\pi_1(M^l(j)) \times \pi_2(M^l(j)) \cap \mathrm{AC} \subseteq M \text{ for any } 1 \leq j \leq 2km.$$

Finally, let $1 \leq j \leq 2km$ and $(x_1, x_2), (y_1, y_2) \in M^s(j)$ with $x_1 \parallel y_2$. To show $x_2 = y_2$, we now assume by contradiction $x_2 < y_2$. Similarly to above, there are at least km elements of C_2^s in the interval $x_2 \uparrow \cap y_2 \downarrow$. Since $|[x]| \leq m$ for any $x \in C_2^s$, the chain $C_2^s \cap x_2\!\uparrow \cap y_2\!\downarrow$ contains k mutually not $\sim$-equivalent elements. Hence, by Lemma 3.3, $C_1 \setminus (x_2\!\uparrow \cap y_2\!\downarrow) = \emptyset$, contradicting $x_1 \in C_1$ and $x_2 \parallel x_1 \parallel y_2$. $\square$

Next, we extend the lemma above to relations of higher arity.

Theorem 3.5 *Let $(P, \leq)$ be a partial order of diabolo width at most m and let $n > 1$. Let $C_i \subseteq P$ be chains for $1 \leq i \leq n$ and let $M \subseteq \prod_{1\leq i\leq n} C_i \cap \mathrm{AC}$. Then M is the intersection of* AC *with $(2m+2)^n$ sets of the form*

$$\bigcap_{1\leq a<b\leq n} \bigcup_{\ell=1}^{4m(2m+3)^2} \left[P^{a-1} \times M_\ell^{a,b} \times P^{b-a-2} \times N_\ell^{a,b} \times P^{n-b-1}\right] \qquad (\star)$$

where $M_\ell^{a,b}, N_\ell^{a,b} \subseteq P$ for all suitable a, b and ℓ.

Proof. By Lemma 3.2, we split the chains C_i into $2m+2$ disjoint subchains $C_i(j)$ for $1 \leq j \leq 2m$ such that for any $x, y \in C_i(j)$ with $x < y$ the interval $\uparrow\! x \cap \downarrow\! y$ contains at least $m+1$ elements from P. For $\vec{\jmath} \in \{1, 2, \ldots, 2m+2\}^n$ let $M_{\vec{\jmath}} = M \cap \prod_{1\leq i\leq n} C_i(j_i)$. Then M is the union of the sets $M_{\vec{\jmath}}$. Since there are $(2m+2)^n$ sets $M_{\vec{\jmath}}$, it suffices to show that any such set is the intersection of AC with a set of the form $(\star)$. Let $1 \leq a < b \leq n$. Applying Lemma 3.4 to the set $\pi_{a,b}(M_{\vec{\jmath}})$ and the chains $C_a(j_a)$ and $C_b(j_b)$, we obtain the existence of sets $M_\ell^{a,b} \subseteq C_a(j_a)$ and $N_\ell^{a,b} \subseteq C_b(j_b)$ for $1 \leq \ell \leq 4m(2m+3)^2$ such that

$$\pi_{a,b}(M_{\vec{\jmath}}) = \bigcup_{\ell=1}^{4m(2m+3)^2} M_\ell^{a,b} \times N_\ell^{a,b} \cap \mathrm{AC}.$$

Let $H_{\vec{\jmath}}$ denote the set of all antichains in

$$\bigcap_{1\leq a<b\leq n} \bigcup_{\ell=1}^{4m(2m+3)^2} \left[P^{a-1} \times M_\ell^{a,b} \times P^{b-a-2} \times N_\ell^{a,b} \times P^{n-b-1}\right].$$

Note that $H_{\vec{\jmath}}$ equals the set of all antichains $(x_1, x_2, \ldots, x_n) \in P^n$ such that $(x_a, x_b) \in \pi_{a,b}(M_{\vec{\jmath}})$ for any $1 \leq a < b \leq n$. Thus, in particular, $H_{\vec{\jmath}}$ is a subset of $\prod_{1\leq i\leq n} C_i(j_i)$ since $\pi_{a,b}(M_{\vec{\jmath}}) \subseteq C_a(j_a) \times C_b(j_b)$. Since $H_{\vec{\jmath}}$ is of the form $(\star)$, it remains to show that $M_{\vec{\jmath}} = H_{\vec{\jmath}}$.

The inclusion $M_{\vec{\jmath}} \subseteq H_{\vec{\jmath}}$ is immediate for any element $(x_1, x_2, \ldots, x_n)$ of $M_{\vec{\jmath}}$ is an antichain satisfying $(x_a, x_b) \in \pi_{a,b}(M_{\vec{\jmath}})$ for all suitable a, b.

We show by induction on the size of $I \subseteq \{1, 2, \ldots, n\}$ that $\pi_I(H_{\vec{\jmath}}) \subseteq \pi_I(M_{\vec{\jmath}})$ which, for $I = \{1, 2, \ldots, n\}$ establishes the claim and therefore the theorem. If I contains precisely two elements, the inclusion $\pi_I(H_{\vec{\jmath}}) \subseteq \pi_I(M_{\vec{\jmath}})$ is immediate by what we said above. Now let I contain at least three elements and assume that $\pi_J(H_{\vec{\jmath}}) \subseteq \pi_J(M_{\vec{\jmath}})$ for any proper subset J of I. For notational simplicity, we assume $I = \{1, 2, \dots, c\}$ for some $3 \leq c \leq n$. Let $(x_1, x_2, \dots, x_c) \in \pi_I(H_{\vec{\jmath}})$. Then, by the induction hypothesis,

$$(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_c) \in \pi_{I\setminus\{i\}}(H_{\vec{\jmath}}) \subseteq \pi_{I\setminus\{i\}}(M_{\vec{\jmath}})$$

for any $1 \leq i \leq c$. Hence there are elements $x_i^i \in C_i(j_i)$ such that

$$(x_1, \dots, x_{i-1}, x_i^i, x_{i+1}, \dots, x_c) \in \pi_I(M_{\vec{\jmath}})$$

for any $i \in I$. If for some $1 \leq i \leq c$ we even have $x_i = x_i^i$, we thus get immediately $(x_1, \dots, x_c) \in \pi_I(M_{\vec{\jmath}})$. Now assume $x_i \neq x_i^i$ for all $1 \leq i \leq c$. Since $x_i, x_i^i \in C_i(j_i)$, they are comparable. Since I contains at least three elements, there are $1 \leq a < b \leq c$ with $x_a < x_a^a$ and $x_b < x_b^b$ or with $x_a > x_a^a$ and $x_b > x_b^b$. By symmetry, it suffices to deal with the first case. Recall that $x_a \nleq\ngeq x_b$ since $(x_1, \dots, x_c)$ (as an element of $\pi_I(H_{\vec{\jmath}})$) is an antichain. Similarly, $x_a^a \nleq\ngeq x_b$ and $x_b^b \nleq\ngeq x_a$ since $(x_1, \dots, x_{a-1}, x_a^a, x_{a+1}, \dots, x_c)$ and $(x_1, \dots, x_{b-1}, x_b^b, x_{b+1}, \dots, x_c)$ are antichains as elements of $\pi_I(M_{\vec{\jmath}})$. Thus x_a^a and x_b^b are incomparable for otherwise $x_a^a \leq x_b^b$ implied $x_a < x_b^b$. Since $\{x_a, x_a^a\}$ and $\{x_b, x_b^b\}$ are incomparable sets, so are the intervals $x_a\!\uparrow \cap x_a^a\!\downarrow$ and $x_b\!\uparrow \cap x_b^b\!\downarrow$. Recall that $x_a < x_a^a$ are both elements of $C_a(j_a)$. Thus, the interval $x_a\!\uparrow \cap x_a^a\!\downarrow$ consists of at least $m + 1$ elements, and similarly the interval $x_b\!\uparrow \cap x_b^b\!\downarrow$. This contradicts the assumption that $(P, \leq)$ has diabolo width at most m. Therefore, it is impossible that $x_i \neq x_i^i$ for all $1 \leq i \leq c$. This finishes the induction step, i.e. we have indeed $\pi_I(H_{\vec{\jmath}}) \subseteq \pi_I(M_{\vec{\jmath}})$. □

The following corollary shows that we can indeed represent any set of antichains in $(P, \leq)$ by some subsets of P. The number of subsets necessary is effectively bounded by the diabolo width of $(P, \leq)$.

Corollary 3.6 *Let $m \in \mathbb{N}$ and $n = 2m$. Then there exists a natural number ℓ and a monadic formula $\varphi(x_1, \dots, x_n, X_1, X_2, \dots, X_\ell)$ such that, for any partially ordered set $(P, \leq)$ of diabolo width at most m and any set R of antichains in P, there are sets $M_1, \dots, M_\ell \subseteq P$ with*

$$R = \{\{x_1, x_2, \dots, x_n\} \subseteq P \mid (P, \leq) \models \varphi(x_1, \dots, x_n, M_1, \dots M_\ell\}.$$

Proof. We explain the idea of the formula and leave the technicalities to the interested reader: We are concerned with partially ordered sets of width at most n, only. Hence any element of R contains at most n elements. By Dilworth' Theorem [Dil50], the partially ordered set $(P, \leq)$ can be covered by n disjoint chains $C_1, \dots, C_n$. For $I \subseteq \{1, 2, \dots, n\}$, let R_I denote the set of all antichains in R that meet a chain C_i iff $i \in I$. In particular, the set R_I contains sets of size $|I|$, only. Thus, we can identify it with an $|I|$-ary relation on P such that $R_I \subseteq \prod_{i \in I} C_i$. Now, applying Theorem 3.5, we easily construct a formula φ_I

with $|I|$ free elementary variables and $(2m+2)^{|I|}n(n-1)4m(2m+3)^2 \cdot 2$ free set variables such that there exist sets M_i with

$$R_I = \{(y_1, \ldots, y_{|I|}) \mid (P, \leq) \models \varphi_I(y_i, M_i)\}.$$

The formula φ is a simple Boolean combination of the formulas φ_I. $\square$

Theorem 3.7 *Let $\mathfrak{P}$ be a set of partially ordered sets and $m \in \mathbb{N}$ such that any $(P, \leq)$ in $\mathfrak{P}$ has diabolo width at most m. Then* $\mathrm{MTh}(\mathbb{H}_f(\mathfrak{P}))$ *can be reduced to* $\mathrm{MTh}(\mathfrak{P})$ *in linear time.*

Proof. Let φ and ℓ denote the formula and the natural number from Corollary 3.6 and let $n = 2m$. The reduction r distributes over the boolean connectives and is defined by

$$\begin{aligned} r(\exists x \alpha) &= (\exists x_1 \exists x_2 \ldots \exists x_n r(\alpha)), \\ r(x \leq y) &= (\bigwedge_{1 \leq i \leq n} \bigvee_{1 \leq j \leq n} x_i \leq y_j), \\ r(\exists M \alpha) &= (\exists M_1 \exists M_2 \ldots \exists M_\ell r(\alpha)), \text{ and} \\ r(x \in M) &= \varphi(x_1, x_2, \ldots, x_n, M_1, M_2, \ldots, M_\ell). \end{aligned}$$

We have to show $\mathbb{H}_f(P, \leq) \models \psi \iff (P, \leq) \models r(\psi)$ for any monadic sentence ψ: Let V be a countable set of individual variables and W that of set variables. We use these variables in monadic formulas that are interpreted over $\mathbb{H}_f(P, \leq)$. By $V' := V \times \{1, 2, \ldots, n\}$ and $W' := W \times \{1, 2, \ldots, \ell\}$ we denote the individual and set variables when speaking on the partial orders in $\mathfrak{P}$. For simplicity, we abbreviate (x, i) by x_i and similarly (A, j) by A_j for $x \in V$ and $A \in W$.

Let $(P, \leq)$ be a partially ordered set in $\mathfrak{P}$. With any $a \in \mathbb{H}_f(P, \leq)$, we associate an n-tuple $f'(a)$ in P with $\{f'(a)_1, f'(a)_2, \ldots, f'(a)_n\} = \max(a)$. Such an n-tuple exists since a is a finitely generated ideal in $(P, \leq)$ implying that it has at most n maximal elements. Furthermore, $a = \bigcup_{1 \leq i \leq n} f'(a)_i{\downarrow}$. Note that the coordinates of the tuple $f'(a)$ are incomparable or equal.

Similarly, we find a function g' that maps subsets of $\mathbb{H}_f(P, \leq)$ to ℓ-tuples of subsets of P as follows: Let $M \subseteq \mathbb{H}_f(P, \leq)$ be a set of finitely generated ideals in $(P, \leq)$. By R, we denote the set of all $(\leq n)$-subsets $\max(a)$ of P for some $a \in M$, i.e. $R = \{\max(a) \mid a \in M\}$. Then R is a set of antichains in the partially ordered set $(P, \leq)$ of diabolo width at most m. Hence, by Corollary 3.6 there exist sets $M_1, M_2, \ldots, M_\ell \subseteq P$ with

$$R = \{\{x_1, x_2, \ldots, x_n\} \subseteq P \mid (P, \leq) \models \varphi(x_1, \ldots, x_n, M_1, \ldots M_\ell\}.$$

For $1 \leq j \leq \ell$, let $g'(M)_j := M_j$. Then we obtain for any $M \subseteq \mathbb{H}_f(P, \leq)$ and any $a \in \mathbb{H}_f(P, \leq)$:

$$a \in M \text{ iff } (P, \leq) \models \varphi(f'(a), g'(M)).$$

Now let (f, g) be an interpretation of the elementary variables V and the set variables W in $\mathbb{H}_f(P, \leq)$, i.e. $f : V \to \mathbb{H}_f(P, \leq)$ and $g : W \to 2^{\mathbb{H}_f(P,\leq)}$. By $f^\star(x_i) := (f' \circ f(x))_i$ and $g^\star(M_j) := (g' \circ g(M))_j$ for $x_i \in V'$ and $M_j \in W'$, we define an interpretation $(f^\star, g^\star)$ of V' and W' in $(P, \leq)$ from (f, g).

To finish the proof, one shows by induction on the monadic formula ψ that $\mathbb{H}_f(P, \leq) \models_{(f,g)} \psi$ iff $(P, \leq) \models_{(f^\star, g^\star)} r(\psi)$. This is an easy exercise which is left to the reader. $\square$

Since undecidable theories cannot be reduced to decidable theories, we obtain as a direct consequence

Corollary 3.8 *Let $\mathfrak{P}$ be a set of partially ordered sets whose diabolo width is bounded such that the monadic theory* $\mathrm{MTh}(\mathfrak{P})$ *is decidable. Then the monadic theory* $\mathrm{MTh}(\mathbb{H}_f(\mathfrak{P}))$ *is decidable as well.*

Recall that the elements of $(P, \leq)$ correspond to the join irreducible elements of $\mathbb{H}_f(P, \leq)$. Hence it is easily seen that the decidability of $\mathrm{MTh}(\mathbb{H}_f(\mathfrak{P}))$ implies that of $\mathrm{MTh}(\mathfrak{P})$. To obtain the inverse implication of the corollary above, it remains to show that the monadic theory of $\mathbb{H}_f(\mathfrak{P})$ can be decidable only in case $\mathfrak{P}$ has bounded diabolo width.

4 Decidable monadic theory implies bounded diabolo width

By contraposition, we will actually show that the monadic theory of $\mathbb{H}_f(\mathfrak{P})$ is undecidable whenever the diabolo width of $\mathfrak{P}$ is unbounded. This will be achieved by reducing the monadic theory of the set of finite two-dimensional grids $\mathcal{G}$ to $\mathrm{MTh}(\mathfrak{P})$. We are even a bit more ambitious and will also consider the monadic antichain and monadic chain theory. In Figure 2, that gives an outline of this section, the arrows denote reductions.

We start with the reductions in the first row, i.e., we first consider the different monadic theories of the set of grids.

4.1 The set of grids

It will be convenient to use the notation $\underline{m}$ for the set $\{0, 1, \ldots, m-1\}$ of non-negative integers properly smaller than m. Let $\mathcal{G}$ denote the set of all finite grids,

$$
\begin{array}{ccccc}
\mathrm{MTh}(\mathcal{G}) & \xrightarrow{\text{Lemma 4.2}} & \mathrm{MATh}(\mathcal{G}) & \xrightarrow{\text{Lemma 4.3}} & \mathrm{MCTh}(\mathcal{G}) \\
 & & \downarrow \text{Lemma 4.6} & & \downarrow \text{Lemma 4.8} \\
 & & \mathrm{MATh}(\mathbb{H}_f(\mathfrak{P})) & & \mathrm{MCTh}(\mathbb{H}_f(\mathfrak{P}))
\end{array}
$$

Figure 2: outline

seen as distributive lattices, i.e. the set of all distributive lattices $(\underline{m}, \leq) \times (\underline{n}, \leq)$ for $m, n > 1$. Given a Turing machine M, one can effectively formulate a monadic sentence μ which is satisfied by the partially ordered set $L = (\underline{m} \times \underline{n}, \leq)$ iff the machine stops after m steps using n cells of the tape. Hence $\mu \in \mathrm{MTh}(\mathcal{G})$ iff M does not stop, i.e. the monadic theory of $\mathcal{G}$ is undecidable. For this encoding, one has to quantify over arbitrary subsets of the grid. Therefore, the undecidability of the monadic chain and antichain theory of $\mathcal{G}$ is not immediate.

The *grid graph of dimension* (m, n) is the Hasse diagram $(\underline{m} \times \underline{n}, E_{m,n})$ of the grid $(\underline{m} \times \underline{n}, \leq)$. Hence $((i, j), (i', j')) \in E_{m,n}$ iff $i = i'$ and $j + 1 = j'$ or $i + 1 = i$ and $j = j'$ for any $(i, j), (i', j') \in \underline{m} \times \underline{n}$. Let $\mathcal{GG}$ denote the set of all structures isomorphic to some grid graph. We first show how a grid graph can be encoded by antichains in a large grid:

Lemma 4.1 *Let $m, n \in \mathbb{N}$. Then there exist antichains X, E_1, E_2 and an element e in the grid $L = (\underline{(mn + 1)}^2, \leq)$ such that*

$$(X, \{(x, y) \in X^2 \mid \sup(x, y) \in E_1 \cup E_2 \wedge \inf(x, e) \leq \inf(y, e)\} \cong (\underline{m} \times \underline{n}, E_{m,n}).$$

Proof. First, we define the following sets:
$X := \{(i, mn - i) \mid 0 \leq i < mn\}$,
$E_1 := \{(i, mn - i - 1) \mid 0 \leq i < mn, i \mod n \neq n - 1\}$, and
$E_2 := \{(i, (m - 1)n - i) \mid 0 \leq i < (m - 1)n\}$. Note that these sets are antichains for increasing i increases the first and decreases the second component of any of their elements. Finally, let $e := (mn, 1) \in \mathbb{J}(L)$. For notational simplicity, let E denote the binary relation

$$\{(x, y) \in X^2 \mid \sup(x, y) \in E_1 \cup E_2 \wedge \inf(x, e) \leq \inf(y, e)\}.$$

We show that (X, E) and $(\underline{m} \times \underline{n}, E_{m,n})$ are isomorphic: Define the bijection $f : \underline{m} \times \underline{n} \to X$ by $f(a_1, a_2) := (a_1 n + a_2, mn - a_1 n - a_2)$. The following sequence of equivalences establishes that f is a graph isomorphism:

$$((a_1, a_2), (b_1, b_2)) \in E_{m,n}$$

$$\begin{aligned}
\iff\ & a_1 + 1 = b_1 \text{ and } a_2 = b_2, \text{ or} \\
& a_1 = b_1 \text{ and } a_2 + 1 = b_2 \\[1ex]
\iff\ & a_1 \leq b_1 \text{ and either} \\
& (a_1 n + a_2, mn - b_1 n - b_2) \in E_1, \text{ or} \\
& (a_1 n + a_2, mn - b_1 n - b_2) \in E_2 \\[1ex]
\iff\ & \inf(f(a_1, a_2), e) = a_1 \leq b_1 = \inf(f(b_1, b_2), e) \text{ and} \\
& \sup(f(a_1, a_2), f(b_1, b_2)) \in E_1 \cup E_2 \\[1ex]
\iff\ & (f(a_1, a_2), f(b_1, b_2)) \in E.
\end{aligned}$$

Hence $(X, E) \cong (\underline{m} \times \underline{n}, E_{m,n})$. □

Lemma 4.2 *The monadic theory of $\mathcal{G}$ can be reduced to the monadic antichain theory of $\mathcal{G}$ in linear time.*

Proof. Since the grid graph of dimension (m, n) is the Hasse diagram of the grid $(\underline{m} \times \underline{n}, \leq)$, it suffices to reduce the monadic theory of the set of all grid graphs to the monadic antichain theory of $\mathcal{G}$.

For a monadic sentence φ over the binary relation symbol E, we construct a monadic formula over the vocabulary $\leq$ as follows: First, we restrict the quantification in φ to the new set variable X. Afterwards, any subformula of the form $(x, y) \in E$ is replaced by

$$\sup(x, y) \in E_1 \cup E_2 \wedge \inf(x, e) \leq \inf(y, e).$$

Let φ' denote the result of this procedure. Then φ' is a monadic formula with free variables contained in $\{X, E_1, E_2, e\}$.

Now we describe the reduction of $\mathrm{MTh}(\mathcal{GG})$ to $\mathrm{MATh}(\mathcal{G})$: It is easily seen that there is a monadic sentence γ such that a graph (X, E) satisfies γ iff it belongs to $\mathcal{GG}$. Let φ be a monadic sentence over the binary relation symbol E. Then define

$$\overline{\varphi} := \forall X \forall E_1 \forall E_2 \forall e((e \in \mathbb{J} \wedge \gamma') \rightarrow \varphi').$$

We show that φ belongs to $\mathrm{MTh}(\mathcal{GG})$ iff $\overline{\varphi}$ belongs to $\mathrm{MATh}(\mathcal{G})$:

First let $\varphi \in \mathrm{MTh}(\mathcal{GG})$. Furthermore, let $m, n > 1$ and $L := (\underline{m} \times \underline{n}, \leq)$. Let $X, E_1, E_2 \subseteq L$ be antichains and $e \in \mathbb{J}(L)$ with $(L, \leq) \models_A \gamma'(X, E_1, E_2, e)$. We have to show that $(L, \leq) \models \varphi'(X, E_1, E_2, e)$. First, define a binary relation $E \subseteq X^2$ by $(x, y) \in E$ iff $\sup(x, y) \in E_1 \cup E_2$ and $\inf(x, e) \leq \inf(y, e)$. By the construction of γ' and the fact that $(L, \leq) \models_A \gamma'(X, E_1, E_2, e)$, the graph (X, E) satisfies γ, i.e. it is isomorphic to a grid graph. Hence $(X, E) \models \varphi$ implying $(L, \leq) \models_A \varphi'(X, E_1, E_2, e)$ as required.

Conversely, let $\overline{\varphi} \in \mathrm{MATh}(\mathcal{G})$ and let $m, n \in \mathbb{N}$. To show $(\underline{m} \times \underline{n}, E_{n,m}) \models \varphi$, we consider the grid $L = (\underline{(mn+1)^2}, \leq)$. By Lemma 4.1, there exist antichains X, E_1, E_2 and a join irreducible element e such that (X, E) is isomorphic to $(\underline{m} \times \underline{n}, E_{n,m})$ (where E is defined as above). Hence $(X, E) \models \gamma$. By the construction of E and of γ', this implies $L \models_A \gamma'(X, E_1, E_2, e)$ and therefore $L \models_A \varphi'(X, E_1, E_2, e)$. By the same argument, $(X, E) \models \varphi$, i.e. $\varphi \in \mathrm{MTh}(\mathcal{GG})$.
$\square$

Next, we reduce the monadic antichain theory of $\mathcal{G}$ to the monadic chain theory of $\mathcal{G}$. The basic idea is depicted in Figure 3: It shows the set $\underline{3} \times \underline{4}$. On this set, we consider two orders: first the natural order $\leq$, i.e., the componentwise order. The second order $\sqsubseteq$ in consideration is obtained by inverting the order on $\underline{4}$ and then considering the componentwise order. Then elements incomparable w.r.t. $\sqsubseteq$ are comparable w.r.t. $\leq$ which is the basis for our transformation of antichains into chains. Unfortunately, the inverse does not hold (consider, e.g., $(0,0)$ and $(1,0)$). Hence, we do not have $(\underline{m} \times \underline{n}, \leq) \models_C \varphi$ iff $(\underline{m} \times \underline{n}, \sqsubseteq) \models_A \varphi$. Therefore, we have to invest slightly more work to reduce the monadic antichain theory to the monadic chain theory.

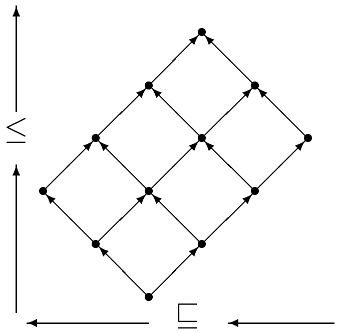

Figure 3: The set $\underline{3} \times \underline{4}$

More formally, our reduction rests on the following definition: Let $n, m > 1$ and $L := (\underline{m} \times \underline{n}, \leq)$. Then $e := (1, m)$ and $\overline{e} := (n, 1)$ are maximal join irreducible and incomparable elements of L. We define a partial order $\sqsubseteq$ on L by $x \sqsubseteq y$ iff $\inf(x, e) \leq \inf(y, e)$ and $\inf(x, \overline{e}) \geq \inf(y, \overline{e})$. For $x = (x_1, x_2)$ and $y = (y_1, y_2)$, it holds $\inf(x, e) = x_2$ and $\inf(x, \overline{e}) = x_1$. Hence $x \sqsubseteq y$ iff $x_2 \leq y_2$ and $x_1 \geq y_1$. In other words, $(\underline{m} \times \underline{n}, \sqsubseteq)$ equals $(\underline{m}, \geq) \times (\underline{n}, \leq)$ and is therefore isomorphic to L. Now let x and y be incomparable with respect to $\sqsubseteq$. Then $x_1 < y_1$ and $x_2 < y_2$ or vice versa. In particular $x \leq y$ or $y \leq x$. Hence antichains in $(\underline{m} \times \underline{n}, \sqsubseteq)$ are chains in $(\underline{m} \times \underline{n}, \leq)$ (the converse implication does not hold).

Lemma 4.3 *The monadic antichain theory* $\mathrm{MATh}(\mathcal{G})$ *can be reduced in linear time to the monadic chain theory* $\mathrm{MCTh}(\mathcal{G})$.

Proof. Let φ be a monadic formula not containing the variables e and $\overline{e}$. In φ, replace any atomic formula $x \leq y$ by

$$\inf(x,e) \leq \inf(y,e) \wedge \inf(x,\overline{e}) \geq \inf(y,\overline{e})$$

and replace any subformula of the form $\exists X\psi$ by $\exists X(\mathrm{antichain}_{\sqsubseteq}(X) \wedge \psi)$ where $\mathrm{antichain}_{\sqsubseteq}(X)$ denotes the formula

$$\forall x, y((x, y \in X \wedge \inf(x,e) \leq \inf(y,e)) \rightarrow \inf(x,\overline{e}) \leq \inf(y,\overline{e})).$$

The subformula $\mathrm{antichain}_{\sqsubseteq}(X)$ is satisfied by a set X iff its elements are mutually incomparable with respect to $\sqsubseteq$. Denote the result of these replacements by φ'.

Let $m, n \geq 1$, $e = (1, m)$ and $\overline{e} = (n, 1)$. As a prerequisite, we show that $(\underline{m}\times\underline{n}, \sqsubseteq) \models_A \varphi$ iff $(\underline{m}\times\underline{n}, \leq) \models_C \varphi'$ by induction: Clearly, $(\underline{m}\times\underline{n}, \sqsubseteq) \models_A x \leq y$ iff $x \sqsubseteq y$ iff $(\underline{m}\times\underline{n}, \leq) \models_C \inf(x,e) \leq \inf(y,e) \wedge \inf(x,\overline{e}) \geq \inf(y,\overline{e})$ which equals $(x \leq y)'$. Now let φ_i $(i = 1, 2)$ be monadic formulas such that for any antichains X_i w.r.t. $\sqsubseteq$ and elements x_j:

$$\begin{gathered}(\underline{m}\times\underline{n}, \sqsubseteq) \models_A \varphi_i(X_1, \ldots, X_k, x_1, \ldots, x_\ell)\\ \Longleftrightarrow\\ (\underline{m}\times\underline{n}, \leq) \models_C \varphi_i'(X_1, \ldots, X_k, x_1, \ldots, x_\ell).\end{gathered}$$

It is straightforward to check that this equivalence holds for $\neg\varphi_1$, $\varphi_1 \wedge \varphi_2$ and for $\exists x_\ell \varphi_1$, too. The only nontrivial case in the induction is the formula $\varphi = \exists X_k \varphi_1$: So let $X_i \subseteq \underline{m}\times\underline{n}$ be antichains w.r.t. $\sqsubseteq$ for $1 \leq i < k$ and let $x_j \in \underline{m}\times\underline{n}$. Then $(\underline{m}\times\underline{n}, \sqsubseteq) \models_A \varphi(X_1, \ldots, X_{k-1}, x_1, \ldots, x_\ell)$ iff there exists an antichain X_k w.r.t. $\sqsubseteq$ such that $(\underline{m}\times\underline{n}, \sqsubseteq) \models_A \varphi_1(X_1, \ldots, X_k, x_1, \ldots, x_\ell)$. By the induction hypothesis, this is equivalent to $(\underline{m}\times\underline{n}, \leq) \models_C \varphi_1'(X_1, \ldots, X_k, x_1, \ldots, x_\ell)$. By the remarks preceding this lemma, X_k is a chain w.r.t. $\leq$. Hence the last statement is equivalent to $(\underline{m}\times\underline{n}, \leq) \models_C (\exists X_k(\mathrm{antichain}_{\sqsubseteq}(X_k) \wedge \varphi_1'))(X_1, \ldots, X_{k-1}, x_1, \ldots, x_\ell)$ which equals φ'.

Now it is straightforward to show that for a monadic sentence φ it holds $(\underline{m}\times\underline{n}, \leq) \models_A \varphi$ iff $(\underline{m}\times\underline{n}, \leq) \models_C \exists e, \overline{e}(e, \overline{e} \in \max(\mathbb{J}) \wedge e \neq \overline{e} \wedge \varphi')$ (where $\mathbb{J}$ denotes the set of join irreducible elements) which is the desired reduction. $\square$

So far, we showed that the monadic theory of $\mathcal{G}$ can be reduced to its monadic chain and antichain theory. Hence these two theories are undeciable.

4.2 Sets of unbounded diabolo width

Next, we show the reductions from $\mathrm{MATh}(\mathcal{G})$ and $\mathrm{MCTh}(\mathcal{G})$ to $\mathrm{MATh}(\mathbb{H}_f(\mathfrak{P}))$ and $\mathrm{MCTh}(\mathbb{H}_f(\mathfrak{P}))$, resp., for $\mathfrak{P}$ a set of partial orders of unbounded diabolo width. Before considering these reductions, we show that $\mathbb{H}_f(P, \leq)$ contains a large grid whenever the diabolo width of $(P, \leq)$ is large:

Lemma 4.4 *Let $(P, \leq)$ be a partially ordered set of diabolo width at least m. Then there exists a semilattice embedding of $\underline{m} \times \underline{m}$ into $(\mathbb{H}_f(P), \subseteq)$. If $(P, \leq)$ is finite, this embedding can be chosen to preserve infima, too.*

Proof. There exist $X, Y \subseteq P$ sets with m elements each such that any $x \in X$ is incomparable with any $y \in Y$. Let $X = \{x_0, x_1 \ldots, x_{m-1}\}$ and $Y = \{y_0, y_1 \ldots, y_{m-1}\}$ be linear extensions of $(X, \leq)$ and $(Y, \leq)$, i.e. $x_i \leq x_j$ or $y_i \leq y_j$ implies $i \leq j$. Then $I(i,j) := \downarrow\{x_1, \ldots, x_i, y_1, \ldots, y_j\}$ is a finitely generated ideal and therefore an element of $\mathbb{H}_f(P, \leq)$. Furthermore, $(\{I(i,j) \mid i \in \underline{m-1}, j \in \underline{m-1}\}, \subseteq)$ is the desired subposet of $\mathbb{H}_f(P, \leq)$. $\square$

Next we show that large grids do not only exist, but can also be defined in monadic antichain logic:

Lemma 4.5 *There exists a monadic formula $\gamma(A, b, x)$ such that for any partially ordered set $(P, \leq)$ the following hold:*

1. *for any antichain $A \subseteq \mathbb{H}_f(P, \leq)$ and any $b \in A$, the set $\{x \in \mathbb{H}_f(P, \leq) \mid \mathbb{H}_f(P, \leq) \models_A \gamma(A, b, x)\}$ is a grid, and*
2. *for any $n \in \mathbb{N}$ and any semilattice-embedding $f : (\underline{2n+1})^2 \to \mathbb{H}_f(P, \leq)$, there exist an antichain $A \subseteq \mathbb{H}_f(P, \leq)$ and an element $b \in A$ such that $\{x \in \mathbb{H}_f(P, \leq) \mid \mathbb{H}_f(P, \leq) \models_A \gamma(A, b, x)\} = \{f(n+i, n+j) \mid i, j \in \underline{n}\}$.*

Proof. Let $(P, \leq)$ be a partially ordered set, $A \subseteq P$ an antichain and $a \in A$. On A, we define a binary relation R_a by $(x, y) \in R_a$ iff $x \vee a \leq y \vee a$. There is a monadic formula φ with free variables A and a such that $(P, \leq) \models_A \varphi(A, a)$ iff

(A) (A, R_a) is a finite linear order, and

(B) for any $x, y, x', y' \in A$ with $(x, y), (x', y') \in R_a$ we have

$$x \vee y \leq x' \vee y' \iff (x, x'), (y', y) \in R_a.$$

Now let γ be the formula

$$\gamma(A, b, x) = b = x \vee \exists a \in A(\varphi(A, a) \wedge \exists a_1, a_2 \in A(b \leq x = \sup(a_1, a_2)).$$

We show the first statement: Let $G = \{x \in \mathbb{H}_f(P, \leq) \mid \mathbb{H}_f(P, \leq) \models_A \gamma(A, b, x)\}$. If there is no $a \in A$ satisfying φ, $G = \{b\}$ which is clearly a grid. So assume $a \in A$ satisfies φ. Let $\{a_0, a_1, \ldots, a_k\}$ be the enumeration of A with $(a_i, a_{i+1}) \in R_a$. Furthermore, let $b = a_m$. Now define $f(i,j) = \sup(a_{m-1}, a_{m+j})$ for $(i,j) \in \underline{m+1} \times \underline{k-m+1}$. We show that this function an order isomorphism from $(\underline{m+1} \times \underline{k-m+1}, \leq)$ onto G: Let $(i,j), (i',j') \in \underline{m+1} \times \underline{k-m+1}$. Then $(i,j) \leq (i',j')$ iff $a_{m-i'} R_a a_{m-i}$ and $a_{m+j} R_a a_{m+j'}$. Since $m - i \leq m + j$, we have in addition $a_{m-i} R_a a_{m+j}$. Hence we can apply (B) and obtain that the last statement is equivalent to $\sup(a_{m-i}, a_{m+j}) \leq \sup(a_{m-i'}, a_{m+j'})$, i.e., to $f(i,j) \leq f(i',j')$. Thus, f is an order embedding into $\mathbb{H}_f(P, \leq)$. Next, we show that its image is G: For $(i,j) \in \underline{m+1} \times \underline{k-m+1}$, we have $a_{m-i} R_a b = a_m R_a a_{m+j}$ implying (by (B)) $b \leq \sup(a_{m-i}, a_{m+j}) = f(i,j)$ which therefore belongs to G. Conversselly, let $x = \sup(a_s, a_t) \in G$. Then (B) implies $s \leq m \leq t$ ensuring $x = f(m-s, t-m)$. Thus, f is indeed an order isomorphism onto G, i.e., G is a grid.

Next, we prove the second statement: Let $A = \{f(i, 2n-i) \mid i \in \underline{2n}\}$, $a = f(0, 2n)$, and $b = f(n,n)$. Then $f(i, 2n) = \sup(a, f(i, 2n-i)) \leq \sup(a, f(j, 2n-j)) = f(j, 2n)$ iff $i \leq j$. Hence the finite antichain $(A, \leq)$ is linearly ordered by the relation R_a, i.e., (A) holds. Next, let $i, j, i', j' \in \underline{2n}$ with $i \leq j$ and $i' \leq j'$. Then $\sup(f(i, 2n-i), f(j, 2n-j)) \leq \sup(f(i', 2n-i'), f(j', 2n-j'))$ iff $\max(i,j) \leq \max(i',j')$ and $\min(i,j) \geq \min(i',j')$. Since $i \leq j$ and $i' \leq j'$, this is equivalent to $i' \leq i$ and $j \leq j'$, i.e., we showed (B). Since $\varphi(A, a)$ is satisfied, $\sup(a_1, a_2) = x$ satisfies $\gamma(A, b, x)$ iff $x \geq b$ for $a_1, a_2 \in A$. Since $b = f(n,n)$, this is equivalent to $x \in \{f(n+i, n+j) \mid i, j \in \underline{n}\}$. $\square$

This lemma enables us to show that indeed the monadic antichain theory of the grids $\mathcal{G}$ can be reduced to the monadic antichain theory of $\mathbb{H}_f(\mathfrak{P})$ whenever $\mathfrak{P}$ has unbounded diabolo width.

Lemma 4.6 *Let $\mathfrak{P}$ be a set of partially ordered sets such that the diabolo width of its members is not bounded above. Then the monadic antichain theory* $\mathrm{MATh}(\mathcal{G})$ *can be reduced to the monadic antichain theory* $\mathrm{MATh}(\mathbb{H}_f(\mathfrak{P}))$.

Proof. For a monadic sentence ψ, let $r(\psi)$ denote the sentence $\forall A \forall b (b \in A \to \psi')$ where ψ' is the restriction of ψ to those elements x that satatisfy $\gamma(A, b, x)$ (cf. Lemma 4.5). By the preceding lemma, these elements form a grid, i.e., $\psi \in \mathrm{MATh}(\mathcal{G})$ implies $r(\psi) \in \mathrm{MATh}(\mathbb{H}_f(\mathfrak{P}))$. Conversselly, let $r(\psi) \in \mathrm{MATh}(\mathbb{H}_f(\mathfrak{P}))$ and let $n \in \mathbb{N}$. Then there exists $(P, \leq) \in \mathfrak{P}$ such that the grid $(\underline{2n}^2, \leq)$ can be embedded into $\mathbb{H}_f(P, \leq)$. Hence $\mathbb{H}_f(P, \leq)$ contains an antichain A and an element b such that $\{x \in \mathbb{H}_f(P, \leq) \mid \mathbb{H}_f(P, \leq) \models \gamma(A, b, x)\}$ is isomorphic to $(\underline{n}^2, \leq)$. Since $r(\psi) \in \mathrm{MATh}(\mathbb{H}_f(\mathfrak{P}))$, we get $\mathbb{H}_f(P, \leq) \models_A \psi'$ and therefore $\psi \in \mathrm{MATh}(\mathcal{G})$. $\square$

To show that also the monadic chain theory of the set of grids can be reduced to the monadic chain theory of $\mathbb{H}_f(\mathfrak{P})$, we proceed similarly to above: First, it is shown that large grids can be defined in the monadic chain logic, and then we prove that this yields the desired reduction:

Lemma 4.7 *There exists a monadic formula $\gamma(C_1, C_2)$ such that for any partially ordered set $(P, \leq)$ the following hold:*

1. *for any chains $C_1, C_2 \subseteq \mathbb{H}_f(P, \leq)$ such that $\mathbb{H}_f(P, \leq) \models_C \gamma(C_1, C_2)$, the set $\{\sup(x, y) \mid x \in C_1, y \in C_2\}$ is a grid, and*
2. *for any grid $G \subseteq \mathbb{H}_f(P, le)$, there exist chains $C_1, C_2 \in \mathbb{H}_f(P, \leq)$ such that $\mathbb{H}_f(P, \leq) \models_C \gamma(C_1, C_2)$ and the set $\{\sup(x, y) \mid x \in C_1, y \in C_2\}$ is isomorphic to G.*

Proof. The formula γ asserts that C_1 and C_2 are finite chains, $\min(C_1) = \min(C_2)$, the elements from $C_1 \setminus \{\min(C_1)\}$ are incomparable with the elements from $C_2 \setminus \{\min(C_2)\}$, and $\sup(x_1, x_2) \leq \sup(y_1, y_2)$ iff $x_1 \leq y_1$ and $x_2 \leq y_2$ for any $x_i, y_i \in C_i$.

Let $f : (\underline{m} \times \underline{n}, \leq) \to \mathbb{H}_f(P, \leq)$ be a semilattice embedding. Then define $C_1 = \{f(i, 0) \mid 0 \leq i < m\}$ and $C_2 = \{f(0, i) \mid) \leq i < n\}$. Then C_1 and C_2 are finite chains, $\min(C_1) = \min(C_2)$, and the elements from $C_1 \setminus \{\min(C_1)\}$ are incomparable with the elements from $C_2 \setminus \{\min(C_2)\}$. To see that $\sup(x_1, x_2) \leq \sup(y_1, y_2)$ iff $x_1 \leq y_1$ and $x_2 \leq y_2$, note that $\sup(f(i_1, 0), f(0, i_2)) = f(i_1, i_2)$. Hence $\sup(f(i_1, 0), f(0, i_2)) \leq \sup(f(j_1, 0), f(0, j_2))$ iff $i_1 \leq j_1$ and $i_2 \leq j_2$, i.e., iff $f(i_1, 0) \leq f(j_1, 0)$ and $f(0, i_2) \leq f(0, j_2)$. Hence the chains C_1 and C_2 satisfy γ. Furthermore, the set $\{\sup(x, y) \mid x \in C_1, y \in C_2\}$ is the image of f and therefore isomorphic to the grid of dimension (m, n). This proves the second statement.

To prove the first statement, let $C_1 = \{f(0, 0), f(1, 0), \ldots, f(m, 0)\})$ and $C_2 = \{f(0, 0), f(0, 1) \ldots, f(0, n)\}$ be finite chains satisfying γ. We can assume $f(0, i) \leq f(0, i + 1)$ and $f(j, 0) \leq f(j + 1, 0)$ for all suitable i and j. We show that $f : \underline{m} \times \underline{n} \to \mathbb{H}_f(P, \leq)$ defined by $f(i, j) = \sup(f(i, 0), f(0, j))$ is a semilattice embedding; for this it actually suffices to show that it is an order embedding: Note that $f(i, j) \leq f(i', j')$ iff $f(i, 0) \leq f(i', 0)$ and $f(0, j) \leq f(0, j')$ by the last requirement expressed by γ. Hence, indeed, f is an order embedding. $\square$

Lemma 4.8 *Let $\mathfrak{P}$ be a set of partially ordered sets such that the diabolo width of its members is not bounded above. Then the monadic chain theory* $\mathrm{MCTh}(\mathcal{G})$ *can be reduced to the monadic chain theory* $\mathrm{MCTh}(\mathbb{H}_f(\mathfrak{P}))$.

Proof. The reduction is given by $r(\psi) = \forall C_1, C_2(\gamma(C_1, C_2) \to \psi')$ where ψ' is the reduction of ψ to the set of suprema of elements of C_1 and C_2. The proof now proceeds similarly to the proof of Lemma 4.6. $\square$

Recall that Theorem 2.1 characterized those classes of partial orders $\mathfrak{P}$ for which $\mathbb{H}_f(\mathfrak{P})$ has a decidable elementary theory. Now we can extend this to the monadic, the monadic chain, and the monadic antichain theory:

Theorem 4.9 *Let $\mathfrak{P}$ be a set of partially ordered sets.*

1. *The following are equivalent:*
   - *(i) The monadic theory* $\mathrm{MTh}(\mathbb{H}_f(\mathfrak{P}))$ *is decidable.*
   - *(ii) The monadic chain theory* $\mathrm{MCTh}(\mathbb{H}_f(\mathfrak{P}))$ *is decidable.*
   - *(iii) the monadic theory* $\mathrm{MTh}(\mathfrak{P})$ *is decidable and the diabolo width of the elements of* $\mathfrak{P}$ *is bounded above.*
   - *(iv) the monadic chain theory* $\mathrm{MCTh}(\mathfrak{P})$ *is decidable and the diabolo width of the elements of* $\mathfrak{P}$ *is bounded above.*
2. *The monadic antichain theory* $\mathrm{MATh}(\mathbb{H}_f(\mathfrak{P}))$ *is decidable if and only if the elementary theory* $\mathrm{Th}(\mathfrak{P})$ *is decidable and the diabolo width of the elements of* $\mathfrak{P}$ *is bounded above.*

Proof. The implication (i)$\Rightarrow$(ii) is trivial. Now assume $\mathrm{MCTh}(\mathbb{H}_f(\mathfrak{P}))$ to be decidable. Next we show the implication (ii)$\Rightarrow$(iv): The monadic chain theory $\mathrm{MCTh}(\mathfrak{P})$ is decidable since it can be interpreted in the monadic chain theory of $\mathbb{H}_f(\mathfrak{P})$. By contradiction, assume that the diabolo width of the elements of $\mathfrak{P}$ is unbounded. By Lemmas 4.2, 4.3, and 4.8 (cf. Figure 2), the monadic theory of the grids $\mathrm{MTh}(\mathcal{G})$ can be reduced to the monadic chain theory of $\mathbb{H}_f(\mathfrak{P})$, contradicting the decidability of this latter theory.

For the implication $(iv) \Rightarrow (iii)$ note that the width of the elements of $\mathfrak{P}$ is bounded by n, say. Hence any subset of P with $(P, \leq) \in \mathfrak{P}$ is the union n chains. Therefore, the monadic theory of $\mathfrak{P}$ can be reduced to the monadic chain theory of $\mathfrak{P}$. Hence (iii) holds. The last implication $(iii) \Rightarrow (i)$ follows from Theorem 3.7.

It remains to show the second statement: Suppose $\mathrm{MATh}(\mathbb{H}_f(\mathfrak{P}))$ is decidable. Then, trivially, $\mathrm{Th}(\mathbb{H}_f(\mathfrak{P}))$ and therefore $\mathrm{Th}(\mathfrak{P})$ are decidable. The diabolo with of the elements of $\mathfrak{P}$ is bounded above by Lemmas 4.2 and 4.6. Thus, we showed one implication.

Conversely, let the diabolo width of the elements of $\mathfrak{P}$ be bounded by n and let $\mathrm{Th}(\mathfrak{P})$ be decidable. Since then the width of the elements of $\mathfrak{P}$ is bounded by $2n$, the elementary theory $\mathrm{Th}(\mathbb{H}_f(\mathfrak{P}))$ is decidable by Corollary 2.2. By Lemma

4.4, the width of the elements of $\mathbb{H}_f(\mathfrak{P})$ is bounded above by some $m \in \mathbb{N}$. Hence any antichain contains at most m elements implying that $\mathrm{MATh}(\mathbb{H}_f(\mathfrak{P}))$ can be reduced to $\mathrm{Th}(\mathbb{H}_f(\mathfrak{P}))$. Hence the monadic antichain theory $\mathrm{MATh}(\mathbb{H}_f(\mathfrak{P}))$ is decidable. $\square$

5 Finite distributive lattices

In this section, we will characterize those classes of finite distributive lattices whose elementary, monadic, monadic chain, and monadic antichain theory are decidable. Before we can actually prove this characterization, we need Ramsey's theorem and Lemma 5.1.

Ramsey's Theorem [Ram30] *Let c, r be positive integers. Then there is a positive integer $R_r(c)$ such that for any mapping d of the two-elements subsets of $\underline{R_r(c)}$ into $\underline{c}$ there exists an r-elements subset $A \subseteq \underline{R_r(c)}$ such that we have $d(B) = d(C)$ for any two-elements subsets B and C of A.*

Lemma 5.1 *Let $(L, \leq)$ be a distributive lattice with $w(L) \geq R_{n+1}(6^n)$ and $(L, \leq) \cong \mathbb{H}_f(\mathbb{J}(L, \leq))$. Then there exists a lattice embedding of $\underline{(n-1)}^2$ into $(L, \leq)$.*

Proof. First we consider the case $w(\mathbb{J}(L)) \geq 2n$. Then there exists an antichain $A \subseteq \mathbb{J}(L)$ containing $2n$ elements. Let M and N be disjoint subsets of A of size n. Then Lemma 4.4 implies the statement.

Now assume $w(\mathbb{J}(L)) = k < 2n$. By Dilworth's Theorem [Dil50], there are chains $C_1, C_2, \ldots, C_k \subseteq \mathbb{J}(L)$ with $\mathbb{J}(L) = \bigcup_{i=1,\ldots k} C_i$. For $x \in L$ and $1 \leq \ell \leq k$ let $\partial_\ell(x)$ denote the maximal element of C_ℓ below x if it exists, and $\perp$ otherwise. To ease the notations in this proof, we will consider $\perp$ as an additional element of $\mathbb{J}(L)$ which is minimal. Since L is distributive, $x = \bigvee_{1 \leq i \leq k} \partial_i(x)$ for any $x \in L$.

Since $w(L) \geq R_{n+1}(6^n)$, there is an antichain $A = \{x_1, x_2, \ldots, x_m\}$ in L with $m \geq R_{n+1}(6^n)$. Now we define a mapping $g_{i,j} : \{1, 2, \ldots, k\} \to \{<, =, >\}$ for $1 \leq i < j \leq m$ by $g_{i,j}(\ell) = \theta$ iff $\partial_\ell(x_i)\theta\partial_\ell(x_j)$ (with $\theta \in \{<, =, >\}$). For $x_i \neq x_j$, we define $g(x_i, x_j) = g_{\min\{i,j\},\max\{i,j\}}$. Thus, g maps the two-elements subsets of A into $\{<, =, >\}^{\{1,2,\ldots,k\}}$. Since this set contains at most $3^{2n} = 6^n$ elements and since $m > R_{n+1}(6^n)$, we can assume $g(x_i, x_j) = g(x_{i'}, x_{j'}) =: f$ for $i, j, i', j' \in \{1, 2, \ldots, n+1\}$ with $i \neq j$ and $i' \neq j'$. Then $f(\ell_1) \neq$"$=$" for some $1 \leq \ell_1 \leq k$ since otherwise $x_1 = x_2$. Similarly, there is an index $1 \leq \ell_2 \leq k$ with $f(\ell_2) \notin \{=, f(\ell_1)\}$ since otherwise x_1 and x_2 are comparable. Without loss of generality, we assume $f(1) =$"$<$" and $f(2) =$"$>$". Then $\partial_1(x_1) < \partial_1(x_2) < \ldots \partial_1(x_{n+1})$ and $\partial_2(x_1) > \partial_2(x_2) > \ldots \partial_2(x_{n+1})$. Thus, $C_j := \{\partial_j(x_i) \mid 1 < i < n+1\}$ for $j = 1, 2$ is a chain in $\mathbb{J}(L)$ containing $n-1$ elements.

Let $1 < i < n+1$ with $\partial_1(x_i) \geq \partial_2(x_i)$. Then $x_{i+1} \geq \partial_1(x_{i+1}) > \partial_1(x_i) \geq \partial_2(x_i) > \partial_2(x_{i+1})$ and $\partial_2(x_i), \partial_2(x_{i+1}) \in C_2$. But this contradicts the definition of $\partial_2(x_{i+1})$ as the maximal element of C_2 below x_{i+1}. Symmetrically, we can argue if $\partial_1(x_i) \leq \partial_2(x_i)$ (with x_{i-1} in place of x_{i+1}). Thus, $\partial_1(x_i)$ and $\partial_2(x_i)$ are incomparable for $1 < i < n+1$.

Now let $1 < i < j < n+1$ with $\partial_1(x_j) \geq \partial_2(x_i)$. Then $\partial_2(x_i) > \partial_2(x_j)$ since $i < j$, i.e. $\partial_1(x_j) > \partial_2(x_j)$, a contradiction to what we showed above. Similarly, we can argue in the cases $\partial_1(x_j) \leq \partial_2(x_i)$, $\partial_1(x_i) \geq \partial_2(x_j)$ and $\partial_1(x_i) \leq \partial_2(x_j)$.

Thus, we found two chains C_1 and C_2 in $\mathbb{J}(L)$ of size $n-1$ whose elements are mutually incomparable. Now Lemma 4.4 implies that $\underline{n-1} \times \underline{n-1}$ can be lattice embedded into $(\mathbb{H}(\mathbb{J}(L)), \subseteq) \cong L$. $\square$

Corollary 5.2 *Let $\mathfrak{L}$ be a set of distributive lattices $(L, \leq)$ satifying $(L, \leq) \cong \mathbb{H}_f(\mathbb{J}(L, \leq))$.*

1. *The elementary theory of $\mathfrak{L}$ is decidable iff the weak monadic antichain theory of $\mathbb{J}(\mathfrak{L})$ is decidable.*
2. *The following are equivalent:*
   - *(i) The monadic theory* $\mathrm{MTh}(\mathfrak{L})$ *is decidable.*
   - *(ii) The monadic chain theory* $\mathrm{MCTh}(\mathfrak{L})$ *is decidable.*
   - *(iii) the monadic theory* $\mathrm{MTh}(\mathbb{J}(\mathfrak{L}))$ *is decidable and the width of the elements of $\mathfrak{L}$ is bounded above.*
   - *(iv) the monadic chain theory* $\mathrm{MCTh}(\mathbb{J}(\mathfrak{L}))$ *is decidable and the width of the elements of $\mathfrak{L}$ is bounded above.*
3. *The monadic antichain theory* $\mathrm{MATh}(\mathfrak{L})$ *is decidable if and only if the elementary theory* $\mathrm{Th}(\mathbb{J}(\mathfrak{L}))$ *is decidable and the width of the elements of $\mathfrak{L}$ is bounded above.*

Proof. Since $\mathbb{H}_f(\mathbb{J}(\mathfrak{L})) = \mathfrak{L}$, the first statement follows immediately from Theorem 2.1. To derive the second statement from Theorem 4.9, it remains to show that the width of the elements of $\mathfrak{L}$ is bounded if and only if the diabolo width of the elements of $\mathbb{J}(\mathfrak{L})$ is bounded.

Suppose the width of the elements of $\mathfrak{L}$ is bounded, but the diabolo width $\mathbb{J}(\mathfrak{L})$ is unbounded. By Lemma 4.4, the unbounded diabolo width implies the embeddability of arbitrary large grids, i.e. an unbounded width. Hence, indeed, a bounded width of $\mathfrak{L}$ implies a bounded diabolo width of $\mathbb{J}(\mathfrak{L})$.

To show the other implication assume $dw(\mathbb{J}(L, \leq)) < n-2$ for any lattice $(L, \leq) \in \mathfrak{L}$. By contradiction, suppose that the width of the elements of $\mathfrak{L}$

is unbounded. Then there exists $(L,\leq)$ in $\mathfrak{L}$ such that $w(L,\leq) \geq R_{n+1}(6^n)$. By Lemma 5.1, there exists a lattice embedding $\eta : \underline{(n-1)}^2 \to L$. Let $A := \mathbb{J}(L) \cap \downarrow\eta(1, n-1)$ and $B := \mathbb{J}(L) \cap \downarrow\eta(n-1, 1)$. Since the elements $\eta(1,i)$ and $\eta(j,1)$ are pairwise incomparable for $i, j > 1$, $A \setminus B$ and $B \setminus A$ both contain at least $n-2$ elements. Furthermore, these two sets are incomparable. Hence the diabolo width of $\mathbb{J}(L,\leq)$ is at least $n-2$, a contradiction. $\square$

Our next aim is to show that the set of finite distributive lattices of width at most n has a decidable monadic second order theory. To this purpose, we show that their Hasse-diagrams have bounded tree width that we define next; this notion was introduced by Robertson and Seymour in their study of graph minors, it is useful in our context since the monadic theory of the sets of directed graphs of bounded tree width is decidable [Cou89]

An undirected graph (T,K) is a *tree* if for any $s,t \in T$, there is a unique path connecting s and t. Now let (V,E) be a finite directed graph and $n \in \mathbb{N}$. Then (V,E) has *tree-width at most* n if there exists a tree (T,K) and a mapping $\psi : T \to 2^V$ such that

1. for any $(x,y) \in E$, there is $t \in T$ with $x, y \in \psi(T)$,

2. for any $s,t,u \in T$ such that t is on the path connecting s and u, we have $\psi(s) \cap \psi(u) \subseteq \psi(t)$,

3. $\bigcup_{s\in T} \psi(i) = V$, and

4. $|\psi(t)| < n$ for any $t \in T$.

Theorem 5.3 *Let $n \in \mathbb{N}$ and let $\mathfrak{L}$ be the set of all finite distributive lattices of width at most n. Then* $\mathrm{MTh}(\mathfrak{L})$ *is decidable.*

Proof. In view of Courcelle's result [Cou89], is suffices to show that any lattice $(L,\leq)$ of width at most n has tree width at most $2n$.

Let m denote the length of L. The tree (T,K) that we construct is (the Hasse diagram of) the linear order on $\{1,2,\ldots,m\}$. Let $\psi(i)$ be the set of all vertices in $(L,\leq)$ of height $i-1$ or i.

Now let $x,y \in L$ with $x \prec y$. Then, since L is distributive, $h(y) - h(x) = 1$, i.e. $x, y \in \psi(h(y))$. Hence the first property is satisfied. For the second note that $\psi(i) \cap \psi(k) = \emptyset$ whenever there is $i < j < k$. Hence it is trivially satisfied. Similarly, the third requirement holds trivially. Finally, $\psi(i)$ consists of two antichains. Since the size of these antichains is bounded by n, the last requirement $|\psi(i)| \leq 2n$ follows. $\square$

Now let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be sets of distributive lattices. Suppose that the elementary theories of $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are decidable. Then, as an easy consequence of the Feferman-Vaught Theorem [FV59], the set of direct products of lattices from $\mathfrak{L}_1$ and lattices from $\mathfrak{L}_2$ has a decidable elementary theory. Next, we want to characterize when this set has a decidable monadic (chain, antichain) theory:

Corollary 5.4 *Let $\mathfrak{L}_1$ and $\mathfrak{L}_2$ be sets of finite distributive lattices satisfying $(L, \leq) \cong \mathbb{H}_f(\mathbb{J}(L, \leq)$. Define $\mathfrak{L} := \{(L_1, \leq) \times (L_2, \leq) \mid (L_i, \leq) \in \mathfrak{L}_i\}$. If* $\mathrm{MTh}(\mathfrak{L}_i)$ *(*$\mathrm{MATh}(\mathfrak{L}_i)$, $\mathrm{MCTh}(\mathfrak{L}_i)$*, resp.) is decidable, then* $\mathrm{MTh}(\mathfrak{L})$ *(*$\mathrm{MATh}(\mathfrak{L})$, $\mathrm{MCTh}(\mathfrak{L})$*, resp.) is decidable iff $\mathfrak{L}_1$ or $\mathfrak{L}_2$ is a finite set of finite lattices.*

Proof. We give the proof for the monadic theories, only. The other cases can be handled similarly. If both $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are infinite, we find for any $n \in \mathbb{N}$ lattices $(L_1, \leq) \in \mathfrak{L}_1$ and $(L_2, \leq) \in \mathfrak{L}_2$ of length at least n. Then the width of the direct product $(L_1, \leq) \times (L_2, \leq)$ is at least n, i.e. the width of the lattices in $\mathfrak{L}$ is not bounded. Hence $\mathrm{MTh}(\mathfrak{L})$ is undecidable by Corollary 5.2.

Conversely let $\mathfrak{L}_1$ be a finite set of finite lattices. Then there is $n \in \mathbb{N}$ with $|L_1| \leq n$ for any lattice $(L_1, \leq) \in \mathfrak{L}_1$. Since $\mathrm{MTh}(\mathfrak{L}_2)$ is decidable, we can assume $w(\mathfrak{L}_2) \leq n$. Note that the width $w(L_1 \times L_2, \leq)$ is at most $|L_1| \cdot w(L_2, \leq)$ for any finite distributive lattices $(L_1, \leq)$ and $(L_2, \leq)$. Hence $w(\mathfrak{L}) \leq n^2$. It remains to show that $\mathbb{J}(\mathfrak{L})$ has a decidable monadic theory: For distributive lattices $(L_1, \leq)$ and $(L_2, \leq)$, one has $\mathbb{J}(L_1 \times L_2, \leq) = \mathbb{J}(L_1, \leq) \dot{\cup} \mathbb{J}(L_2, \leq)$. Thus, we have to show that the monadic theory of $\{(P_1, \leq) \dot{\cup} (P_2, \leq) \mid (P_i, \leq) \in \mathbb{J}(\mathfrak{L}_i)\}$ is decidable. This follows from the composition theorem from Shelah [She75, Theorem 2.4] (cf. [Gur85] for the proof of this result) since $\mathrm{MTh}(\mathbb{J}(\mathfrak{L}_i))$ is decidable. $\square$

Note that in the corollary above we assumed from the very beginning that $\mathrm{MTh}(\mathfrak{L}_i)$ is decidable for $i = 1, 2$. Actually, the finiteness of $\mathfrak{L}_1$ or $\mathfrak{L}_2$ follows without this assumption from the decidability of $\mathrm{MTh}(\mathfrak{L})$. We finish this section with an example of classes $\mathfrak{L}_1$, $\mathfrak{L}_2$ and $\mathfrak{L}$ as in the corollary above such that $\mathfrak{L}_1$ is finite, $\mathfrak{L}$ has a decidable monadic theory but the monadic theory of $\mathfrak{L}_2$ is undecidable:

Example 5.5 For simplicity, let $\underline{2}$ denote the Boolean lattice $(\{1, 2\}, \leq)$. Let $\mathfrak{L}_1$ consist of the lattices $\underline{2}^i$ for $0 \leq i \leq 2$ (i.e. $\mathfrak{L}_1$ contains the one-point-lattice, the Boolean lattice and the diamond). Let Lin denote the set of finite linear orders and let $\mathfrak{P} \subseteq \mathrm{Lin}$ be an undecidable set of linear orders. We define a set of finite distributive lattices $\mathfrak{L}_2 \subseteq \{\underline{2}^i \times (L, \leq) \mid 0 \leq i \leq 2, (L, \leq) \in \mathrm{Lin}\}$ by $\underline{2}^i \times (L, \leq) \in \mathfrak{L}_2$ iff

1. $i \in \{0, 2\}$ and $(L, \leq) \in \mathrm{Lin}$, or
2. $i = 1$ and $(L, \leq) \in \mathfrak{P}$.

Then $\mathfrak{L}_2$ is a set of finite distributive lattices. It is undecidable since the subset of $\mathfrak{L}_2$ of lattices of width 2 corresponds to $\mathfrak{P}$ which was chosen to be undecidable. Hence, in particular, $\mathfrak{L}_2$ has an undecidable monadic theory. It is straightforward to show that the set of direct products of lattices from $\mathfrak{L}_1$ and $\mathfrak{L}_2$ equals the set $\{\underline{2}^i \times (L, \leq) \mid 1 \leq i \leq 5, (L, \leq) \in \mathrm{Lin}\} = \mathfrak{L}$. Since Lin has a decidable monadic theory, we can apply the corollary above and obtain that $\mathrm{MTh}(\mathfrak{L})$ is decidable.

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