THE INTERSECTION OF OPPOSED BIG CELLS IN THE REAL FLAG VARIETY OF TYPE G_2

R. J. MARSH AND K. RIETSCH

Abstract. We compute the Euler characteristics of the individual connected components of the intersection of two opposed big cells in the real flag variety of type G_2, verifying a conjecture from [6].

1. Introduction

Let G be a connected linear algebraic group. Any flag variety (homogeneous projective variety G/B) has a myriad of cell decompositions, so–called 'Bruhat decompositions'. For every Borel subgroup there is precisely one, and the cells in the decomposition are simply the orbits under this group. Furthermore there is always a unique open dense orbit called the 'big cell'. In the present paper we fix two Borel subgroups opposite to one another and study the intersection of the two resulting big cells. More precisely we are interested in the real points of this variety, in the case where everything is split over $\mathbb{R}$.

We let $\mathcal{B}^*$ denote the real points of the intersection of two opposed big cells, endowed with the Hausdorff topology coming from $\mathbb{R}$. There has been some recent work on determining the number of connected components and the Euler characteristics of these varieties $\mathcal{B}^*$, see e.g. [6, 5], and for the type A case [7, 8]. In particular it was open for a long time whether the connected components are always contractible. This has proved not to be the case. The first example is in type G_2 found in [6] (but it is also false in type A for rank > 4 as follows from the explicit formula for the number of connected components in [8] and a computation of the Euler characteristics e.g. using [6]).

Therefore to the graph defined in [6] parameterizing the connected components of $\mathcal{B}^*$ one can in principle add another nontrivial datum. That is, every component of the graph has an integer associated to it given simply by the Euler characteristic of the corresponding connected component of $\mathcal{B}^*$. Lusztig showed in [4] that the graphs arising in this setting are 'mod 2' quotients of the graphs parameterizing the canonical basis of the corresponding quantum enveloping algebra $\mathcal{U}^-$. Therefore by taking the Euler characteristics one is assigning in a natural way an integer to every canonical basis element. No direct canonical basis interpretation of these integers is known. Moreover they have not up to now been computed in any nontrivial examples.

We focus our attention here on the G_2 case. In Figure 2 we have reproduced the parameterization [6] of the connected components of $\mathcal{B}^*$ in type G_2. There are 11 connected components, and one is conjectured to have Euler characteristic 2 (the

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last one), while the others should have Euler characteristic 1. By an independent method the total Euler characteristic of $\mathcal{B}^*$ was worked out in [6] as 12.

The aim of this paper is to compute the individual Euler characteristics of the connected components of $\mathcal{B}^*$ to verify the above conjecture. We use two main tools. The first one is a cell decomposition of $\mathcal{B}^*$ due to Deodhar [2], which is recalled in the next section. And the second ingredient is Berenstein and Zelevinsky's Chamber Ansatz, explained in Section 3, which we require to be able to tell which connected component each of the various cells of Deodhar's decomposition lie in. The resulting decompositions for the connected components are indicated in Figure 6. The Euler characteristics are the expected ones as can be read off from that figure.

2. Decomposition of $\mathcal{B}^*$

2.1. **Preliminaries.** We begin by introducing some notation and standard facts as can be found e.g. in [9]. Let $G_{\mathbb{C}}$ be a complex simple linear algebraic group. After this general section we will choose it specifically as the one of type G_2. In any case we always focus on its split real form $G_{\mathbb{R}} = G$. All the varieties in this paper will be defined over $\mathbb{R}$, and we identify them with their $\mathbb{R}$–valued points. The topology we consider is the usual Hausdorff topology coming from $\mathbb{R}$ (rather than Zariski topology). We write $\mathbb{R}^*$ for $\mathbb{R} \setminus \{0\}$.

Let $\mathcal{B}$ denote the (real) flag variety of G. The elements of $\mathcal{B}$ are the Borel subgroups of G. We sometimes write $[B]$ for the Borel subgroup B considered as a point in the flag variety. The transitive action of G on $\mathcal{B}$ is denoted by $g \cdot [B] := [gBg^{-1}]$.

Let $T \subset G$ be a fixed split maximal torus, and B^+ a fixed Borel subgroup which contains T. We also automatically have given the opposite Borel subgroup B^- (such that $B^+ \cap B^- = T$) and the unipotent radicals U^+ and U^- of these two Borel subgroups.

Let $X^*(T)$ and $X_*(T)$ be the character, respectively cocharacter lattices of T with their canonical perfect pairing

$$\langle\ ,\ \rangle :\ X_*(T) \times X^*(T) \to \mathbb{Z}.$$

And let $A = (A_{ij})$ be the Cartan matrix of G. We denote the positive simple roots (corresponding to B^+) by $\alpha_1, \dots, \alpha_r \in X^*(T)$. We will write $\alpha > 0$ if $\alpha \in X^*(T)$ is a positive root, i.e. a nonnegative linear combination of the simple roots. We also have the simple coroots $\alpha_1^\vee, \dots, \alpha_r^\vee \in X_*(T)$ which are determined by $\langle \alpha_i^\vee, \alpha_j \rangle = A_{ij}$.

We will make extensive use of the 1–parameter simple root subgroups: Let us fix for $i = 1, \dots, r$, Chevalley generators e_i and $f_i \in Lie(G)$ of the (real) Lie algebra, with the e_i's lying in positive simple root spaces. Then we define

$$\begin{aligned} x_i &: \mathbb{R} \to B^+, \qquad & x_i(t) &:= \exp(te_i), \\ y_i &: \mathbb{R} \to B^-, \qquad & y_i(t) &:= \exp(tf_i). \end{aligned}$$

The Weyl group $N_G(T)/T$ of G is denoted W and is generated by the usual simple reflections $s_1, \dots, s_r$ (r being the rank of G). We fix a representative $\dot{s}_i \in G$ for the Weyl group element s_i by defining $\dot{s}_i := x_i(1)y_i(-1)x_i(1)$. For any other $w \in W$ choose a reduced (minimal number of factors) expression $w = s_{i_1} \dots s_{i_k}$ and set $\dot{w} := \dot{s}_{i_1} \dots \dot{s}_{i_k}$ to get a representative. It is well known that this definition of $\dot{w}$ is independent of the reduced expression. Note that $x_i(t) \in U^+ \cap B^- \dot{s}_i B^-$ and

$y_i(t) \in U^- \cap B^+\dot{s}_iB^+$, whenever $t \neq 0$. Explicitly,

$$\begin{aligned} x_i(t) &= y_i(t^{-1})\ \dot{s}_i\alpha_i^\vee(-t)\ y_i(t^{-1}), \\ y_i(t) &= x_i(t^{-1})\ \alpha_i^\vee(-t)\dot{s}_i\ x_i(t^{-1}). \end{aligned}$$

The Weyl group acts on T and hence naturally also on $X_*(T)$ and $X^*(T)$. The length of (a reduced expression of) an element $w \in W$ is denoted $\ell(w)$. We also consider the Bruhat order $<$ on W. The statement $ws_i < w$ is equivalent to $w \cdot \alpha_i < 0$, that w sends α_i to a negative root.

There is a unique longest element of W which is called w_0. We propose to study the open subvariety of the flag variety

$$\mathcal{B}^* := B^+\dot{w}_0 \cdot [B^+] \cap B^-\dot{w}_0 \cdot [B^-] \ \subset\ \mathcal{B},$$

in other words the intersection of the two big cells for the Bruhat decompositions relative to B^-, respectively B^+.

Note that $\mathcal{B}^*$ is an affine variety and can be identified naturally with open subvarieties both of U^+ and of U^-. We have two isomorphisms

$$\begin{array}{ccccc} & & \mathcal{B}^* & & \\ & i_+ \nearrow & & \nwarrow i_- & \\ U^+ \cap (B^-\dot{w}_0B^-) & & & & U^- \cap (B^+\dot{w}_0B^+), \end{array} \tag{2.1}$$

where i_+ takes $u \in U^+$ to $u \cdot [B^-]$ while the right hand map i_- takes $u \in U^-$ to $u \cdot [B^+]$.

2.2. **Deodhar's decomposition of $\mathcal{B}^*$.** We begin by making the following definition (see [3, Appendix]).

Definition 2.1. (relative position in $\mathcal{B}$) Let $w \in W$ and consider the action of G on $\mathcal{B} \times \mathcal{B}$ by simultaneous conjugation. We say that two Borel subgroups $B, B' \in \mathcal{B}$ are in relative position w if the pair (B, B') lies in the G–orbit of $(B^+, \dot{w} \cdot B^+)$. From Bruhat decomposition it follows that such w is unique and exists for any pair (B, B'). We write $B \xrightarrow{w} B'$.

If $B \xrightarrow{w} B'$ and $w = s_{i_1} \dots s_{i_n}$ is a reduced expression, then it also follows from Bruhat decomposition that there exist uniquely determined Borel subgroups $B_0, B_1, \dots, B_n = B'$ such that

$$B = B_0 \xrightarrow{s_{i_1}} B_1 \xrightarrow{s_{i_2}} B_2 \longrightarrow \cdots \longrightarrow B_{n-1} \xrightarrow{s_{i_n}} B_n = B'$$

Explicitly, $B \xrightarrow{w} B'$ means $B = g \cdot B^+$ and $B' = g\dot{w} \cdot B^+$ for some $g \in G$. And the B_j's are then given by $B_j = g\dot{s}_{i_1} \dots \dot{s}_{i_j} \cdot B^+$.

Using the above notation, we have $\mathcal{B}^* = \{B \in \mathcal{B} \mid B^+ \xrightarrow{w_0} B \xleftarrow{w_0} B^-\}$. The following definition is a special case of [2, Definition 2.3].

Definition 2.2 (distinguished subexpressions for 1)**.** Let $s_{i_1} \dots s_{i_n} = w \in W$ be a reduced expression for w (so $n = \ell(w)$). Then by a *subexpression* of this reduced expression we mean a sequence of Weyl group elements $\sigma = (\sigma_0, \sigma_1, \dots, \sigma_n)$ such that

$$\sigma_0 = 1, \qquad \sigma_j = \begin{cases} \text{either} & \sigma_{j-1}s_{i_j} \\ \text{or} & \sigma_{j-1}, \end{cases} \qquad \text{for all } j = 1, \dots n.$$

In particular the "empty" subexpression $\sigma = (1, \dots, 1)$ is allowed. We call σ a *subexpression for* 1 if $\sigma_n = 1$.

A subexpression is called *distinguished* if we have

$$\sigma_j \leq \sigma_{j-1}\ s_{i_j}, \quad \text{for all } j \in \{1, \dots, n\}.$$

This means that if right multiplication by s_{i_j} decreases the length of σ_{j-1}, then we must choose $\sigma_j = \sigma_{j-1}s_{i_j}$ to get a distinguished subexpression.

The following result is stated by Deodhar over an algebraically closed field, but it extends trivially to any split form, so we state it here in our present setting over the reals.

Theorem 2.3 ([2]). *Let $s_{i_1}\dots s_{i_N}$ be a fixed reduced expression for w_0.*

1. *Suppose $B = x\dot{w}_0 \cdot B^+$ is an element of $\mathcal{B}^*$, where $x \in B^+$. Then the sequence $(\sigma_0, \dots, \sigma_N) =: \sigma(B)$ defined by*
$$x s_{i_1} \dots s_{i_k} \in B^- \sigma_k B^+$$
is a well–defined distinguished subexpression for 1 *(of $s_{i_1}\dots s_{i_N}$).*
2. *Let $D_\sigma := \{B \in \mathcal{B}^* \mid \sigma(B) = \sigma\}$. Then*

$$D_\sigma \cong (\mathbb{R}^*)^{|I(\sigma)|} \times \mathbb{R}^{|K(\sigma)|}, \quad \textit{where} \quad I(\sigma) = \{j \in \{1, \dots, N\} \mid \sigma_j = \sigma_{j-1}\},$$
$$\textit{and} \quad K(\sigma) = \{j \in \{1, \dots, N\} \mid \sigma_j < \sigma_{j-1}\}.$$

An isomorphism as in part (2) of the theorem (albeit not identical to the one in [2]) will be constructed explicitly below. We continue to fix the reduced expression $s_{i_1}\dots s_{i_N}$ for w_0 in what follows.

Remark 2.4. The definition of D_σ is natural to state using relative position. If $B \in \mathcal{B}^*$, and $B_1, \dots, B_N$ are defined by

$$B^+ \xrightarrow{s_{i_1}} B_1 \xrightarrow{s_{i_2}} B_2 \longrightarrow \cdots \longrightarrow B_{N-1} \xrightarrow{s_{i_N}} B_N = B,$$

then $B \in D_\sigma$ precisely if $B^- \xrightarrow{w_0\sigma_j} B_j$ for all j. So the σ controls the relative positions of the intermediate B_i's with respect to B^-.

2.3. **Inductive construction.** Let σ be a fixed distinguished subexpression for 1 of $s_{i_1}\dots s_{i_N}$. We now want to describe $D_\sigma^- \subset U^- \cap B^+w_0B^+$, the preimage of D_σ under the isomorphism i_- in (2.1). We will do this by building up the elements $B \in D_\sigma$ from the B_j's defined in the previous remark. To begin with note that if $B_{j-1} = g \cdot B^+$, then $B_{j-1} \xrightarrow{s_{i_j}} B_j$ just says that $B_j = gx_{i_j}(t)\dot{s}_{i_j} \cdot B^+$ for some $t \in \mathbb{R}$, or equivalently

$$B_j = \begin{cases} \text{either} & gy_{i_j}(t) \cdot B^+ \quad \text{some } t \in \mathbb{R}^*, \\ \text{or} & g\dot{s}_{i_j} \cdot B^+. \end{cases}$$

Let us first determine the possible B_1's. There are two cases.

- If $\sigma_1 = 1$, then we have $B^+ \xrightarrow{s_{i_1}} B_1 \xleftarrow{w_0} B^-$. Therefore $B_1 = y_{i_1}(t) \cdot B^+$ for some $t \in \mathbb{R}^*$.
- If on the other hand $\sigma_1 = s_{i_1}$ then $B^+ \xrightarrow{s_{i_1}} B_1 \xleftarrow{w_0 s_{i_1}} B^-$ and we get $B_1 = \dot{s}_{i_1} \cdot B^+$.

Suppose in general we have $B_{j-1} = g \cdot B^+$ given, where $g = y\dot{\sigma}_{j-1}$ for some $y \in U^-$. We then want to construct all possible $B_j = g' \cdot B^+$ from this B_{j-1}. There are three cases.

1. Suppose first that $\sigma_j = \sigma_{j-1}$. Then we have the setting
$$\begin{tikzcd} B^- \arrow[rr, "w_0\sigma_j"] \arrow[dr, "w_0\sigma_j"'] & & B_{j-1} \arrow[dl, "s_{i_j}"] \\ & B_j & \end{tikzcd}$$
So if $B_{j-1} = g \cdot B^+$ where $g = y\dot{\sigma}_{j-1}$, then it is easy to rule out $g\dot{s}_{i_j} \cdot B^+$ for B_j and the only possible solutions are of the form $B_j = gy_{i_j}(t) \cdot B^+$ for some $t \in \mathbb{R}^*$. Note that $w_0\sigma_j s_{i_j} < w_0\sigma_j$ (since σ is distinguished). We claim that therefore $B_j = gy_{i_j}(t) \cdot B^+$ has the correct relative positions for any $t \in \mathbb{R}^*$ and in fact $g' := gy_{i_j}(t) \in U^-\dot{\sigma}_j$. All of this follows since we have $\sigma_{j-1} \cdot \alpha_{i_j} > 0$ and therefore
$$gy_{i_j}(t) = y\dot{\sigma}_{j-1}y_{i_j}(t) = y\dot{\sigma}_{j-1}y_{i_j}(t)\dot{\sigma}_{j-1}^{-1}\dot{\sigma}_j \in U^-\dot{\sigma}_j,$$
using $\dot{\sigma}_{j-1} = \dot{\sigma}_j$.
2. Suppose next that $\sigma_j > \sigma_{j-1}$. Then
$$\begin{tikzcd} B^- \arrow[rr, "w_0\sigma_{j-1}"] \arrow[dr, "w_0\sigma_j"'] & & B_{j-1} \\ & B_j \arrow[ur, "s_{i_j}"'] & \end{tikzcd}$$
and since the lengths add, $\ell(w_0\sigma_j) + \ell(s_{i_j}) = \ell(w_0\sigma_{j-1})$, we get that B_j is uniquely determined by B_{j-1} and equals to $g\dot{s}_{i_j} \cdot B^+$. We immediately have that $g' := g\dot{s}_{i_j} \in U^-\dot{\sigma}_j$.
3. Finally we have the case $\sigma_j < \sigma_{j-1}$. In this case the other two lengths add, $\ell(w_0\sigma_{j-1}) + \ell(s_{i_j}) = \ell(w_0\sigma_j)$, and the diagram
$$\begin{tikzcd} B^- \arrow[rr, "w_0\sigma_{j-1}"] \arrow[dr, "w_0\sigma_j"'] & & B_{j-1} \arrow[dl, "s_{i_j}"] \\ & B_j & \end{tikzcd}$$
is automatically satisfied for any B_j in position s_{i_j} relative to B_{j-1}. Therefore we can take $B_j = gx_{i_j}(m)\dot{s}_{i_j} \cdot B^+$ for any $m \in \mathbb{R}$.

   We claim that in this case $g' := gx_{i_j}(m)\dot{s}_{i_j}^{-1} \in U^-\dot{\sigma}_j$. This holds because we have $\sigma_{j-1} \cdot \alpha_{i_j} < 0$. Therefore $\dot{\sigma}_{j-1}x_{i_j}(t)\dot{\sigma}_{j-1}^{-1} \in U^-$ and
$$gx_{i_j}(m)\dot{s}_{i_j}^{-1} = y\dot{\sigma}_{j-1}x_{i_j}(m)\dot{s}_{i_j}^{-1} = y\dot{\sigma}_{j-1}x_{i_j}(m)\dot{\sigma}_{j-1}^{-1}\dot{\sigma}_j \in U^-\dot{\sigma}_j.$$

Applying this procedure recursively to express finally B_N we get the following proposition (and we also recover Deodhar's result Theorem 2.3).

Proposition 2.5. *For $w_0 = s_{i_1} \dots s_{i_N}$ a fixed reduced expression, and σ a distinguished subexpression for 1 let*
$$I(\sigma) = \{j \,:\, 1 \le j \le N \text{ and } \sigma_{j-1} = \sigma_j\},$$
$$J(\sigma) = \{j \,:\, 1 \le j \le N \text{ and } \sigma_{j-1} < \sigma_j\},$$
$$K(\sigma) = \{j \,:\, 1 \le j \le N \text{ and } \sigma_{j-1} > \sigma_j\}.$$
Then we have explicitly
$$D_\sigma^- = \left\{ z_1 z_2 \dots z_N \;\middle|\; z_j = \begin{cases} y_{i_j}(t_j) & \text{if } j \in I(\sigma) \\ \dot{s}_{i_j} & \text{if } j \in J(\sigma) \\ x_{i_j}(m_j)\dot{s}_{i_j}^{-1} & \text{if } j \in K(\sigma) \end{cases}, \text{ where } m_j \in \mathbb{R},\ t_j \in \mathbb{R}^* \right\}.$$

Proof. By the inductive construction in 2.3 above we showed that any $B \in D_\sigma$ is of the form $z \cdot B^+$ for $z = z_1 \dots z_N$ as in the proposition (and that these $z \cdot B$'s all lie in D_σ). We also showed that $z_1 \dots z_k \in U^- \dot\sigma_k$ for all k, so in particular $z_1 \dots z_N \in U^-$, since $\sigma_N = 1$. Therefore D_σ^- is precisely the set of these $z = z_1 \dots z_N$. $\square$

Remark 2.6. Note that the map

$$(\mathbb{R}^*)^{I(\sigma)} \times \mathbb{R}^{K(\sigma)} \to D_\sigma^-$$

arising from the proposition is an isomorphism, with inverse basically constructed in 2.3.

2.4. **Further refinement.** Over $\mathbb{R}$ it is natural to consider the connected components of the D_σ's to get a cell decomposition. Let $h : I(\sigma) \to \{1, -1\}$ be a choice of signs for the elements of $I(\sigma)$. Then we define

$$D_\sigma^-(h) = \left\{ z_1 z_2 \dots z_N \;\middle|\; z_j = \begin{cases} y_{i_j}(t_j) \text{ with } h(j)t_j \in \mathbb{R}_{>0} & \text{if } j \in I(\sigma), \\ \dot s_{i_j} & \text{if } j \in J(\sigma), \\ x_{i_j}(m_j)\dot s_{i_j}^{-1} \text{ with } m_j \in \mathbb{R} & \text{if } j \in K(\sigma). \end{cases} \right\}.$$

$D_\sigma^-(h)$ is a (real) semi–algebraic cell in $U^- \cap B^+ \dot w_0 B^+$ of dimension $|I(\sigma)| + |K(\sigma)|$. Its image $i_-(D_\sigma^-(h))$ in $\mathcal{B}^*$ is denoted $D_\sigma(h)$.

3. Type G_2

From now on let G be of type G_2. Then the Cartan matrix $A = (A_{ij})$ for type G_2 is given by

$$A = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

The Weyl group W has two generators s_1, s_2 corresponding to reflection by the short root α_1 and the long root α_2, respectively (see Figure 3). The fundamental weights are $\omega_1 = \varepsilon_1$, giving rise to a 7–dimensional representation, and $\omega_2 = 2\varepsilon_1 + \varepsilon_2$, the highest weight of the 14–dimensional adjoint representation.

The longest element is

$$w_0 = s_1 s_2 s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1 s_2 s_1.$$

We let $\mathbf{i} = (1,2,1,2,1,2)$ and $\widetilde{\mathbf{i}} = (2,1,2,1,2,1)$ stand for these two reduced expressions of w_0.

To give Deodhar's decomposition of $\mathcal{B}^*$ in this case we first need to fix a reduced expression of w_0, so let us pick $\mathbf{i} = (1,2,1,2,1,2)$. Next we list all of the distinguished subexpressions for 1 of $\mathbf{i}$:

$$\Sigma_{\mathbf{i}} := \left\{ \begin{array}{ll} xxxxxx := (1,1,1,1,1,1,1), & xxx2x2 := (1,1,1,1,s_2,s_2,1), \\ 1x1xxx := (1,s_1,s_1,1,1,1,1), & 1x12x2 := (1,s_1,s_1,1,s_2,s_2,1), \\ x2x2xx := (1,1,s_2,s_2,1,1,1), & 12x21x := (1,s_1,s_1s_2,s_1s_2,s_1,1,1), \\ xx1x1x := (1,1,1,s_1,s_1,1,1), & x21x12 := (1,1,s_2,s_2s_1,s_2s_1,s_2,1) \end{array} \right\}.$$

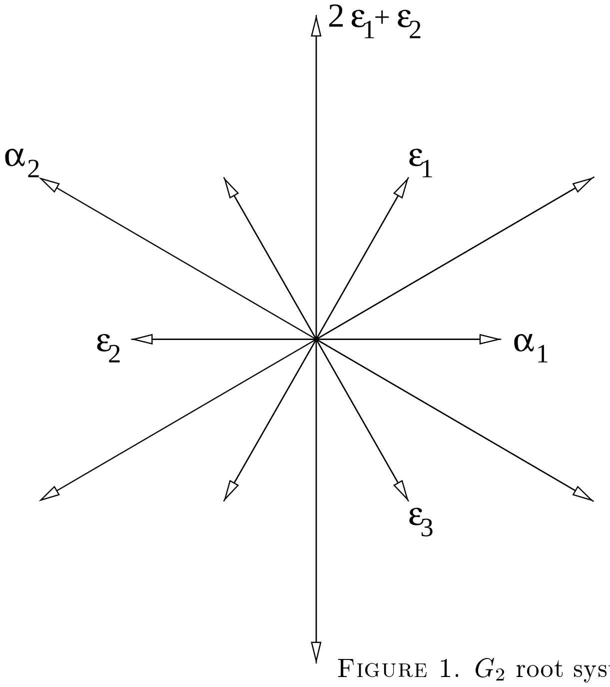

FIGURE 1. G_2 root system

Furthermore, define

$$\begin{aligned}
y_{xxxxxx}(t_1,\ldots,t_6) := y_{\mathbf{i}}(t_1,\ldots,t_6) &:= y_1(t_1)y_2(t_2)y_1(t_3)y_2(t_4)y_1(t_5)y_2(t_6)\\
y_{1x1xxx}(t_1,m_1,t_2,t_3,t_4) &:= \dot{s}_1 y_2(t_1)x_1(m_1)\dot{s}_1^{-1}y_2(t_2)y_1(t_3)y_2(t_4)\\
y_{x2x2xx}(t_1,t_2,m_1,t_3,t_4) &:= y_1(t_1)\dot{s}_2 y_1(t_2)x_2(m_1)\dot{s}_2^{-1}y_1(t_3)y_2(t_4)\\
y_{xx1x1x}(t_1,t_2,t_3,m_1,t_4) &:= y_1(t_1)y_2(t_2)\dot{s}_1 y_2(t_3)x_1(m_1)\dot{s}_1^{-1}y_2(t_4)\\
y_{xxx2x2}(t_1,t_2,t_3,t_4,m_1) &:= y_1(t)y_2(t_2)y_1(t_3)\dot{s}_2 y_1(t_4)x_2(m_1)\dot{s}_2^{-1}\\
y_{1x12x2}(t_1,m_1,t_2,m_2) &:= \dot{s}_1 y_2(t_1)x_1(m_1)\dot{s}_1^{-1}\dot{s}_2 y_1(t_2)x_2(m_2)\dot{s}_2^{-1}\\
y_{12x21x}(t_1,m_1,m_2,t_2) &:= \dot{s}_1\dot{s}_2 y_1(t_1)x_2(m_1)\dot{s}_2^{-1}x_1(m_2)\dot{s}_1^{-1}y_2(t_2)\\
y_{x21x12}(t_1,t_2,m_1,m_2) &:= y_1(t_1)\dot{s}_2\dot{s}_1 y_2(t_2)x_1(m_1)\dot{s}_1^{-1}x_2(m_2)\dot{s}_2^{-1}.
\end{aligned}$$

By Proposition 2.5 we have

$$\begin{aligned}
D_{\mathbf{i}}^- := D_{xxxxxx}^- &= \{y_{\mathbf{i}}(t_1,t_2,t_3,t_4,t_5,t_6) \mid t_1,\ldots,t_6 \in \mathbb{R}^*\}\\
D_{1x1xxx}^- &= \{y_{1x1xxx}(t_1,m_1,t_2,t_3,t_4) \mid t_1,\ldots,t_4 \in \mathbb{R}^*,\ m_1 \in \mathbb{R}\}\\
&\ \ \vdots\\
D_{x21x12}^- &= \{y_{x21x12}(t_1,t_2,m_1,m_2) \mid t_1,t_2 \in \mathbb{R}^*, m_1,m_2 \in \mathbb{R}\},
\end{aligned}$$

with the property that

$$U^- \cap B^+\dot{w}_0 B^+ = \bigsqcup_{\sigma\in\Sigma_{\mathbf{i}}} D_\sigma^-.$$

From these we also get the $D_\sigma^-(h)$'s and $D_\sigma(h)$'s defined in 2.4, where $h : I(\sigma) \to \{1,-1\}$ determines the signs of the $t_i \in \mathbb{R}^*$.

4. Parameterization of the connected components of $\mathcal{B}^*$: using the opposite reduced expression

We recall the parameterization of the set of connected components of $\mathcal{B}^*$ from [6]. Consider the open subset $D_{\mathbf{i}} = \bigsqcup_{h:\{1,\dots,6\}\to\{\pm 1\}} D_{\mathbf{i}}(h)$ of $\mathcal{B}^*$ from above. Explicitly,

$$D_{\mathbf{i}}(h) = \left\{ y_{\mathbf{i}}(t_1,\dots,t_6)\cdot[B^+] = y_1(t_1)y_2(t_2)\dots y_1(t_5)y_2(t_6)\cdot[B^+] \mid h(i)t_i \in \mathbb{R}_{>0} \right\}.$$

We also have analogs of these for $\widetilde{\mathbf{i}} = (2,1,2,1,2,1)$,

$$\begin{aligned} D_{\widetilde{\mathbf{i}}}(h) &:= \left\{ y_{\widetilde{\mathbf{i}}}(t_1,\dots,t_6)\cdot[B^+] := y_2(t_1)\dots y_1(t_6)\cdot[B^+] \mid h(i)t_i \in \mathbb{R}_{>0} \right\} \\ D_{\widetilde{\mathbf{i}}} &:= \bigsqcup_{h:\{1,\dots,6\}\to\{\pm 1\}} D_{\widetilde{\mathbf{i}}}(h). \end{aligned}$$

It was proved in [6] that the union $D^* := D_{\mathbf{i}} \cup D_{\widetilde{\mathbf{i}}}$ has complement of codimension ≥ 2 in $\mathcal{B}^*$. Hence the connected components of $\mathcal{B}^*$ correspond bijectively to the connected components of D^*. And these were determined roughly by checking which of the $D_{\widetilde{\mathbf{i}}}(h')$'s overlap with which $D_{\mathbf{i}}(h)$'s. Figure 2 adapted from [6] shows which $D_{\mathbf{i}}(h)$'s and $D_{\widetilde{\mathbf{i}}}(h')$'s lie in the same connected component, and these components are numbered for later use. The sequences of signs in the figure stand for the values of $h : \{1,\dots,6\} \to \{\pm 1\}$, and the columns indicate whether $D_{\mathbf{i}}(h)$ or $D_{\widetilde{\mathbf{i}}}(h)$ is meant. So for example the sequence of signs in the third row first column stands for $D_{\mathbf{i}}((1,1,-1,1,-1,1))$, and the figure says that this cell lies in connected component 5.

5. Berenstein–Zelevinsky's generalized chamber Ansatz for G_2

We wish to determine which connected component of $\mathcal{B}^*$ each $D_\sigma(h)$ lies in. We shall do this by proving that each $D_\sigma(h)$ actually intersects one of the cells $D_{\widetilde{\mathbf{i}}}(h')$ (for some choice h' of signs) listed in the previous section. It then follows that $D_\sigma(h)$ lies in the same connected component as this cell.

In order to show that an element $y \in D_\sigma^-$ also lies in one of the cells $D_{\widetilde{\mathbf{i}}}^-(h')$, we need to show that y can be expressed in the form $y = y_{\widetilde{\mathbf{i}}}(a_1,\dots,a_6) = y_2(a_1)y_1(a_2)y_2(a_3)y_1(a_4)y_2(a_5)y_1(a_6)$ for some non–zero real numbers $a_1,a_2,a_3,a_4,a_5,a_6 \in \mathbb{R}^*$ with signs $h'(1)$, $h'(2)$, $h'(3)$, $h'(4)$, $h'(5)$, $h'(6)$. We can then deduce which connected component $D_\sigma(h)$ lies in.

There already exists in the literature a beautiful method for determining a_1, a_2, a_3, a_4, a_5, a_6, should they exist — this is the Chamber Ansatz of Berenstein, Fomin and Zelevinsky; we therefore employ this method.

There is a natural map $\varepsilon : U^+ \cap B^-\dot{w}_0B^- \to U^- \cap B^+\dot{w}_0B^+$, given explicitly by $\varepsilon = i_-^{-1}\circ i_+$ (see 2.1); similarly we define $\alpha : U^- \cap B^+wB^+ \to U^+ \cap B^-w^{-1}B^-$ to be $\alpha = i_+^{-1}\circ i_-$, the inverse of ε.

$$U^+ \cap B^-\dot{w}_0B^- \underset{\varepsilon}{\longrightarrow} U^- \cap B^+\dot{w}_0B^+$$

$$U^+ \cap B^-\dot{w}_0B^- \underset{\alpha}{\longleftarrow} U^- \cap B^+\dot{w}_0B^+$$

	$D_{\mathbf{i}}(h)$	$D_{\tilde{\mathbf{i}}}(h)$	$D_{\mathbf{i}}(h)$	$D_{\tilde{\mathbf{i}}}(h)$	
1	++++++	++++++	−−−−−−	−−−−−−	2
3	+−+−+−	−+−+−+	−+−+−+	+−+−+−	4
5	++−+−+	+−+−++	−−+−+−	−+−+−−	6
	−−++−+	+−++−−	++−−+−	−+−−++	
	−+−−−+	+−−−+−	+−+++−	−+++−+	
	−−−−++	++−−−−	++++−−	−−++++	
	−+−++−	−++−+−	+−+−−+	+−−+−+	
	−−−+−−	−−+−−−	+++−++	++−+++	
	−++−−−	−−−++−	+−−+++	+++−−+	
	+−−−−−	−−−−−+	−+++++	+++++−	
7	+−+−++	++−+−+	−+−+−−	−−+−+−	8
	+−++−−	−−++−+	−+−−++	++−−+−	
	+−−−+−	−+−−−+	−+++−+	+−+++−	
	++−−−−	−−−−++	−−++++	++++−−	
	−++−+−	−+−++−	+−−+−+	+−+−−+	
	−−+−−−	−−−+−−	++−+++	+++−++	
	−−−++−	−++−−−	+++−−+	+−−+++	
	−−−−−+	+−−−−−	+++++−	−+++++	
9	−+−+++	++++−+	+−+−−−	−−−−+−	
	+−−++−	++−+−−	−++−−+	−−+−++	
	−++++−	++−−++	+−−−−+	−−++−−	
	−−−+−+	+−++++	+++−+−	−+−−−−	
10	++++−+	−+−+++	−−−−+−	+−+−−−	
	++−+−−	+−−++−	−−+−++	−++−−+	
	++−−++	−++++−	−−++−−	+−−−−+	
	+−++++	−−−+−+	−+−−−−	+++−+−	
11	−−−+++	+++−−−	+++−−−	−−−+++	
	−++−++	++−++−	+−−+−−	−−+−−+	
	+−−−++	++−−−+	−+++−−	−−+++−	
	+−++−+	+−++−+	−+−−+−	−+−−+−	
	−−+−−+	+−−+−−	++−++−	−++−++	
	++−−−+	+−−−++	−−+++−	−+++−−	

FIGURE 2. Connected components of $\mathcal{B}^*$ identified with $U^- \cap B^+\dot{w}_0B^+$, following [6]

Let $y \in U^- \cap B^+\dot{w}_0B^+$. Then Theorem 1.4 in [1] can be used to calculate $\alpha(y) \in U^+ \cap B^-\dot{w}_0B^-$ as an element of the form

$$x = x_{\tilde{\mathbf{i}}}(a, b, c, d, e, f) := x_2(a)x_1(b)x_2(c)x_1(d)x_2(e)x_1(f).$$

A second application of this Theorem can be used to calculate $y = \varepsilon(x) \in U^- \cap B^+\dot{w}_0B^+$ as an element of the form $y_{\tilde{\mathbf{i}}}(a', b', c', d', e', f')$. If the minors in the Chamber Ansatz do not vanish, then we have found $a', b', c', d', e', f' \in \mathbb{R}^*$ such that $y = y_{\tilde{\mathbf{i}}}(a', b', c', d', e', f')$ as required.

We shall therefore need a description of the Chamber Ansatz in type G_2. If $w \in W$ and $i = 1, 2$, then Berenstein and Zelevinsky define $w\omega_i$ to be a *chamber weight of level* i, and define a corresponding 'generalised minor' $\Delta^{w\omega_i}$, which is a function on G. These minors reduce to usual minors of a matrix in type A. The generalised minors can be described (using section 6 of [1]) in the following way. Let $g \in G$, $w \in W$, and ω a dominant weight. Then there is a module V_ω for G of highest weight ω. Fix a vector $v_\omega \in V_\omega$ of highest weight ω, fix a reduced expression $w = s_{j_1}s_{j_2}\cdots s_{j_l}$ for w, and for $k = 1, \ldots, l$ set $b_k = \langle \alpha_{j_k}^\vee, s_{j_{k-1}} \cdots s_{j_1}\omega \rangle$. Then

$$v_{w\omega} := f_{j_l}^{(b_l)} f_{j_{l-1}}^{(b_{(l-1)})} \cdots f_{j_1}^{(b_1)} v_\omega$$

is an extremal weight vector of weight $w\omega$. Then $\Delta^{w\omega}(g)$ is defined to be the coefficient of v_ω in $g \cdot v_{w\omega}$. We have:

Theorem 5.1. *(Berenstein and Zelevinsky) Let $x \in U^+ \cap B^-\dot{w}_0B^-$, and suppose that $\mathbf{j} = (j_1, \ldots, j_N)$ is a reduced expression for w_0. Then*

$$\varepsilon(x) = y = y_{j_1}(a_1)y_{j_2}(a_2)\cdots y_{j_N}(a_N),$$

where $a_1, a_2, \ldots, a_N$ are given by:

$$a_{N+1-k} = \frac{1}{\Delta^{w_k\omega_{j_k}}(x)\Delta^{w_{k+1}\omega_{j_k}}(x)} \prod_{j \neq j_k} \Delta^{w_k\omega_j}(x)^{-A_{j,j_k}},$$

where $w_k = s_{j_N}s_{j_{N-1}}\cdots s_{j_k}$.

If we are in the situation of the Theorem, we write $\varepsilon^{\mathbf{j}}(x) = (a_1, a_2, \ldots, a_N)$.

Again using [1, Theorem 1.4], we can describe α using the Chamber Ansatz. Define $\Delta_-^{w\omega}(g)$ to be the coefficient of $v_{-\omega}$ in $g \cdot v_{w\omega}$, where $v_{w\omega}$ is the extremal weight vector of weight $w\omega$ as defined above.

Theorem 5.2. *(Berenstein and Zelevinsky) Let $y \in U^- \cap B^+\dot{w}_0B^+$, and suppose that $\mathbf{j} = (j_1, \ldots, j_N)$ is a reduced expression for w_0. Then*

$$\alpha(y) = x = x_{j_1}(a_1)x_{j_2}(a_2)\cdots x_{j_N}(a_N),$$

where $a_1, a_2, \ldots, a_N$ are given by:

$$a_{N+1-k} = \frac{1}{\Delta_-^{-w_k\omega_{j_k}}(y)\Delta_-^{-w_{k+1}\omega_{j_k}}(y)} \prod_{j \neq j_k} \Delta_-^{-w_k\omega_j}(y)^{-A_{j,j_k}},$$

where $w_k = s_{j_N}s_{j_{N-1}}\cdots s_{j_k}$.

If we are in the situation of this Theorem, we write $\alpha^{\mathbf{j}}(x) = (a_1, a_2, \ldots, a_N)$.

We can use a wiring diagram to describe these maps in type G_2, much as in type A. Recall that our two reduced expressions of w_0 are $\mathbf{i} = (1, 2, 1, 2, 1, 2)$ and $\tilde{\mathbf{i}} = (2, 1, 2, 1, 2, 1)$. We begin by calculating need $\varepsilon^{212121} = \varepsilon^{\tilde{\mathbf{i}}}$; this can be done using the Chamber Ansatz in Figure 5.

Suppose that a_j is one of the parameters in Theorem 5.1, and suppose that adjacent to the jth crossing from the left in Figure 5, chamber A is above, D below, and B and C on the same horizontal level. Let $\Delta(A)$, $\Delta(B)$, $\Delta(C)$, $\Delta(D)$

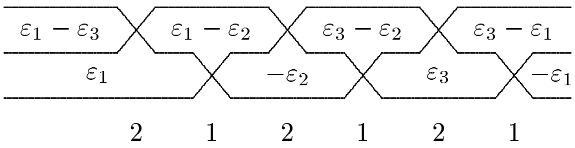

FIGURE 3. The Chamber Ansatz for ε^{212121}.

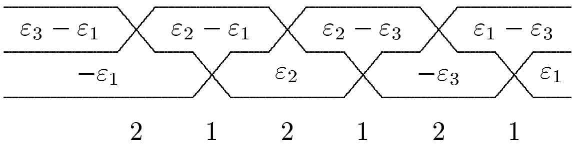

FIGURE 4. The Chamber Ansatz for α^{212121}.

be the generalized minors corresponding to these four chambers (taking the value 1 on unbounded chambers). Then Theorem 5.1 states that:

$$a_j = \frac{\Delta(A)^3\Delta(D)^3}{\Delta(B)\Delta(C)}.$$

We also need to calculate $\alpha^{\tilde{\mathbf{i}}} = \alpha^{212121}$, which can be done using the Chamber Ansatz in Figure 5. The value of the component at a crossing is calculated as for ε, but using this second picture.

6. THE MAP i_+

Note that our approach to parameterizing $\mathcal{B}^*$ in Section 4 was inherently asymmetric, suited to the map $i_- : U^- \cap B^+\dot{w}_0B^+ \to \mathcal{B}^*$. That is, the cells $D_{\mathbf{j}}(h)$ came naturally from cells $D^-_{\mathbf{j}}(h)$ defined in U^-. We now need to consider another set of cells in $\mathcal{B}^*$ coming from the isomorphism $i^+ : U^+ \cap B^-\dot{w}_0B^- \to \mathcal{B}^*$. Let us define

$$\begin{aligned} x_{\tilde{\mathbf{i}}}(a_1,\ldots,a_6) &:= x_2(a_1)x_1(a_2)x_2(a_3)x_1(a_4)x_2(a_5)x_1(a_6), \\ D^+_{\tilde{\mathbf{i}}}(h) &:= \{x_{\tilde{\mathbf{i}}}(a_1,\ldots,a_6) \mid a_i h(i) \in \mathbb{R}_{>0}\}. \end{aligned}$$

for any map $h : \{1,2,3,4,5,6\} \to \{\pm 1\}$. It is clear that two cells $D^+_{\tilde{\mathbf{i}}}(h)$ and $D^+_{\tilde{\mathbf{i}}}(h')$ lie in the same connected component of $U^+ \cap B^+\dot{w}_0B^+$ precisely if $D^-_{\tilde{\mathbf{i}}}(h)$ and $D^-_{\tilde{\mathbf{i}}}(h')$ lie in the same component, by symmetry between U^+ and U^-. Reading off from Figure 2 when this happens, we obtain the new Figure 5, which groups together all sequences of signs (determining maps $h : \{1,2,3,4,5,6\} \to \{\pm 1\}$) such that the corresponding $D^+_{\tilde{\mathbf{i}}}(h)$'s lie in the same connected component of $U^+ \cap B^-\dot{w}_0B^-$. We have labeled these connected components by letters A–K.

Our aim is now to identify which connected components of $U^+\cap B^-\dot{w}_0B^-$ labeled by A–K correspond to which components labeled 1–11 of $\mathcal{B}^*$ under the map i_+. To do this we simply need to evaluate the map ε^{212121} on 11 test points, one from each component of $U^+ \cap B^-\dot{w}_0B^-$.

It suffices to calculate ε^{212121} on elements x of the form $x = x_{\tilde{\mathbf{i}}}(a,b,c,d,e,f)$. From the Chamber Ansatz we have

$$\varepsilon(x_{\tilde{\mathbf{i}}}(a,b,c,d,e,f)) = y_{\tilde{\mathbf{i}}}(a',b',c',d',e',f'),$$

A	B	C	D
++++++	------	-+-+-+	+-+-+-

E	F	G	H
+-+-++	-+-+--	++-+-+	--+-+-
+-++--	-+--++	--++-+	++--+-
+---+-	-+++-+	-+---+	+-+++-
++----	--++++	----++	++++--
-++-+-	+--+-+	-+-++-	+-+--+
--+---	++-+++	---+--	+++-++
---++-	+++--+	-++---	+--+++
-----+	+++++-	+-----	-+++++

I		J	
++++-+	----+-	-+-+++	+-+---
++-+--	--+-++	+--++-	-++--+
++--++	--++--	-++++-	+----+
+-++++	-+----	---+-+	+++-+-

K	
+++---	---+++
++-++-	--+--+
++---+	--+++-
+-++-+	-+--+-
+--+--	-++-++
+---++	-+++--

FIGURE 5. The connected components of $U^+ \cap B^- \dot{w}_0 B^-$ in terms of the cells $D^+_{\tilde{\mathbf{i}}}(h)$.

where

$$
\begin{aligned}
a' &= \frac{1}{\Delta^{\varepsilon_1-\varepsilon_3}(x)\Delta^{\varepsilon_1-\varepsilon_2}(x)},\\
b' &= \frac{\Delta^{\varepsilon_1-\varepsilon_2}(x)}{\Delta^{\varepsilon_1}(x)\Delta^{-\varepsilon_2}(x)},\\
c' &= \frac{\Delta^{-\varepsilon_2}(x)^3}{\Delta^{\varepsilon_1-\varepsilon_2}(x)\Delta^{\varepsilon_3-\varepsilon_2(x)}},\\
d' &= \frac{\Delta^{\varepsilon_3-\varepsilon_2}(x)}{\Delta^{-\varepsilon_2}(x)\Delta^{\varepsilon_3}(x)},\\
e' &= \frac{\Delta^{\varepsilon_3}(x)^3}{\Delta^{\varepsilon_3-\varepsilon_2}(x)\Delta^{\varepsilon_3-\varepsilon_1}(x)},\\
f' &= \frac{\Delta^{\varepsilon_3-\varepsilon_1}(x)}{\Delta^{\varepsilon_3}(x)\Delta^{-\varepsilon_1}(x)}.
\end{aligned}
$$

By considering the usual action of the Chevalley generators of the Lie algebra in the two fundamental representations V_{ω_1} and V_{ω_2} of G_2, the action of $x_1(t)$,$x_2(t)$, $y_1(t)$,$y_2(t)$,$\dot{s}_1$ and $\dot{s}_2$ can be computed explicitly. We list below the relevant matrix coefficients for the element $x = x_{\tilde{\mathbf{i}}}(a,b,c,d,e,f)$.

$$
\begin{aligned}
\Delta^{\varepsilon_1}(x) &= 1,\\
\Delta^{-\varepsilon_3}(x) &= f+d+b,\\
\Delta^{-\varepsilon_2}(x) &= ed+eb+bc,\\
\Delta^{\varepsilon_2}(x) &= f^2ed+f^2eb+f^2bc+2\,bcdf+bcd^2,\\
\Delta^{\varepsilon_3}(x) &= bcd^2e,\\
\Delta^{-\varepsilon_1}(x) &= bcd^2ef,\\
\Delta^{\varepsilon_1-\varepsilon_3}(x) &= 1,\\
\Delta^{\varepsilon_1-\varepsilon_2}(x) &= e+c+a,\\
\Delta^{\varepsilon_2-\varepsilon_3}(x) &= f^3e+f^3c+f^3a+3\,f^2cd+3\,f^2ad+3\,f^2ab+3\,fd^2c+3\,fd^2a+\\
&\quad +6fabd+3\,fab^2+d^3c+d^3a+3\,abd^2+3\,ab^2d+ab^3,\\
\Delta^{\varepsilon_3-\varepsilon_2}(x) &= e^2d^3c+e^2d^3a+3\,e^2abd^2+3\,e^2ab^2d+e^2ab^3+3\,eab^2cd+2\,eab^3c\\
&\quad +ab^3c^2,\\
\Delta^{\varepsilon_2-\varepsilon_1}(x) &= f^3e^2d^3c+f^3e^2d^3a+3\,f^3e^2abd^2+3\,f^3e^2ab^2d+f^3e^2ab^3+\\
&\quad +3\,f^3eab^2cd+2\,f^3eab^3c+f^3ab^3c^2+3\,f^2eab^2cd^2+3\,f^2eab^3cd+\\
&\quad +3\,f^2ab^3c^2d+3\,ab^3c^2d^2f+ab^3c^2d^3,\\
\Delta^{\varepsilon_3-\varepsilon_1}(x) &= ab^3c^2d^3e.
\end{aligned}
$$

Applying the Chamber Ansatz, we obtain the components a', b', c', d', e', f':

$$
\begin{aligned}
a' &= \frac{1}{e+c+a},\\
b' &= \frac{e+c+a}{ed+eb+bc},\\
c' &= \frac{(ed+eb+bc)^3}{u\,(e+c+a)},\\
d' &= \frac{u}{bcd^2e\,(ed+eb+bc)},\\
e' &= \frac{e^2d^3c}{au},\\
f' &= \frac{ab}{def},
\end{aligned}
$$

where

$$u=\left(e^2d^3c+e^2d^3a+3\,e^2abd^2+3\,e^2ab^2d+e^2ab^3+3\,eab^2cd+2\,eab^3c+ab^3c^2\right).$$

The next step is to substitute values for a, b, c, d, e, f with specific combinations of signs, one choice from each component of Figure 5, ensuring that the matrix coefficients in the Chamber Ansatz do not vanish. It turns out that this can be achieved by setting $a=\pm1, b=\pm2, c=\pm3, d=\pm5, e=\pm7$ and $f=\pm11$. For any choice of signs of a, b, c, d, e, f, we obtain the signs of a', b', c', d', e', f', and thus the connected component using Figure 2. The resulting bijection between connected components of $U^+ \cap B^- \dot{w}_0 B^-$ and $\mathcal{B}^*$ is shown below.

$$
\begin{array}{cccc}
A \longleftrightarrow 1 & B \longleftrightarrow 2 & C \longleftrightarrow 3 & D \longleftrightarrow 4 \\
E \longleftrightarrow 5 & F \longleftrightarrow 6 & G \longleftrightarrow 7 & H \longleftrightarrow 8 \\
 & I \longleftrightarrow 10 & J \longleftrightarrow 9 & \\
 & \multicolumn{2}{c}{K \longleftrightarrow 11} &
\end{array} \tag{6.1}
$$

7. Calculation of the Euler Characteristics

We need to determine which connected components of $U^- \cap B^+ \dot{w}_0 B^+$ the $D_\sigma^-(h)$ belong to. If one point in such a $D_\sigma^-(h)$ lies in a particular connected component, then the whole of it does. So a similar approach to that in the previous section will work. We start with a general point in $D_\sigma^-(h) \subseteq U^- \cap B^+ \dot{w}_0 B^+$, apply α to it, to get a point in $U^+ \cap B^- \dot{w}_0 B^-$; we ensure this is of the form $x_{212121}(a', b', c', d', e', f')$ for some a', b', c', d', e', f'; then it lies in one of the components A–K of Figure 5; the bijection (6.1) above will then give us the connected component of $D_\sigma^-(h)$. Again, not every point in D_σ^- need be in the domain of α^{212121} but we can always find one which is. We use the Chamber Ansatz for α^{212121} as described above. Our computation is as follows:

The following are the components of α^{212121} applied to $y_{x21x12}(t_1, t_2, m_1, m_2)$ (provided this is well-defined).

$$
\begin{aligned}
a &= -{m_2}^{-1}, \\
b &= \frac{m_2}{t_1\, m_2 + m_1}, \\
c &= \frac{(t_1\, m_2 + m_1)^3}{m_2\, (-m_2\, t_2 + {m_1}^3)}, \\
d &= \frac{-m_2\, t_2 + {m_1}^3}{(t_1\, m_2 + m_1)\,(t_1\, {m_1}^2 + t_2)}, \\
e &= -\frac{\left(t_1\, {m_1}^2 + t_2\right)^3}{(-m_2\, t_2 + {m_1}^3)\, {t_2}^2}, \\
f &= \frac{t_2}{(t_1\, {m_1}^2 + t_2)\, t_1}.
\end{aligned}
$$

The following are the components of α^{212121} applied to $y_{12x21x}(t_1, m_1, m_2, t_2)$.

$$
\begin{aligned}
a &= {t_2}^{-1}, \\
b &= -{m_2}^{-1}, \\
c &= \frac{{m_2}^3}{3\, t_1\, m_2 + m_1}, \\
d &= \frac{3\, t_1\, m_2 + m_1}{m_2\, (2\, t_1\, m_2 + m_1)}, \\
e &= \frac{(2\, t_1\, m_2 + m_1)^3}{(3\, t_1\, m_2 + m_1)\, {t_1}^3}, \\
f &= -\frac{t_1}{2\, t_1\, m_2 + m_1}.
\end{aligned}
$$

The following are the components of α^{212121} applied to $y_{1x12x2}(t_1, m_1, t_2, m_2)$.

$$\begin{aligned}
a &= -{m_2}^{-1}, \\
b &= -\frac{m_2}{m_1\, m_2 + t_2}, \\
c &= -\frac{(m_1\, m_2 + t_2)^3}{m_2\left(t_1\, {m_2}^2 - {t_2}^3\right)}, \\
d &= \frac{t_1\, {m_2}^2 - {t_2}^3}{(m_1\, m_2 + t_2)\left(t_1\, m_2 + m_1\, {t_2}^2\right)}, \\
e &= \frac{\left(t_1\, m_2 + m_1\, {t_2}^2\right)^3}{\left(t_1\, {m_2}^2 - {t_2}^3\right) t_1\, {t_2}^3}, \\
f &= \frac{{t_2}^2}{t_1\, m_2 + m_1\, {t_2}^2}.
\end{aligned}$$

The following are the components of α^{212121} applied to $y_{xx1x1x}(t_1, t_2, t_3, m_1, t_4)$.

$$\begin{aligned}
a &= (t_2 + t_4)^{-1}, \\
b &= \frac{t_2 + t_4}{t_1\, t_2 - m_1\, t_4 + t_4\, t_1}, \\
c &= -\frac{(t_1\, t_2 - m_1\, t_4 + t_4\, t_1)^3}{(t_2 + t_4)\, t_4\, (-t_2\, t_3 + t_4\, {m_1}^3 t_2 - t_4\, t_3)}, \\
d &= -\frac{-t_2\, t_3 + t_4\, {m_1}^3 t_2 - t_4\, t_3}{(t_1\, t_2\, {m_1}^2 - t_3)(t_1\, t_2 - m_1\, t_4 + t_4\, t_1)}, \\
e &= \frac{\left(t_1\, t_2\, {m_1}^2 - t_3\right)^3 t_4}{(-t_2\, t_3 + t_4\, {m_1}^3 t_2 - t_4\, t_3)\, t_2\, {t_3}^2}, \\
f &= -\frac{t_3}{(t_1\, t_2\, {m_1}^2 - t_3)\, t_1}.
\end{aligned}$$

The following are the components of α^{212121} applied to $y_{1x1xxx}(t_1, m_1, t_2, t_3, t_4)$.

$$\begin{aligned}
a &= (t_2 + t_4)^{-1}, \\
b &= -\frac{t_2 + t_4}{m_1\, t_2 + m_1\, t_4 - t_4\, t_3}, \\
c &= -\frac{(m_1\, t_2 + m_1\, t_4 - t_4\, t_3)^3}{(t_2 + t_4)\left(t_1\, {t_2}^2 + 2\, t_1\, t_2\, t_4 + {t_4}^2 t_1 + t_2\, {t_3}^3 {t_4}^2\right)}, \\
d &= \frac{t_1\, {t_2}^2 + 2\, t_1\, t_2\, t_4 + {t_4}^2 t_1 + t_2\, {t_3}^3 {t_4}^2}{(m_1\, t_2 + m_1\, t_4 - t_4\, t_3)\left(t_1\, t_2 + t_4\, t_1 + t_2\, {t_3}^2 t_4\, m_1\right)}, \\
e &= -\frac{\left(t_1\, t_2 + t_4\, t_1 + t_2\, {t_3}^2 t_4\, m_1\right)^3}{\left(t_1\, {t_2}^2 + 2\, t_1\, t_2\, t_4 + {t_4}^2 t_1 + t_2\, {t_3}^3 {t_4}^2\right) t_1\, {t_2}^2 {t_3}^3 t_4}, \\
f &= \frac{t_2\, {t_3}^2 t_4}{t_1\, t_2 + t_4\, t_1 + t_2\, {t_3}^2 t_4\, m_1}.
\end{aligned}$$

The following are the components of α^{212121} applied to $y_{xxx2x2}(t_1, t_2, t_3, t_4, m_1)$.

$$
\begin{aligned}
a &= -\left(m_1 - t_2\right)^{-1},\\
b &= \frac{m_1 - t_2}{t_1\, m_1 + m_1\, t_3 - t_1\, t_2 - t_4},\\
c &= \frac{\left(t_1\, m_1 + m_1\, t_3 - t_1\, t_2 - t_4\right)^3}{\left(m_1 - t_2\right)\left(t_2\, t_3{}^3 m_1{}^2 - 3\, t_2\, t_3{}^2 t_4\, m_1 + 3\, t_2\, t_3\, t_4{}^2 - t_4{}^3\right)},\\
d &= \frac{t_2\, t_3{}^3 m_1{}^2 - 3\, t_2\, t_3{}^2 t_4\, m_1 + 3\, t_2\, t_3\, t_4{}^2 - t_4{}^3}{\left(t_1\, m_1 + m_1\, t_3 - t_1\, t_2 - t_4\right)\left(t_1\, t_2\, t_3{}^2 m_1 - 2\, t_1\, t_2\, t_3\, t_4 + t_4{}^2 t_1 + t_3\, t_4{}^2\right)},\\
e &= -\frac{\left(t_1\, t_2\, t_3{}^2 m_1 - 2\, t_1\, t_2\, t_3\, t_4 + t_4{}^2 t_1 + t_3\, t_4{}^2\right)^3}{\left(t_2\, t_3{}^3 m_1{}^2 - 3\, t_2\, t_3{}^2 t_4\, m_1 + 3\, t_2\, t_3\, t_4{}^2 - t_4{}^3\right) t_2\, t_3{}^3 t_4{}^3},\\
f &= \frac{t_3\, t_4{}^2}{\left(t_1\, t_2\, t_3{}^2 m_1 - 2\, t_1\, t_2\, t_3\, t_4 + t_4{}^2 t_1 + t_3\, t_4{}^2\right) t_1}.
\end{aligned}
$$

The following are the components of α^{212121} applied to $y_{x2x2xx}(t_1, t_2, m_1, t_3, t_4)$.

$$
\begin{aligned}
a &= -\left(m_1 - t_4\right)^{-1},\\
b &= \frac{m_1 - t_4}{t_1\, m_1 - t_2 - t_4\, t_1 - t_4\, t_3},\\
c &= -\frac{\left(t_1\, m_1 - t_2 - t_4\, t_1 - t_4\, t_3\right)^3}{\left(m_1 - t_4\right)\left(t_2{}^3 + 3\, t_2{}^2 t_3\, t_4 + 3\, t_2\, t_3{}^2 t_4{}^2 + t_4{}^2 m_1\, t_3{}^3\right)},\\
d &= -\frac{t_2{}^3 + 3\, t_2{}^2 t_3\, t_4 + 3\, t_2\, t_3{}^2 t_4{}^2 + t_4{}^2 m_1\, t_3{}^3}{\left(t_1\, m_1 - t_2 - t_4\, t_1 - t_4\, t_3\right)\left(t_1\, t_2{}^2 + 2\, t_1\, t_2\, t_3\, t_4 + t_4\, t_3{}^2 t_1\, m_1 - t_2\, t_3{}^2 t_4\right)},\\
e &= -\frac{\left(t_1\, t_2{}^2 + 2\, t_1\, t_2\, t_3\, t_4 + t_4\, t_3{}^2 t_1\, m_1 - t_2\, t_3{}^2 t_4\right)^3}{\left(t_2{}^3 + 3\, t_2{}^2 t_3\, t_4 + 3\, t_2\, t_3{}^2 t_4{}^2 + t_4{}^2 m_1\, t_3{}^3\right) t_2{}^3 t_3{}^3 t_4},\\
f &= -\frac{t_2\, t_3{}^2 t_4}{\left(t_1\, t_2{}^2 + 2\, t_1\, t_2\, t_3\, t_4 + t_4\, t_3{}^2 t_1\, m_1 - t_2\, t_3{}^2 t_4\right) t_1}.
\end{aligned}
$$

We now substitute in values for the t_i and the m_i in order to obtain points in the $D^-_\sigma(h)$ on which α^{212121} is defined. In the first three cases we set $t_1 = \pm 1$, $t_2 = \pm 2$, $m_1 = 3$, $m_2 = 5$, and in the last 4 cases, we set $t_1 = \pm 1$, $t_2 = \pm 2$, $t_3 = \pm 3$, $t_4 = \pm 5$, and $m_1 = 7$. In this way, we obtain the connected component for each $D^-_\sigma(h)$. To denote a choice of sign we write a list of 6 symbols, with $+$ or $-$ indicating the sign associated to an element j of $I(\sigma)$, 0 indicating an element of $J(\sigma)$, and $*$ indicating elements of $K(\sigma)$.

1. $y_{1x12x2}(t_1, m_1, t_2, m_2)$.

Sign choice	Signs of a, b, c, d, e, f	Connected Component
$0+*0+*$	$---+++$	$K \longleftrightarrow 11$
$0+*0-*$	$---+-+$	$J \longleftrightarrow 9$
$0-*0+*$	$--+-++$	$I \longleftrightarrow 10$
$0-*0-*$	$--+--+$	$K \longleftrightarrow 11$

2. $y_{xx1x1x}(t_1, t_2, t_3, m_1, t_4)$.

Sign choice	Signs of a, b, c, d, e, f	Connected Component
$++0+*+$	$+-+++-$	$H \longleftrightarrow 8$
$++0+*-$	$--+++-$	$K \longleftrightarrow 11$
$++0-*+$	$+-++++$	$I \longleftrightarrow 10$
$++0-*-$	$--++++$	$F \longleftrightarrow 6$
$+-0+*+$	$+--+-+$	$F \longleftrightarrow 6$
$+-0+*-$	$---+-+$	$J \longleftrightarrow 9$
$+-0-*+$	$+--+--$	$K \longleftrightarrow 11$
$+-0-*-$	$---+--$	$G \longleftrightarrow 7$
$-+0+*+$	$+-+---$	$J \longleftrightarrow 9$
$-+0+*-$	$--+---$	$E \longleftrightarrow 5$
$-+0-*+$	$+-+--+$	$H \longleftrightarrow 8$
$-+0-*-$	$--+--+$	$K \longleftrightarrow 11$
$--0+*+$	$+---++$	$K \longleftrightarrow 11$
$--0+*-$	$----++$	$G \longleftrightarrow 7$
$--0-*+$	$+---+-$	$E \longleftrightarrow 5$
$--0-*-$	$----+-$	$I \longleftrightarrow 10$

3. $y_{1x1xxx}(t_1, m_1, t_2, t_3, t_4)$.

Sign choice	Signs of a, b, c, d, e, f	Connected Component
$0+*+++$	$+--+-+$	$F \longleftrightarrow 6$
$0+*++-$	$---+-+$	$J \longleftrightarrow 9$
$0+*+-+$	$+-+--+$	$H \longleftrightarrow 8$
$0+*+--$	$--+--+$	$K \longleftrightarrow 11$
$0+*-++$	$+-++-+$	$K \longleftrightarrow 11$
$0+*-+-$	$--++-+$	$G \longleftrightarrow 7$
$0+*--+$	$+----+$	$J \longleftrightarrow 9$
$0+*---$	$-----+$	$E \longleftrightarrow 5$
$0-*+++$	$+--+++$	$H \longleftrightarrow 8$
$0-*++-$	$---+++$	$K \longleftrightarrow 11$
$0-*+-+$	$+-+-++$	$E \longleftrightarrow 5$
$0-*+--$	$--+-++$	$I \longleftrightarrow 10$
$0-*-++$	$+-++++$	$I \longleftrightarrow 10$
$0-*-+-$	$--++++$	$F \longleftrightarrow 6$
$0-*--+$	$+---++$	$K \longleftrightarrow 11$
$0-*---$	$----++$	$G \longleftrightarrow 7$

4. $y_{12x21x}(t_1, m_1, m_2, t_2)$.

Sign choice	Signs of a, b, c, d, e, f	Connected Component
$00+**+$	$+-+++-$	$H \longleftrightarrow 8$
$00+**-$	$--+++-$	$K \longleftrightarrow 11$
$00-**+$	$+--+--$	$K \longleftrightarrow 11$
$00-**-$	$---+--$	$G \longleftrightarrow 7$

5. $y_{xxx2x2}(t_1, t_2, t_3, t_4, m_1)$.

Sign choice	Signs of a, b, c, d, e, f	Connected Component
$++ +0 + *$	$-++ + -+$	$F \longleftrightarrow 6$
$++ +0 - *$	$-++ + ++$	$H \longleftrightarrow 8$
$++ -0 + *$	$--+ + --$	$I \longleftrightarrow 10$
$++ -0 - *$	$--+ + +-$	$K \longleftrightarrow 11$
$+- +0 + *$	$-+- - -+$	$G \longleftrightarrow 7$
$+- +0 - *$	$-+- + --$	$F \longleftrightarrow 6$
$+- -0 + *$	$--- + ++$	$K \longleftrightarrow 11$
$+- -0 - *$	$--- + -+$	$J \longleftrightarrow 9$
$-+ +0 + *$	$-++ - ++$	$K \longleftrightarrow 11$
$-+ +0 - *$	$-++ - -+$	$J \longleftrightarrow 9$
$-+ -0 + *$	$--+ - +-$	$H \longleftrightarrow 8$
$-+ -0 - *$	$--+ - --$	$E \longleftrightarrow 5$
$-- +0 + *$	$-+- - --$	$I \longleftrightarrow 10$
$-- +0 - *$	$-+- - +-$	$K \longleftrightarrow 11$
$-- -0 + *$	$--- - -+$	$E \longleftrightarrow 5$
$-- -0 - *$	$--- + --$	$G \longleftrightarrow 7$

6. $y_{x2x2xx}(t_1, t_2, m_1, t_3, t_4)$.

Sign choice	Signs of a, b, c, d, e, f	Connected Component
$+0 + * + +$	$--+ + --$	$I \longleftrightarrow 10$
$+0 + * + -$	$-+- + --$	$F \longleftrightarrow 6$
$+0 + * - +$	$-++ + --$	$K \longleftrightarrow 11$
$+0 + * - -$	$--- + --$	$G \longleftrightarrow 7$
$+0 - * + +$	$--+ + ++$	$F \longleftrightarrow 6$
$+0 - * + -$	$-+- + ++$	$J \longleftrightarrow 9$
$+0 - * - +$	$-++ + ++$	$H \longleftrightarrow 8$
$+0 - * - -$	$--- + ++$	$K \longleftrightarrow 11$
$-0 + * + +$	$--+ - +-$	$H \longleftrightarrow 8$
$-0 + * + -$	$-+- - +-$	$K \longleftrightarrow 11$
$-0 + * - +$	$-++ - +-$	$E \longleftrightarrow 5$
$-0 + * - -$	$--- - +-$	$I \longleftrightarrow 10$
$-0 - * + +$	$--+ - -+$	$K \longleftrightarrow 11$
$-0 - * + -$	$-+- - -+$	$G \longleftrightarrow 7$
$-0 - * - +$	$-++ - -+$	$J \longleftrightarrow 9$
$-0 - * - -$	$--- - -+$	$E \longleftrightarrow 5$

7. $y_{x21x12}(t_1, t_2, m_1, m_2)$.

Sign choice	Signs of a, b, c, d, e, f	Connected Component
$+00 + **$	$-++ + -+$	$F \longleftrightarrow 6$
$+00 - **$	$-++ + --$	$K \longleftrightarrow 11$
$-00 + **$	$--- + ++$	$K \longleftrightarrow 11$
$-00 - **$	$--- + +-$	$E \longleftrightarrow 5$

We now have a decomposition of each connected component of $\mathcal{B}^*$ into a disjoint union of subsets of form $D_\sigma(h)$. This allows us to calculate the Euler characteristic (for compactly supported cohomology) of each connected component X as an alternating sum:

$$\chi(X) = \sum_{c=0}^{\dim(X)} (-1)^c n_c,$$

where n_c is the number of subsets of form $D_\sigma(h)$ of codimension c contained in X (note that the dimension of X is even). Since the components X are smooth (open in $\mathcal{B}$) this compactly supported Euler characteristic coincides with the usual one by Poincaré duality. We note that each $D_\sigma(h)$ has codimension at most 2.

All the required information is displayed in Figure 6, using a format similar to that used in [6], grouping the subsets of form $D_\sigma(h)$ by connected component. We denote each $D_\sigma(h)$ by the sign choice string listed above (first column); note that the number of 0's gives the codimension.

Remarks 7.1. Fix σ such that D_σ is of codimension 2 (cases 1–3). Then, as $|I(\sigma)| = 2$ there are $2^2 = 4$ possible choices of sign, giving rise to 4 subsets $D_\sigma(h)$. Two of these lie in component 11, and the other two lie in either components 5 and 6, components 7 and 8, or components 9 and 10 respectively.

Fix σ such that D_σ is of codimension 1 (cases 4–7). Then, as $|I(\sigma)| = 4$ there are $2^4 = 16$ possible choices of sign, giving rise to 16 subsets $D_\sigma(h)$. Each connected component 5–10 contains precisely two of these subsets, while connected component 11 contains precisely 4 of these.

Theorem 7.2. *The Euler characteristic of each connected component of $\mathcal{B}^*$ is given in Table II. We also give, for each connected component, and for codimension* 0, 1, *and* 2, *the number of subsets of the form $D_\sigma(h)$ of that codimension contained in that component. We thus conclude that there are* 10 *connected components of Euler characteristic* 1 *and one of Euler characteristic* 2, *confirming a total Euler characteristic of $\mathcal{B}^*$ of* 12, *the sum of these.*

Table II. Euler characteristics of the connected components.

Connected component	Codim 0	Codim 1	Codim 2	Euler characteristic
1	1	0	0	1
2	1	0	0	1
3	1	0	0	1
4	1	0	0	1
5	8	8	1	1
6	8	8	1	1
7	8	8	1	1
8	8	8	1	1
9	8	8	1	1
10	8	8	1	1
11	12	16	6	2

1	++++++	−−−−−−	2
3	−+−+−+	+−+−+−	4

5	++−+−+	−−++−+	−−+−+−	++−−+−	6
	−+−−−+	−−−−++	+−+++−	++++−−	
	−+−++−	−−−+−−	+−+−−+	+++−++	
	−++−−−	+−−−−−	+−−+++	−+++++	
	−+0+∗−	−−0−∗+	++0−∗−	+−0+∗+	
	0+∗−−−	0−∗+−+	0+∗+++	0−∗−+−	
	−+−0−∗	−−−0+∗	+++0+∗	+−+0−∗	
	−0+∗−+	−0−∗−−	+0+∗+−	+0−∗++	
	−00−∗∗		+00+∗∗		

7	+−+−++	+−++−−	−+−+−−	−+−−++	8
	+−−−+−	++−−−−	−+++−+	−−++++	
	−++−+−	−−+−−−	+−−+−+	++−+++	
	−−−++−	−−−−−+	+++−−+	+++++−	
	+−0−∗−	−−0+∗−	++0+∗+	−+0−∗+	
	0+∗−+−	0−∗−−−	0+∗+−+	0−∗+++	
	+−+0+∗	−−−0−∗	+++0−∗	−+−0+∗	
	+0+∗−−	−0−∗+−	+0−∗−+	−0+∗++	
	00−∗∗−		00+∗∗+		

9	−+−+++	+−−++−	−++++−	−−−+−+
	+−+−−−	−++−−+	+−−−−+	+++−+−
	+−0+∗−	−+0+∗+	0+∗++−	0+∗−−+
	+−−0−∗	−++0−∗	+0−∗+−	−0−∗−+
	0+∗0−∗			

10	++++−+	++−+−−	++−−++	+−++++
	−−−−+−	−−+−++	−−++−−	−+−−−−
	++0−∗+	−−0−∗−	0−∗+−−	0−∗−++
	++−0+∗	−−+0+∗	+0+∗++	−0+∗−−
	0−∗0+∗			

11	−−−+++	−++−++	+−−−++	+−++−+
	−−+−−+	++−−−+	+++−−−	+−−+−−
	−+++−−	−+−−+−	++−++−	−−+++−
	++0+∗−	+−0−∗+	−+0−∗−	−−0+∗+
	0+∗+−−	0+∗−++	0−∗++−	0−∗−−+
	++−0−∗	+−−0+∗	−++0+∗	−−+0−∗
	+0+∗−+	+0−∗−−	−0+∗+−	−0−∗++
	+00−∗∗	−00+∗∗	00+∗∗−	00−∗∗+
	0+∗0+∗	0−∗0−∗		

FIGURE 6. The connected components of $\mathcal{B}^*$ with their cell decompositions.

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University of Leicester
E-mail address: R.Marsh@mcs.le.ac.uk

DPMMS, Cambridge University and Newnham College
E-mail address: rietsch@dpmms.cam.ac.uk