Singular Dirac Systems and Sturm-Liouville Problems Nonlinear in the Spectral Parameter

Harald Schmid and Christiane Tretter*

NWF I - Mathematik, Universität Regensburg, 93040 Regensburg, Germany

Dept. of Mathematics and Computer Science, University of Leicester, University Road, Leicester LE1 7RH, United Kingdom

E-mail: harald.schmid@mathematik.uni-regensburg.de; c.tretter@mcs.le.ac.uk

For the Dirac operator with spherically symmetric potential $V:(0,\infty)\to\mathbb{R}$ we investigate the problem whether the boundary points of the essential spectrum are accumulation points of discrete eigenvalues or not. Our main result shows that the accumulation of such eigenvalues is essentially determined by the asymptotic behaviour of V at 0 and ∞. We obtain this result by using a Levinson type theorem for asymptotically diagonal systems depending on some parameter, a comparison theorem for the principal solutions of singular Dirac systems, and some criteria on the eigenvalue accumulation (respectively, non-accumulation) of λ-nonlinear singular Sturm-Liouville problems.

1. INTRODUCTION

In this paper we study the Dirac operator

$$H = -\mathrm{i}\,\alpha\cdot\nabla+\beta+V$$

in $\mathrm{L}^2(\mathbb{R}^3)^4$ with a spherically symmetric potential V. Here the units are chosen such that $\hbar = m = c = 1$, where m is the mass of the particle and c the velocity of light. Under some assumptions on V, the Dirac operator H is self-adjoint on the first order Sobolev space $\mathrm{H}^1(\mathbb{R}^3)^4$, and the essential spectrum of H coincides with the essential spectrum of the free Dirac operator $H_0 = -\mathrm{i}\,\alpha\cdot\nabla+\beta$ (see [11]):

$$\sigma_{ess}(H) = \sigma_{ess}(H_0) = \mathbb{R}\setminus(-1,1).$$

* The authors gratefully acknowledge the support for this work by the German Research Foundation, DFG, under Grant TR 368/4.

Since V is spherically symmetric, H can be decomposed into a direct sum of radial Dirac operators H_κ of the form

$$H_\kappa y(x) = Jy'(x) + \begin{pmatrix} -1+V(x) & \frac{\kappa}{x} \\ \frac{\kappa}{x} & 1+V(x) \end{pmatrix} y(x), \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

which have the same essential spectrum as H.

Our aim is to find criteria for the potential V which allow to decide whether the endpoints -1 and 1 of the gap $(-1,1)$ in the essential spectrum of H_κ are accumulation points of eigenvalues. We will show that e.g. 1 is an accumulation point of discrete eigenvalues of H_κ if and only if the differential equation

$$\left(\frac{x^{-2\kappa}u'(x)}{2-V(x)}\right)' - x^{-2\kappa}V(x)u(x) = 0, \quad x \in (0,\infty),$$

is oscillatory at ∞. It then follows by Sturm's Comparison Theorem that accumulation or non-accumulation occur if

$$\limsup_{x\to\infty} x^2V(x) < -\frac{1}{8}(2\kappa+1)^2 \quad \text{or} \quad \liminf_{x\to\infty} x^2V(x) > -\frac{1}{8}(2\kappa+1)^2,$$

respectively, i.e., $-\frac{1}{8}(2\kappa+1)^2$ is the critical value for the so-called coupling constant.

An analogous result has been established by Griesemer and Lutgen in [3], but under much stronger assumptions on the potential V requiring, e.g., that V is continuously differentiable and bounded at 0, whereas in the present paper it is only assumed that the potential V is locally essentially bounded and that V "behaves like" a Coulomb potential $\frac{\rho}{x}$, $0 \le |\rho| < \frac{1}{2}\sqrt{3}$, in a neighbourhood of 0.

In order to derive this criterion, we establish results for two different problems which are of independent interest.

First, in Section 3, we study singular Dirac systems

$$Jy'(x) + Q(x,\lambda)y(x) = 0, \quad x \in \Omega,$$

on an interval $\Omega = (a,b]$ which depend on parameter λ varying in some interval $\Lambda \subset \mathbb{R}$ with

$$Q(x,\lambda) = r(x)A + B(x,\lambda), \quad (x,\lambda) \in \Omega \times \Lambda.$$

Here r is locally integrable, A is a hermitian 2×2 matrix and B is an integrable function with values in the hermitian 2×2 matrices satisfying some additional assumptions. We prove that such a singular Dirac system possesses a fundamental matrix of a special form depending continuously both on x and on λ which allows us to select the principal solutions. To show this, we develop a Levinson type theorem for asymptotically diagonal systems of first order differential equations depending on a parameter in Section 2.

Secondly, in Section 4, we consider singular Sturm–Liouville problems

$$\begin{aligned}(p(x,\lambda)u'(x))' - q(x,\lambda)u(x) = 0, \quad x \in I,\\ \alpha(\lambda)u(a) + \beta(\lambda)v(a) = 0,\end{aligned}$$

on an interval $I = [a, b)$ which depend nonlinearly on a parameter $\lambda \in \Lambda = [\omega, \nu)$. Here the coefficients p and q are real-valued functions on $I \times \overline{\Lambda}$, $p > 0$, such that $p(\,\cdot\,,\lambda)^{-1}$, $q(\,\cdot\,,\lambda)$ are locally integrable on I for all $\lambda \in \overline{\Lambda}$ and some additional conditions are satisfied. For these problems we derive theorems about the accumulation and non-accumulation of eigenvalues at the right endpoint ν of Λ, and in the latter case we give an estimate for the number of eigenvalues in Λ. As a result, if the above problem satisfies certain monotonicity conditions (with regard to the coefficients of the differential equation and of the boundary conditions), then ν is a point of accumulation of eigenvalues if and only if the limit equation for $\lambda = \nu$ is oscillatory.

The results of Section 4 should be compared with analogous statements proved by Lutgen in [5] by using "locally" a technique inspired by the earlier work [6] which goes back to ideas of Glazman for the λ–linear case. The main difference to [5] is that we do not require the coefficients p and q to be continuous, and our monotonicity assumptions for the theorem about non-accumulation of eigenvalues are weaker.

When considering the radial Dirac operators H_κ, the result of Section 3 enables us to select a solution y_1 of $H_\kappa y = \lambda y$ which is square-integrable on $(0, a]$ for some positive a. By means of the transformation

$$y(x) = \begin{pmatrix} x^{\kappa} v(x) \\ x^{-\kappa} u(x) \end{pmatrix}, \quad x \in [a, \infty),$$

we arrive at a Sturm–Liouville differential equation for u of the form mentioned above, and the corresponding boundary condition is induced by an interface condition matching y_1 and y at the point a. Now the results of Section 4 can be applied and yield the desired criterion for the accumulation of eigenvalues for H_κ.

2. ASYMPTOTICALLY DIAGONAL SYSTEMS DEPENDING ON A PARAMETER

In this section we consider *asymptotically diagonal* $n \times n$ systems of first order differential equations

$$y'(x) = (D(x) + C(x,\lambda))\, y(x), \quad x \in \Omega, \tag{1}$$

on an interval $\Omega = [a,b)$, $-\infty < a < b \leq \infty$, which depend on a parameter λ varying in some interval $\Lambda \subset \mathbb{R}$. We assume that $D \in \mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathrm{M_n}(\mathbb{C}))$ is a diagonal matrix valued function,

$$D(x) = \operatorname{diag}(\alpha_1(x), \ldots, \alpha_n(x)), \quad x \in \Omega,$$

and that the functions $C(\,\cdot\,,\lambda) \in \mathrm{L}^1(\Omega, \mathrm{M_n}(\mathbb{C}))$ satisfy

$$\int_\Omega |C(t,\mu) - C(t,\lambda)|\, dt \to 0, \quad \mu \to \lambda,$$

for all $\lambda \in \Lambda$. Here $\mathrm{M_n}(\mathbb{C})$ denotes the space of all complex $n \times n$ matrices, and $|\cdot|$ the maximum norm in $\mathrm{M_n}(\mathbb{C})$. Moreover, with

$$K_{i,j}(s,t) := \int_s^t \operatorname{Re}\,(\alpha_i(x) - \alpha_j(x))\, dx, \quad a \leq s < t < b, \tag{2}$$

for $i, j \in \{1, \ldots, n\}$, we assume that D satisfies the following *dichotomy condition*:

(D) There exists a constant $\rho > 0$ such that for all $i, j \in \{1, \ldots, n\}$ $(i \neq j)$ either of the two inequalities $K_{i,j}(s,t) \geq -\rho$ or $K_{i,j}(s,t) \leq \rho$ holds for all $s \leq t$ in Ω.

For a fixed $\lambda \in \Lambda$, a function $y : \Omega \longrightarrow \mathbb{C}^n$ is called a *solution* of (1), if (every component of) y is absolutely continuous and (1) holds almost everywhere (a.e.) in Ω. Further, a *fundamental matrix* of (1) is a function $Y : \Omega \longrightarrow \mathrm{M_n}(\mathbb{C})$ with the property that every solution y of this system can be expressed as $y(x) = Y(x)c$, $x \in \Omega$, with some vector $c \in \mathbb{C}^n$. By the Levinson Theorem (see [2, Theorem 1.3.1]), for every $\lambda \in \Lambda$ there exists a fundamental matrix $Y(\,\cdot\,,\lambda)$ of (1) which has the form

$$Y(x,\lambda) = H(x,\lambda)\, \mathrm{e}^{\Delta(x)}, \quad x \in \Omega.$$

Here

$$\Delta(x) := \operatorname{diag}(\beta_1(x), \ldots, \beta_n(x))$$

and

$$\beta_i(x) := \gamma_i + \int_a^x \alpha_i(t)\,dt, \quad x \in \Omega,$$

with some constants $\gamma_i \in \mathbb{C}$, $i \in \{1, \dots, n\}$, and $H(\,\cdot\,, \lambda) \in \mathrm{AC}(\Omega, \mathrm{M_n}(\mathbb{C}))$ satisfies $\lim_{x \to b} H(x, \lambda) = I$, where $\mathrm{AC}(\Omega, \mathrm{M_n}(\mathbb{C}))$ denotes the class of matrix valued functions which are absolutely continuous on Ω. In the following we will prove that there even exist fundamental matrices of (1) with the additional property that H (and hence Y) is continuous on $\Omega \times \Lambda$.

Theorem 2.1. *The system* (1) *has a fundamental matrix*

$$Y(x, \lambda) = H(x, \lambda)\, \mathrm{e}^{\Delta(x)}, \quad x \in \Omega,$$

where $H(\,\cdot\,, \lambda) \in \mathrm{AC}(\Omega, \mathrm{M_n}(\mathbb{C}))$, $\lim_{x \to b} H(x, \lambda) = I$ *and*

$$\max_{x \in \Omega} |H(x, \mu) - H(x, \lambda)| \to 0, \quad \mu \to \lambda,$$

holds for all $\lambda \in \Lambda$.

Proof. We begin with some notations and preparations which will be useful in this proof. First, let

$$\phi_{i,j}(x) := \exp\left(\beta_i(x) - \beta_j(x)\right), \quad x \in \Omega, \quad i, j \in \{1, \dots, n\}.$$

If we fix an index $j \in \{1, \dots, n\}$, then hypothesis (D) implies that for each $i \in \{1, \dots, n\}$ exactly one of the following conditions holds (compare [2, Section 1.3]):

(a) $K_{i,j}(s, t) \geq -\sigma$ for $a \leq s \leq t < b$, or

(b) $K_{i,j}(s, t) \leq \sigma$ for $a \leq s \leq t < b$ and $K_{i,j}(a, t) \to -\infty$ as $t \to b$,

where $\sigma > 0$ is a constant which may be different from ρ. Hence, we can divide the integers $i \in \{1, \dots, n\}$ into two classes according to whether (a) or (b) holds. If $i \in \{1, \dots, n\}$ satisfies (a), then

$$\left| \frac{\phi_{i,j}(x)}{\phi_{i,j}(t)} \right| = \mathrm{e}^{-K_{i,j}(x,t)} \leq \mathrm{e}^{\sigma}, \quad a \leq x \leq t < b, \tag{3}$$

and if $i \in \{1, \dots, n\}$ satisfies (b), then

$$\left| \frac{\phi_{i,j}(x)}{\phi_{i,j}(t)} \right| = \mathrm{e}^{K_{i,j}(t,x)} \leq \mathrm{e}^{\sigma}, \quad a \leq t \leq x < b. \tag{4}$$

Moreover, in case of (b) we have

$$|\phi_{i,j}(x)| = \exp\left(\mathrm{Re}\,(\gamma_i - \gamma_j) + K_{i,j}(a,x)\right) \to 0, \quad x \to b. \tag{5}$$

Further, let P_j, Q_j be the projection matrices

$$P_j := \mathrm{diag}\,(\psi_{1,j}, \dots, \psi_{n,j}), \quad Q_j := I - P_j,$$

where $\psi_{i,j} := 1$ in case of (b) and $\psi_{i,j} := 0$ otherwise. Now, if we define

$$\Phi_j(x) := \mathrm{diag}\,(\phi_{1,j}(x), \dots, \phi_{n,j}(x)), \quad x \in \Omega,$$

then $\mathrm{e}^{\beta_j(x)}\Phi_j(x) = \Phi(x) := \mathrm{e}^{\Delta(x)}$ for all $x \in \Omega$, and from (5) it follows that $\Phi_j(x)P_j \to 0$ as $x \to b$. In addition, the estimates (3) and (4) imply

$$|\Phi_j(x)P_j\Phi_j(t)^{-1}| \le \mathrm{e}^{\sigma}, \quad a \le t \le x < b, \tag{6}$$

$$|\Phi_j(x)Q_j\Phi_j(t)^{-1}| \le \mathrm{e}^{\sigma}, \quad a \le x \le t < b. \tag{7}$$

Moving on to the main part of the proof, we will first verify the assertion for all λ in a compact subinterval Λ_0 of Λ. In this case, there exists a point $c \in \Omega$ such that

$$\int_c^b |C(t,\lambda)|\, dt \le \frac{1}{2}\,\mathrm{e}^{-\sigma}, \quad \lambda \in \Lambda_0. \tag{8}$$

Let us fix some $\lambda \in \Lambda_0$ and consider a particular $j \in \{1, \dots, n\}$. In [2, Section 1.4] it is proved by the method of successive approximations that the integral equation

$$\begin{aligned} h(x) = e_j + \Phi_j(x) &\int_c^x P_j\Phi_j(t)^{-1}C(t,\lambda)h(t)\, dt \\ - \Phi_j(x) &\int_x^b Q_j\Phi_j(t)^{-1}C(t,\lambda)h(t)\, dt, \quad x \in [c,b), \end{aligned}$$

has an absolutely continuous solution $h_j(\,\cdot\,,\lambda)$ with $\lim_{x\to b} h_j(x,\lambda) = e_j$, where e_j denotes the j-th unit vector in $\mathbb{C}^n$. If we define

$$y_j(x,\lambda) := \mathrm{e}^{\beta_j(x)}h_j(x,\lambda), \quad x \in [c,b),$$

then $y(\,\cdot\,,\lambda)$ is an absolutely continuous function on $[c,b)$ which solves the integral equation

$$\begin{aligned} y_j(x,\lambda) = \Phi(x)e_j + \Phi(x) &\int_c^x P_j\Phi(t)^{-1}C(t,\lambda)y_j(t,\lambda)\, dt \\ - \Phi(x) &\int_x^b Q_j\Phi(t)^{-1}C(t,\lambda)y_j(t,\lambda)\, dt, \quad x \in [c,b). \end{aligned}$$

Since $\Phi'(x) = D(x)\Phi(x)$ and $P_j + Q_j = I$, we obtain

$$\begin{aligned} y_j'(x,\lambda) = D(x)\Phi(x)e_j \; &+ \; D(x)\Phi(x)\int_c^x P_j\Phi(t)^{-1}C(t,\lambda)y_j(t,\lambda)\,dt \\ &+ \; \Phi(x)P_j\Phi(x)^{-1}C(x,\lambda)y_j(x,\lambda) \\ &- \; D(x)\Phi(x)\int_x^b Q_j\Phi(t)^{-1}C(t,\lambda)y_j(t,\lambda)\,dt \\ &+ \; \Phi(x)Q_j\Phi(x)^{-1}C(x,\lambda)y_j(x,\lambda) \\ = (D(x) + C(x,\lambda))\, y_j(x,\lambda), \quad & x \in [c,b). \end{aligned}$$

Summarizing, the functions $y_j(\,\cdot\,,\lambda)$, $j \in \{1,\dots,n\}$, are solutions of the differential equation (1) on the interval $[c,b)$ for all $\lambda \in \Lambda_0$. If we set

$$H(x,\lambda) := (h_1(x,\lambda),\dots,h_n(x,\lambda)), \quad (x,\lambda) \in [c,b) \times \Lambda_0,$$

then $H(\,\cdot\,,\lambda) \in \mathrm{AC}([c,b),\mathrm{M_n}(\mathbb{C}))$ and $\lim_{x\to b} H(x,\lambda) = I$ for all $\lambda \in \Lambda_0$. In particular, a Neumann series argument shows that for every $\lambda \in \Lambda_0$ there exists a point $\xi(\lambda) \in [c,b)$ such that $H(\xi(\lambda),\lambda)$ is invertible. Now, if we define

$$Y(x,\lambda) := (y_1(x,\lambda),\dots,y_n(x,\lambda)), \quad (x,\lambda) \in [c,b) \times \Lambda_0,$$

then $Y(\,\cdot\,,\lambda)$ is a solution of the matrix differential equation

$$Y'(x,\lambda) = (D(x) + C(x,\lambda))\, Y(x,\lambda), \quad x \in [c,b), \tag{9}$$

which has the form $Y(x,\lambda) = H(x,\lambda)\,\mathrm{e}^{\Delta(x)}$. Moreover, since $Y(\xi(\lambda),\lambda)$ is invertible, $Y(\,\cdot\,,\lambda)$ is actually a fundamental matrix of (9) for all $\lambda \in \Lambda_0$.

Now we prove that $H(\,\cdot\,,\lambda)$ depends continuously on $\lambda \in \Lambda_0$, that is,

$$\max_{x\in[c,b)} |H(x,\mu) - H(x,\lambda)| \to 0, \quad \mu \to \lambda.$$

Let μ, $\lambda \in \Lambda_0$ and $j \in \{1,\dots,n\}$. Further, denote by $\|\cdot\|_0$ the maximum norm in the space of bounded functions $f : [c,b) \longrightarrow \mathbb{C}^{\mathrm{n}}$. From

$$\begin{aligned} h_j(x,\mu) - h_j(x,\lambda) = \Phi_j(x) &\int_c^x P_j\Phi_j(t)^{-1}C(t,\mu)\,(h_j(t,\mu) - h_j(t,\lambda))\; dt \\ - \Phi_j(x) &\int_x^b Q_j\Phi_j(t)^{-1}C(t,\mu)\,(h_j(t,\mu) - h_j(t,\lambda))\; dt \\ + \Phi_j(x) &\int_c^x P_j\Phi_j(t)^{-1}\,(C(t,\mu) - C(t,\lambda))\, h_j(t,\lambda)\, dt \\ - \Phi_j(x) &\int_x^b Q_j\Phi_j(t)^{-1}\,(C(t,\mu) - C(t,\lambda))\, h_j(t,\lambda)\, dt \end{aligned}$$

and the estimates (6), (7), (8) we deduce

$$\begin{aligned}|h_j(x,\mu)-h_j(x,\lambda)| &\le \mathrm{e}^{\sigma}\|h_j(\,\cdot\,,\mu)-h_j(\,\cdot\,,\lambda)\|_0\int_c^b |C(t,\mu|\,dt\\ &\qquad + \mathrm{e}^{\sigma}\|h_j(\,\cdot\,,\lambda)\|_0\int_c^b |C(t,\mu)-C(t,\lambda)|\,dt\\ &\le \frac{1}{2}\,\|h_j(\,\cdot\,,\mu)-h_j(\,\cdot\,,\lambda)\|_0\\ &\qquad + \mathrm{e}^{\sigma}\|h_j(\,\cdot\,,\lambda)\|_0\int_c^b |C(t,\mu)-C(t,\lambda)|\,dt\end{aligned}$$

for all $x \in [c,b)$. This yields

$$\|h_j(\,\cdot\,,\mu)-h_j(\,\cdot\,,\lambda)\|_0 \le 2\,\mathrm{e}^{\sigma}\|h_j(\,\cdot\,,\lambda)\|_0\int_c^b |C(t,\mu)-C(t,\lambda)|\,dt$$

for all μ, $\lambda \in \Lambda_0$. Since $\int_c^b |C(t,\mu)-C(t,\lambda)|\,dt \to 0$ as $\mu \to \lambda$ due to our basic assumptions, we conclude that $\|h_j(\,\cdot\,,\mu)-h_j(\,\cdot\,,\lambda)\|_0 \to 0$ for all $j \in \{1,\dots,n\}$, and this implies $\max_{x\in[c,b)}|H(x,\mu)-H(x,\lambda)| \to 0$ for $\mu \to \lambda$.

In the next step we extend the fundamental matrices $Y(\,\cdot\,,\lambda)$ to the whole interval Ω. Note that the matrix function $Y(c,\,\cdot\,)$ is continuous on Λ_0. Hence, by the Existence and Uniqueness Theorem (see, for example, [14, Theorem 2.1]), there exist fundamental matrices $\Psi(\,\cdot\,,\lambda)$ of (1) on $[a,c]$ satisfying the initial condition $\Psi(c,\lambda) = Y(c,\lambda)$ for all $\lambda \in \Lambda_0$ and $\max_{x\in[a,c]}|\Psi(x,\mu)-\Psi(x,\lambda)| \to 0$ as $\mu \to \lambda$. If we define $H(\,\cdot\,,\lambda)$ on the interval $[a,c)$ by

$$H(x,\lambda) := \Psi(x,\lambda)\,\mathrm{e}^{-\Delta(x)}, \quad x \in [a,c),$$

then, for all $\lambda \in \Lambda_0$, $H(\,\cdot\,,\lambda)$ is an absolutely continuous function on Ω, and H has the properties stated in Theorem 2.1. In addition, $Y(x,\lambda) = H(x,\lambda)\,\mathrm{e}^{\Delta(x)}$ is a fundamental matrix of (1) on $\Omega \times \Lambda_0$, which proves Theorem 2.1 for all λ in a compact subinterval $\Lambda_0 \subset \Lambda$.

Finally, it remains to show the assertion in case of a non-compact interval $\Lambda \subset \mathbb{R}$. For this purpose, we have to investigate the problem of uniqueness of the fundamental matrices in Theorem 2.1. Denote by $G(\Delta)$ the set of all matrices $\Gamma \in \mathrm{M_n}(\mathbb{C})$ which satisfy $\lim_{x\to b}\mathrm{e}^{\Delta(x)}\Gamma\,\mathrm{e}^{-\Delta(x)} = I$. Obviously, $G(\Delta)$ is a group with respect to matrix multiplication.

Proposition 2.1. *For a fixed $\lambda \in \Lambda$, let $Y_1(x) = H_1(x)\,\mathrm{e}^{\Delta(x)}$ be a fundamental matrix of* (1) *such that $\lim_{x\to b} H_1(x) = I$. Then $Y_2(x) = H_2(x)\,\mathrm{e}^{\Delta(x)}$ is a fundamental matrix of* (1) *satisfying $\lim_{x\to b} H_2(x) = I$ if* *and only if there exists a matrix* $\Gamma \in G(\Delta)$ *such that* $Y_2(x) = Y_1(x)\,\Gamma$ *for all* $x \in \Omega$.

Proof. If Y_2 is such a fundamental matrix, then there exists an invertible matrix $\Gamma \in \mathrm{M_n}(\mathbb{C})$ with the property that $Y_2(x) = Y_1(x)\,\Gamma$ for all $x \in \Omega$. Hence,

$$\lim_{x\to b} \mathrm{e}^{\Delta(x)}\Gamma\,\mathrm{e}^{-\Delta(x)} = \lim_{x\to b} H_1(x)^{-1}H_2(x) = I,$$

and therefore $\Gamma \in G(\Delta)$. Conversely, let $\Gamma \in G(\Delta)$. If we define

$$H_2(x) := H_1(x)\,\mathrm{e}^{\Delta(x)}\Gamma\,\mathrm{e}^{-\Delta(x)}, \quad x \in \Omega,$$

then $Y_2(x) := Y_1(x)\,\Gamma = H_2(x)\,\mathrm{e}^{\Delta(x)}$ is a fundamental matrix of (1), and $\lim_{x\to b} H_2(x) = I$. ▮

In order to complete the proof of Theorem 2.1, we first assume that $\Lambda = [\mu,\nu)$ with some $-\infty < \mu < \nu \le \infty$. Let $\mu = \mu_0 < \mu_1 < \mu_2 < \cdots < \nu$ be a sequence in Λ such that $\mu_k \to \nu$ as $k \to \infty$. If we define $\Lambda_k := [\mu_{k-1},\mu_k]$ for all $k \in \mathbb{N}$, then $\Lambda = \bigcup_{k\in\mathbb{N}} \Lambda_k$, and, by what has been shown before, there exist fundamental matrices $Y_k(\,\cdot\,,\lambda)$ of (1) having the form $Y_k(x,\lambda) = H_k(x,\lambda)\,\mathrm{e}^{\Delta(x)}$, $x \in \Omega$, where $H_k(\,\cdot\,,\lambda) \in \mathrm{AC}(\Omega,\mathrm{M_n}(\mathbb{C}))$, $\lim_{x\to b} H_k(x,\lambda) = I$ and $\max_{x\in\Omega} |H_k(x,\mu) - H_k(x,\lambda)| \to 0$ as $\mu \to \lambda$ for all $\lambda \in \Lambda_k$. Now, according to Proposition 2.1, there exist matrices $\Gamma_k \in G(\Delta)$ such that $Y_k(x,\mu_k) = Y_{k+1}(x,\mu_k)\,\Gamma_{k+1}$ holds for all $x \in \Omega$ and $k \in \mathbb{N}$. In addition, let $\Gamma_1 := I$. Then the matrix valued function

$$Y(x,\lambda) := Y_k(x,\lambda)\prod_{j=1}^{k}\Gamma_{k-j+1}, \quad (x,\lambda) \in \Omega \times \Lambda_k,$$

is well-defined on $\Omega \times \Lambda$ and for every $\lambda \in \Lambda$ a fundamental matrix of (1) with the required properties. A similar reasoning can be used if $\Lambda = (\mu,\nu]$ or $\Lambda = (\mu,\nu)$. ▮

Remark 2. 1. If $\Lambda = \{\lambda_0\}$, then (1) is a usual system of differential equations (without parameter dependence) and Theorem 2.1 is equivalent to the Levinson Theorem (see [2, Theorem 1.3.1]). Moreover, if $D \equiv 0$, Theorem 2.1 and Proposition 2.1 reduce to the Existence and Uniqueness Theorem (note that $G(\Delta) = \{I\}$ in this case).

3. SINGULAR DIRAC SYSTEMS DEPENDING ON A PARAMETER

In this section we consider 2×2 *Dirac systems*

$$Jy'(x) + Q(x, \lambda)y(x) = 0, \quad x \in \Omega, \tag{10}$$

on an interval $\Omega = (a, b]$, $-\infty \leq a < b < \infty$, which depend on a parameter λ varying in some interval $\Lambda \subset \mathbb{R}$. Here the constant matrix J is defined by

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Further we assume that $Q(\,\cdot\,, \lambda) \in \mathrm{L}^1_{\mathrm{loc}}(\Omega, \mathrm{M}_2(\mathbb{R}))$ for all $\lambda \in \Lambda$ and that $Q(x, \lambda)$ is hermitian for all $x \in \Omega$.

If $y : \Omega \longrightarrow \mathbb{R}^2$ is a nontrivial solution of (10),

$$y(x) = \begin{pmatrix} v(x) \\ u(x) \end{pmatrix}, \quad x \in \Omega,$$

then we can write the components of y in polar coordinates:

$$u(x) = \rho(x) \sin \phi(x), \quad v(x) = \rho(x) \cos \phi(x), \quad x \in \Omega,$$

with $\rho(x)^2 = u(x)^2 + v(x)^2 \neq 0$ and

$$\phi(x) = \begin{cases} \arctan \frac{u(x)}{v(x)} & \text{if } v(x) \neq 0, \\ \operatorname{arccot} \frac{v(x)}{u(x)} & \text{if } u(x) \neq 0, \end{cases} \tag{11}$$

where the branches of arctan and arccot are chosen such that $\phi : \Omega \longrightarrow \mathbb{R}$ is absolutely continuous. The function ϕ is called *angle function* of y and it is uniquely defined up to an additive constant $k\pi$ $(k \in \mathbb{Z})$.

Definition 3.1. A nontrivial solution $y : \Omega \longrightarrow \mathbb{R}^2$ of (10) is called *principal* at a, if there exists a real-valued solution y_0 of (10),

$$y_0(x) = \begin{pmatrix} v_0(x) \\ u_0(x) \end{pmatrix}, \quad x \in \Omega,$$

which is linearly independent of y, and either of the conditions $u_0(x) \neq 0$, $\lim_{x\to a} \frac{u(x)}{u_0(x)} = 0$ or $v_0(x) \neq 0$, $\lim_{x\to a} \frac{v(x)}{v_0(x)} = 0$ holds in a neighbourhood of a.

Remark 3. 1. The concept of principal solutions was first introduced for Sturm-Liouville equations (see Appendix) and generalized by Hartman and Reid for systems (10) which are identically normal on Ω (compare [9, Chapter V, Section 9]). An extension of this concept for general systems (10) are the notions of principal and coprincipal solutions due to Ahlbrandt (see [1]): a nontrivial solution $y : \Omega \longrightarrow \mathbb{R}^2$ of (10) is said to be principal at a if y_0 in Definition 3.1 satisfies $\lim_{x\to a} \frac{v(x)}{v_0(x)} = 0$, while it is called *coprincipal* at a if $\lim_{x\to a} \frac{u(x)}{u_0(x)} = 0$ (both definitions are equivalent under certain assumptions on Q, especially for identically normal systems). In the present paper we will not distinguish between principal and coprincipal solutions.

In the following we present a continuity property and a comparison theorem for the principal solutions of (10) in the case that the coefficient matrix $Q(x, \lambda)$ has a special form. We suppose that

$$Q(x, \lambda) = r(x)A + B(x, \lambda), \quad (x, \lambda) \in \Omega \times \Lambda, \tag{12}$$

where $r : \Omega \longrightarrow \mathbb{R}$ is a locally integrable function satisfying

$$\lim_{x\to a} \int_x^b r(t)\,dt = -\infty, \quad \int_{\Omega_0} r(t)\,dt \le \rho_0$$

on any subinterval $\Omega_0 \subset \Omega$ with some constant $\rho_0 > 0$. Furthermore we assume that $A \in \mathrm{M}_2(\mathbb{R})$ is a hermitian matrix with $\det A < 0$ and that $B(\,\cdot\,, \lambda) : \Omega \longrightarrow \mathrm{M}_2(\mathbb{R})$ are integrable functions with the property that $B(x, \lambda)$ is hermitian for all $x \in \Omega$ and

$$\int_\Omega |B(t, \mu) - B(t, \lambda)|\,dt \to 0, \quad \mu \to \lambda, \tag{13}$$

holds for all $\lambda \in \Lambda$.

We begin with some notations in order to specify the fundamental solutions of (10). Let $\alpha := \sqrt{-\det A} > 0$. Then $-\alpha$ and α are the eigenvalues of JA. Hence, we can find an invertible matrix $T \in \mathrm{M}_2(\mathbb{R})$,

$$T = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix},$$

such that $T^{-1}JAT = \mathrm{diag}\,(-\alpha, \alpha)$. Moreover, we define

$$s(x) := \sigma - \int_x^b r(t)\,dt, \quad x \in \Omega,$$

with some constant $\sigma \in \mathbb{R}$.

THEOREM 3.1. *The system* (10) *has a fundamental matrix of the form*

$$Y(x,\lambda) = H(x,\lambda)\,\mathrm{diag}\,(\mathrm{e}^{-\alpha s(x)}, \mathrm{e}^{\alpha s(x)}), \quad x \in \Omega,$$

where $H : \Omega \times \Lambda \longrightarrow \mathrm{M}_2(\mathbb{R})$ *is a continuous function which satisfies* $\lim_{x\to a} H(x,\lambda) = T$ *for all* $\lambda \in \Lambda$.

Proof. By the transformation $y(x,\lambda) = Tz(-x,\lambda)$, the system (10) is equivalent to the asymptotically diagonal system

$$z'(x,\lambda) = (D(x) + C(x,\lambda))\,z(x,\lambda), \quad x \in \Omega^*, \tag{14}$$

where $\Omega^* := [-b,-a)$ and

$$D(x) := \mathrm{diag}\,(\alpha\, r(-x), -\alpha\, r(-x)), \quad C(x,\lambda) := -T^{-1}JB(-x,\lambda)\,T.$$

Obviously, $D : \Omega^* \longrightarrow \mathrm{M}_2(\mathbb{R})$ satisfies the dichotomy condition (D) with $\rho = 2\,\alpha\rho_0$, and from (13) it follows that

$$\int_{\Omega^*} |C(t,\mu) - C(t,\lambda)|\,dt \to 0, \quad \mu \to \lambda,$$

for all $\lambda \in \Lambda$. Hence, we can apply Theorem 2.1 which implies that for every $\lambda \in \Lambda$ there exists a fundamental matrix

$$Z(x,\lambda) = G(x,\lambda)\,\mathrm{diag}\,(\mathrm{e}^{-\alpha s(-x)}, \mathrm{e}^{\alpha s(-x)}), \quad x \in \Omega^*,$$

of (14), where $G(\,\cdot\,,\lambda) : \Omega^* \longrightarrow \mathrm{M}_2(\mathbb{C})$ are absolutely continuous functions satisfying $\lim_{x\to -a} G(x,\lambda) = I$ and

$$\max_{x\in\Omega^*} |G(x,\mu) - G(x,\lambda)| \to 0, \quad \mu \to \lambda,$$

for all $\lambda \in \Lambda$. If we define

$$H(x,\lambda) := \frac{1}{2}\,T\left(G(-x,\lambda) + \overline{G(-x,\lambda)}\right), \quad (x,\lambda) \in \Omega \times \Lambda,$$

then $Y(x,\lambda) = H(x,\lambda)\,\mathrm{diag}\,(\mathrm{e}^{-\alpha s(x)}, \mathrm{e}^{\alpha s(x)})$ is a fundamental matrix of (10), and H has the required properties. ▮

Now, we consider the linearly independent solutions

$$y_1(x,\lambda) = \mathrm{e}^{-\alpha s(x)} h_1(x,\lambda), \quad y_2(x,\lambda) = \mathrm{e}^{\alpha s(x)} h_2(x,\lambda) \tag{15}$$

of (10) where $h_i(x,\lambda) := H(x,\lambda)e_i$, $i \in \{1,2\}$. The next result follows immediately from the above theorem and from $s(x) \to \infty$ as $x \to a$.

COROLLARY 3.1. *The function y_1 is continuous on $\Omega \times \Lambda$, and $y_1(\,\cdot\,,\lambda)$ is a principal solution of* (10) *at a for every $\lambda \in \Lambda$. Moreover, for a fixed $\lambda \in \Lambda$, a solution y of* (10) *is principal at a if and only if $y = \gamma\, y_1(\,\cdot\,,\lambda)$ with some constant $\gamma \in \mathbb{R} \setminus \{0\}$.*

We can also characterize principal solutions of (10) by the asymptotic behaviour of the angle functions at the boundary point a. In the proposition below, let

$$\psi_1 := \begin{cases} \arctan \frac{\tau_{21}}{\tau_{11}} & \text{if } \tau_{11} \neq 0, \\ \operatorname{arccot} \frac{\tau_{11}}{\tau_{21}} & \text{if } \tau_{21} \neq 0, \end{cases} \qquad \psi_2 := \begin{cases} \arctan \frac{\tau_{22}}{\tau_{12}} & \text{if } \tau_{12} \neq 0, \\ \operatorname{arccot} \frac{\tau_{12}}{\tau_{22}} & \text{if } \tau_{22} \neq 0, \end{cases}$$

and $\psi_i \in [-\frac{\pi}{2}, \frac{\pi}{2})$, $i \in \{1,2\}$. Note that $\psi_1 \neq \psi_2$ since T is invertible.

PROPOSITION 3.1. *If, for a fixed $\lambda \in \Lambda$, y is a nontrivial solution of* (10) *and ϕ is an angle function of y, then $\lim_{x\to a} \phi(x) =: \psi$ exists. Moreover, $\psi \equiv \psi_1 \bmod \pi$ if and only if y is principal at a.*

Proof. For a fixed $\lambda \in \Lambda$, let

$$H(x,\lambda) =: \begin{pmatrix} h_{11}(x,\lambda) & h_{12}(x,\lambda) \\ h_{21}(x,\lambda) & h_{22}(x,\lambda) \end{pmatrix}, \quad x \in \Omega,$$

and let $y : \Omega \longrightarrow \mathbb{R}^2$ be a nontrivial solution of (10),

$$y(x) = \begin{pmatrix} v(x) \\ u(x) \end{pmatrix}, \quad x \in \Omega.$$

Then there exist constants $c_1, c_2 \in \mathbb{R}$, $|c_1| + |c_2| > 0$, such that $y(x) = c_1 y_1(x,\lambda) + c_2 y_2(x,\lambda)$ for all $x \in \Omega$. If $c_2 = 0$, then $\lim_{x\to a} H(x,\lambda) = T$ and (11) readily implies $\lim_{x\to a} \phi(x) = \psi_1 + k\pi$ with some $k \in \mathbb{Z}$. Now, let $c_2 \neq 0$. If $\tau_{12} \neq 0$, then $v(x) \neq 0$ in a neighbourhood $\mathfrak{V}$ of a, and we have

$$\frac{u(x)}{v(x)} = \frac{\frac{c_1}{c_2} h_{21}(x,\lambda) \mathrm{e}^{-2\alpha s(x)} + h_{22}(x,\lambda)}{\frac{c_1}{c_2} h_{11}(x,\lambda) \mathrm{e}^{-2\alpha s(x)} + h_{12}(x,\lambda)}, \quad x \in \mathfrak{V}.$$

Hence, from $\lim_{x\to a} h_{ij}(x,\lambda) = \tau_{ij}$, $i,j \in \{1,2\}$, and $\lim_{x\to a} s(x) = \infty$ it follows that $\lim_{x\to a} \frac{u(x)}{v(x)} = \frac{\tau_{22}}{\tau_{12}}$. Similarily, we get $\lim_{x\to a} \frac{v(x)}{u(x)} = \frac{\tau_{12}}{\tau_{22}}$ in case of $\tau_{22} \neq 0$. Therefore, $\lim_{x\to a} \phi(x) = \psi_2 + k\pi$ with some $k \in \mathbb{Z}$. By Corollary 3.1, this completes the proof. ∎

The following result is a comparison theorem (with respect to the parameter λ) for the principal solutions of (10).

THEOREM 3.2. *Suppose that Q has the form* (12) *and that $B(\cdot,\lambda_1) \geq B(\cdot,\lambda_2)$ holds a.e. in Ω for all $\lambda_1 < \lambda_2$ in Λ. Moreover, let $y_1(\cdot,\lambda)$ be the principal solutions of* (10) *specified in* (15),

$$y_1(x,\lambda) = \begin{pmatrix} v_1(x,\lambda) \\ u_1(x,\lambda) \end{pmatrix}, \quad (x,\lambda) \in I \times \Lambda.$$

Finally, let $\xi \in \Omega$, and assume that $u_1(\xi,\lambda_0) \neq 0$ for some $\lambda_0 \in \Lambda$. Then there exists a neigbourhood Λ_0 of λ_0 such that $u_1(\xi,\lambda) \neq 0$ for all $\lambda \in \Lambda_0$, and the function

$$\lambda \longmapsto \frac{v_1(\xi,\lambda)}{u_1(\xi,\lambda)}$$

is decreasing on Λ_0.

Proof. Let $(\xi,\lambda_0) \in \Omega \times \Lambda$ be fixed as above. Since $u_1(\xi,\lambda_0) \neq 0$ and $u_1(\xi,\cdot)$ is continuous on Λ, there exists a neighbourhood $\Lambda_0 \subset \Lambda$ of λ_0 such that $u_1(\xi,\lambda) \neq 0$ for all $\lambda \in \Lambda_0$. Now, let $\phi(\cdot,\lambda)$ be the angle functions of $y_1(\cdot,\lambda)$ which satisfy $\lim_{x\to a} \phi(x,\lambda) = \psi_1$ for all $\lambda \in \Lambda$. In order to prove the assertion we show that $\lambda \longmapsto \phi(\xi,\lambda)$ is increasing on Λ_0.

To this end, we assume to the contrary that $\phi(\xi,\lambda_1) > \phi(\xi,\lambda_2)$ holds for some $\lambda_1 < \lambda_2$ in Λ_0. Let

$$\theta := \min\left\{\frac{1}{2}(\phi(\xi,\lambda_1) + \phi(\xi,\lambda_2)),\, \phi(\xi,\lambda_1) - \frac{\pi}{2}\right\}$$

and let y be the solution of (10) for $\lambda = \lambda_1$ which satisfies

$$y(\xi) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}.$$

It is not difficult to verify that the functions y and $y_1(\cdot,\lambda_1)$ are linearly independent due to the choice of θ. Moreover, if Φ denotes the angle function of y with $\Phi(\xi) = \theta$, then $\phi(\xi,\lambda_1) > \Phi(\xi) > \phi(\xi,\lambda_2)$. Since $-Q(\cdot,\lambda_1) \leq -Q(\cdot,\lambda_2)$ holds a.e. in Ω, we can apply the Comparison Theorem [14, Theorem 16.1] which yields $\phi(x,\lambda_1) \geq \Phi(x) \geq \phi(x,\lambda_2)$ for all $x \in (a,\xi]$. From $\lim_{x\to a} \phi(x,\lambda_i) = \psi_1$, $i \in \{1,2\}$, it follows that $\lim_{x\to a} \Phi(x) = \psi_1$. Hence, by Proposition 3.1, y is a principal solution of (10), and Corollary 3.1 implies that y is a constant multiple of $y_1(\cdot,\lambda_1)$, a contradiction. ▌

4. SINGULAR STURM-LIOUVILLE PROBLEMS NONLINEAR IN THE SPECTRAL PARAMETER

In the following we consider a Sturm-Liouville problem

$$(p(x,\lambda)u'(x))' - q(x,\lambda)u(x) = 0, \quad x \in I, \qquad \text{(SL)}(\lambda)$$
$$\alpha(\lambda)u(a) + \beta(\lambda)v(a) = 0, \qquad \text{(BC)}(\lambda)$$

on a half-open interval $I = [a,b)$, $-\infty < a < b \le \infty$, which depends nonlinearly on the parameter $\lambda \in \overline{\Lambda}$, where $\Lambda = [\omega,\nu)$, $-\infty < \omega < \nu < \infty$. The function $v := p(\,\cdot\,,\lambda)u'$ is called the first *quasi-derivative* of u, and we assume that the coefficients and boundary conditions satisfy the following hypotheses (the basic definitions and results for Sturm-Liouville equations are collected in the Appendix):

(i) The functions p, q are real-valued on $I \times \overline{\Lambda}$, $p(\,\cdot\,,\lambda) > 0$ holds a.e. in I, and $p(\,\cdot\,,\lambda)^{-1}$, $q(\,\cdot\,,\lambda)$ are locally integrable on I for all $\lambda \in \overline{\Lambda}$.

(ii) For every $\lambda \in \overline{\Lambda}$ and any compact subinterval $K \subset I$,

$$\int_K \left| \frac{1}{p(t,\mu)} - \frac{1}{p(t,\lambda)} \right| dt \to 0, \quad \int_K |q(t,\mu) - q(t,\lambda)|\, dt \to 0, \quad \mu \to \lambda.$$

(iii) There exists a continuous function $c : \Lambda \longrightarrow (a,b)$ such that (SL)(λ) is disconjugate on $[c(\lambda), b)$ for all $\lambda \in \Lambda$.

(iv) The functions α, $\beta : \overline{\Lambda} \longrightarrow \mathbb{R}$ are continuous, $|\alpha(\lambda)| + |\beta(\lambda)| > 0$, and either $\beta \equiv 0$ or $\beta(\lambda) \ne 0$ for all $\lambda \in \overline{\Lambda}$.

Further we introduce the following notations: For $\lambda_1 \ne \lambda_2$ in $\overline{\Lambda}$ we write (SL)(λ_1) $\succ$ (SL)(λ_2) if $p(\,\cdot\,,\lambda_1) \ge p(\,\cdot\,,\lambda_2)$ and $q(\,\cdot\,,\lambda_1) > q(\,\cdot\,,\lambda_2)$ holds a.e. in I, and we write (BC)(λ_1) $\preceq$ (BC)(λ_2) if either $\beta \equiv 0$ or $\frac{\alpha(\lambda_1)}{\beta(\lambda_1)} \le \frac{\alpha(\lambda_2)}{\beta(\lambda_2)}$. If the differential equation (SL)(λ) is non-oscillatory for some $\lambda \in \overline{\Lambda}$ and u is a principal solution of (SL)(λ) at b, then we denote by $N(\lambda)$ the number of zeros of u in (a,b).

Definition 4.1. A point $\lambda \in \overline{\Lambda}$ is said to be an *eigenvalue* of the spectral problem (SL)(λ), (BC)(λ) if there exists a principal solution u of (SL)(λ) at b which satisfies the boundary condition (BC)(λ) at a.

In the sequel we will present some criteria for the eigenvalue accumulation at the point ν. For this purpose, we introduce some additional assumptions on the spectral problem (SL)(λ), (BC)(λ) used in the following theorems.

(P) For every $\lambda \in \Lambda$ there exist principal solutions $u(\,\cdot\,,\lambda)$ of (SL)(λ) at b such that u and $v := pu'$ are continuous on $I \times \Lambda$.

(M) For all $\lambda_1 < \lambda_2$ in Λ, $(\mathrm{SL})(\lambda_1) \succ (\mathrm{SL})(\lambda_2)$ and $(\mathrm{BC})(\lambda_1) \preceq (\mathrm{BC})(\lambda_2)$.

(L) For every λ in Λ, $(\mathrm{SL})(\lambda) \succ (\mathrm{SL})(\nu)$ and $(\mathrm{BC})(\lambda) \preceq (\mathrm{BC})(\nu)$.

A sufficient condition for (P) to hold is established at the end of this section.

The following theorems show that the accumulation and non-accumulation of eigenvalues at the right endpoint ν of the interval Λ is essentially determined by the oscillatory properties of the limit equation $(\mathrm{SL})(\nu)$.

THEOREM 4.1 (Eigenvalue Accumulation). *Suppose that the spectral problem* $(\mathrm{SL})(\lambda)$, $(\mathrm{BC})(\lambda)$ *satisfies the assumptions* (i) – (iv) *and that the limit equation* $(\mathrm{SL})(\nu)$ *is oscillatory on* I.
If, in addition, (P) *holds, then* ν *is an accumulation point of eigenvalues of* $(\mathrm{SL})(\lambda)$, $(\mathrm{BC})(\lambda)$. *Moreover, every left neighbourhood of* ν *contains an interval which consists of non-eigenvalues.*

THEOREM 4.2 (Non-Accumulation). *Suppose that the spectral problem* $(\mathrm{SL})(\lambda)$, $(\mathrm{BC})(\lambda)$ *satisfies the assumptions* (i) – (iv) *and that the limit equation* $(\mathrm{SL})(\nu)$ *is non-oscillatory on* I.
If, in addition, (L) *or* (M) *holds, then* ν *is not an accumulation point of eigenvalues of* $(\mathrm{SL})(\lambda)$, $(\mathrm{BC})(\lambda)$. *If* $(\mathrm{SL})(\lambda) \succ (\mathrm{SL})(\nu)$ *for all* $\lambda \in \Lambda$ *and* ν *is an accumulation point of eigenvalues, then* ν *is an eigenvalue of this spectral problem.*

COROLLARY 4.1. *Suppose that the spectral problem* $(\mathrm{SL})(\lambda)$, $(\mathrm{BC})(\lambda)$ *satisfies the assumptions* (i) – (iv), (P) *and one of the conditions* (L) *or* (M). *Then* ν *is an accumulation point of eigenvalues of* $(\mathrm{SL})(\lambda)$, $(\mathrm{BC})(\lambda)$ *if and only if the limit equation* $(\mathrm{SL})(\nu)$ *is oscillatory on* I.

THEOREM 4.3 (Number of Eigenvalues). *Suppose that the spectral problem* $(\mathrm{SL})(\lambda)$, $(\mathrm{BC})(\lambda)$ *satisfies the assumptions* (i) – (iv) *and that the limit equation* $(\mathrm{SL})(\nu)$ *is non-oscillatory on* I.
If (L) *and* (M) *are satisfied, then the interval* $\Lambda = [\omega, \nu)$ *contains at most* $N(\nu) - N(\omega) + 1$ *eigenvalues; if* (L), (M) *and* (P) *are satisfied, then* Λ *contains at least* $N(\nu) - N(\omega) - 1$ *eigenvalues of* $(\mathrm{SL})(\lambda)$, $(\mathrm{BC})(\lambda)$.

Proof (of Theorems 4.1, 4.2 and 4.3). In the following, let $u(\,\cdot\,,\lambda)$ be a principal solution of $(\mathrm{SL})(\lambda)$ at b for every $\lambda \in \Lambda$. Since the function c in assumption (iii) is continuous and $(\mathrm{SL})(\lambda)$ is disconjugate on $[c(\lambda), b)$, there exists a continuous function ξ on Λ, $\xi : \lambda \longmapsto \xi(\lambda) \in (c(\lambda), b)$, such that $u(x,\lambda) \neq 0$, $x \in [\xi(\lambda), b)$, for all $\lambda \in \Lambda$ (see [9, p. 208]). If we write $u(x,\lambda)$ and $v(x,\lambda) := p(x,\lambda)u'(x,\lambda)$ in the polar coordinate form (*Prüfer's transformation*),

$$u(x,\lambda) = \rho(x,\lambda)\sin\phi(x,\lambda), \quad v(x,\lambda) = \rho(x,\lambda)\cos\phi(x,\lambda), \tag{16}$$

where $\rho(x,\lambda)^2 = u(x,\lambda)^2 + v(x,\lambda)^2$, then $\phi(\,\cdot\,,\lambda)$ is a solution of the differential equation

$$\phi'(x,\lambda) = \frac{1}{p(x,\lambda)}\cos^2\phi(x,\lambda) - q(x,\lambda)\sin^2\phi(x,\lambda), \quad x \in I, \tag{17}$$

see [14, Section 13]. Moreover, we have

$$\phi(x,\lambda) = \begin{cases} \arctan\frac{u(x,\lambda)}{v(x,\lambda)} & \text{if } v(x,\lambda) \neq 0, \\ \operatorname{arccot}\frac{v(x,\lambda)}{u(x,\lambda)} & \text{if } u(x,\lambda) \neq 0, \end{cases} \tag{18}$$

where the branches of arctan and arccot are chosen such that $\phi(\,\cdot\,,\lambda)$ is absolutely continuous on I. If we fix $\phi(\xi(\lambda),\lambda) \in (-\pi,0)$, then $\phi(\,\cdot\,,\lambda)$ is uniquely defined and called the *angle function* of $u(\,\cdot\,,\lambda)$. Since $\rho(x,\lambda) \neq 0$ for all $x \in I$, we obtain from (16) that $\phi(x,\lambda) = k\pi$ for some $k \in \mathbb{Z}$ if and only if $u(x,\lambda) = 0$. Further, $\phi(\,\cdot\,,\lambda)$ is strictly increasing in every $x_0 \in I$ with $u(x_0,\lambda) = 0$ by (17). Using these properties of ϕ, one can show that

$$\phi(a,\lambda) \in [-(N(\lambda)+1)\pi, -N(\lambda)\pi), \quad \lambda \in \Lambda. \tag{19}$$

If, in addition, the differential equation (SL)(ν) is non-oscillatory on I, let $u(\,\cdot\,,\nu)$ be a principal solution of (SL)(ν) at b and define the angle function $\phi(\,\cdot\,,\nu)$ of $u(\,\cdot\,,\nu)$ as the solution of (17) for $\lambda = \nu$ with $\phi(\xi(\nu),\nu) \in (-\pi,0)$, where $\xi(\nu)$ is a point in (a,b) such that $u(x,\nu) \neq 0$ for all $x \in [\xi(\nu),b)$. Again, we have $\phi(a,\nu) \in [-(N(\nu)+1)\pi, -N(\nu)\pi)$. Finally, we define

$$\psi(\lambda) := \begin{cases} -\pi & \text{if } \beta(\lambda) = 0, \\ \operatorname{Arccot}\left(-\frac{\alpha(\lambda)}{\beta(\lambda)}\right) & \text{if } \beta(\lambda) \neq 0, \end{cases}$$

where $\operatorname{Arccot}: \mathbb{R} \longrightarrow (-\pi,0)$ denotes the main branch of arccot. Note that assumption (iv) implies that the function $\psi : \overline{\Lambda} \longrightarrow [-\pi,0)$ is continuous, and either $\psi \equiv -\pi$ or $\psi(\lambda) \in (-\pi,0)$ for all $\lambda \in \overline{\Lambda}$. If now $\beta(\lambda) \neq 0$ for all $\lambda \in \Lambda$, then $u(\,\cdot\,,\lambda)$ satisfies the boundary condition (BC)(λ) if and only if

$$u(a,\lambda) \neq 0, \quad \frac{v(a,\lambda)}{u(a,\lambda)} = -\frac{\alpha(\lambda)}{\beta(\lambda)}.$$

If $\beta \equiv 0$, then (BC)(λ) holds if and only if $u(a,\lambda) = 0$. Hence, by the definitions of ϕ and ψ, a point $\lambda \in \Lambda$ is an eigenvalue of (SL)(λ), (BC)(λ) if and only if

$$\phi(a,\lambda) \equiv \psi(\lambda) \bmod \pi. \tag{20}$$

In the following we formulate some lemmas which are needed subsequently in the proof.

LEMMA 4.1. *If u and v are continuous on $I\times\Lambda$, then the function $\phi(a,\,\cdot\,)$ is continuous on Λ.*

Proof. Let $\lambda_0 \in \Lambda$ and $\Lambda_0 \subset \Lambda$ be a compact neighbourhood of λ_0. By the Separation Theorem [9, Chapter II, Theorem 2.3], the zeros of $u(\,\cdot\,,\lambda_0)$ are isolated in I. Hence, we can find a point $\gamma \in (a,b)$ such that $u(x,\lambda_0) \neq 0$ for all $x \in (a,\gamma]$. Moreover, if $u(a,\lambda_0) = 0$, then $v(a,\lambda_0) \neq 0$ by the Existence and Uniqueness Theorem, and we can choose $\gamma \in (a,b)$ such that, in addition, $v(x,\lambda_0) \neq 0$ for all $x \in [a,\gamma]$. Now, let $x_1 < \cdots < x_n$ be the zeros of $u(\,\cdot\,,\lambda_0)$ in $I_0 := [\gamma,\eta]$, where $\eta := \max_{\lambda\in\Lambda_0} \xi(\lambda)$ (the function ξ has been defined at the beginning of the main proof). Since

$$u(x,\lambda_0) = \int_{x_i}^{x} \frac{1}{p(t,\lambda_0)} v(t,\lambda_0)\,dt, \quad x \in I,$$

and $v(x_i,\lambda_0) \neq 0$ according to the Existence and Uniqueness Theorem, there exists a constant $\varepsilon > 0$ and a compact neighbourhood $\Delta_i = [a_i,b_i] \subset (\gamma,\eta)$ of x_i for all $i \in \{1,\dots,n\}$ such that $u(\,\cdot\,,\lambda_0) \neq 0$ on $\Delta_i \setminus \{x_i\}$, $\operatorname{sign} u(a_i,\lambda_0) = -\operatorname{sign} u(b_i,\lambda_0)$ and $|v(\,\cdot\,,\lambda_0)| > \varepsilon$ on Δ_i. Additionally, we have $u(\,\cdot\,,\lambda_0) \neq 0$ on the compact set $\Delta_0 := I_0 \setminus \bigcup_{i=1}^{n} (a_i,b_i)$. From the continuity of u, v it follows that there exists a constant $\delta > 0$ such that $\operatorname{sign} u(a_i,\lambda) = -\operatorname{sign} u(b_i,\lambda)$, $|v(\,\cdot\,,\lambda)| > \frac{1}{2}\varepsilon$ on Δ_i and $u(\,\cdot\,,\lambda) \neq 0$ on Δ_0 for all $\lambda \in K_\delta(\lambda_0) := \{\lambda \in \Lambda : |\lambda - \lambda_0| < \delta\}$. Moreover,

$$u(x,\lambda) = u(a_i,\lambda) + \int_{a_i}^{x} \frac{1}{p(t,\lambda)} v(t,\lambda)\,dt, \quad x \in I,$$

implies that $u(\,\cdot\,,\lambda)$ is either increasing or decreasing on Δ_i. Thus, for all $\lambda \in K_\delta(\lambda_0)$, $u(\,\cdot\,,\lambda)$ has no zeros in Δ_0 and exactly one zero in Δ_i for every $i \in \{1,\dots,n\}$. Consequently, the functions $u(\,\cdot\,,\lambda)$ have exactly n zeros in I_0 and hence in $[\gamma,b)$ for all $\lambda \in K_\delta(\lambda_0)$. In particular, we get $\phi(\gamma,\lambda) \in [-(n+1)\pi, -n\pi)$ for all $\lambda \in K_\delta(\lambda_0)$, and from

$$\phi(\gamma,\lambda) = \operatorname{Arccot} \frac{v(\gamma,\lambda)}{u(\gamma,\lambda)} - n\pi, \quad \lambda \in K_\delta(\lambda_0),$$

and the continuity of $u(\gamma,\,\cdot\,)$, $v(\gamma,\,\cdot\,)$ it follows that $\phi(\gamma,\,\cdot\,)$ is continuous on $K_\delta(\lambda_0)$. Now, if $u(a,\lambda_0) \neq 0$, then there exists a neighbourhood $\mathfrak{U} \subset K_\delta(\lambda_0)$ of λ_0 such that $u(\,\cdot\,,\lambda) \neq 0$ on $[a,\gamma]$ for all $\lambda \in \mathfrak{U}$. Since the functions u, v are continuous on $I \times \Lambda$ and

$$\phi(x,\lambda) = \operatorname{Arccot}\frac{v(x,\lambda)}{u(x,\lambda)} - \operatorname{Arccot}\frac{v(\gamma,\lambda)}{u(\gamma,\lambda)} + \phi(\gamma,\lambda), \quad (x,\lambda) \in [a,\gamma] \times \mathfrak{U},$$

we deduce that $\phi(a,\,\cdot\,)$ is continuous on $\mathfrak{U}$. If $u(a,\lambda_0) = 0$, then $v(x,\lambda_0) \neq 0$ for all $x \in [a,\gamma]$ by the choice of γ, and there exists a neighbourhood $\mathfrak{V} \subset K_\delta(\lambda_0)$ of λ_0 with the property that $v(\,\cdot\,,\lambda) \neq 0$ on $[a,\gamma]$ for all $\lambda \in \mathfrak{V}$. From the continuity of u, v and

$$\phi(x,\lambda) = \operatorname{Arctan}\frac{u(x,\lambda)}{v(x,\lambda)} - \operatorname{Arctan}\frac{u(\gamma,\lambda)}{v(\gamma,\lambda)} + \phi(\gamma,\lambda), \quad (x,\lambda) \in [a,\gamma] \times \mathfrak{V},$$

where $\operatorname{Arctan} : \mathbb{R} \longrightarrow (-\frac{\pi}{2},\frac{\pi}{2})$ denotes the main branch of the function arctan, we conclude that $\phi(a,\,\cdot\,)$ is continuous on $\mathfrak{V}$. Altogether, $\phi(a,\,\cdot\,)$ is continuous in every point $\lambda_0 \in \Lambda$. ▮

LEMMA 4.2. *If* (SL)(ν) *is oscillatory on* I, *then* $\phi(a,\lambda) \to -\infty$ *as* $\lambda \to \nu$.

Proof. By (19), it suffices to prove that $N(\lambda) \to \infty$ as $\lambda \to \nu$. Let $n \in \mathbb{N}$. If (SL)(ν) is oscillatory on I, then there exists a point $\gamma \in (a,b)$ and a nontrivial solution $u_0(\,\cdot\,,\nu)$ of (SL)(ν) such that $u_0(\,\cdot\,,\nu)$ has at least $n+1$ zeros $x_1 < \cdots < x_{n+1}$ in (a,γ).

Now, suppose that $u_0(\,\cdot\,,\lambda)$ are the solutions of (SL)(λ) with the initial values

$$u_0(a,\lambda) = u_0(a,\nu), \quad v_0(a,\lambda) = v_0(a,\nu)$$

for all $\lambda \in \Lambda$, where $v_0(x,\lambda) := p(x,\lambda)u_0'(x,\lambda)$. From the Existence and Uniqueness Theorem it follows that u_0, v_0 are continuous on $I \times \overline{\Lambda}$. Moreover, since

$$u_0(x,\nu) = \int_{x_i}^{x} \frac{1}{p(t,\nu)} v_0(t,\nu)\,dt, \quad x \in I,$$

and $v_0(x_i,\nu) \neq 0$, there exists a constant $\varepsilon > 0$ and an open neighbourhood $\Delta_i \subset [a,\gamma]$ of x_i for all $i \in \{1,\dots,n+1\}$ such that $u_0(\,\cdot\,,\nu) \neq 0$ on $\Delta_i \setminus \{x_i\}$ and $|v_0(\,\cdot\,,\nu)| > \varepsilon$ on Δ_i. Arguing as in the proof of Lemma 4.1 (with u_0, v_0 instead of u, v and ν instead of λ_0), we conclude that there exists a constant $\delta > 0$ such that for every $\lambda \in (\nu-\delta,\nu]$ the function $u_0(\,\cdot\,,\lambda)$ has exactly one zero in Δ_i for all $i \in \{1,\dots,n+1\}$ and therefore at least $n+1$ zeros in I. Hence, by the Separation Theorem (see [9, Chapter II, Theorem 2.3]), it follows that $N(\lambda) \geq n$ for all $\lambda \in (\nu-\delta,\nu)$. ▮

Now, we are able to prove Theorem 4.1. Because of (P), we can suppose that u and v are continuous on $I \times \Lambda$. From Lemma 4.1 it follows that $\phi(a, \cdot)$ is continuous on Λ. Moreover, since $(\mathrm{SL})(\nu)$ is assumed to be oscillatory, we have $\phi(a,\lambda) \to -\infty$ as $\lambda \to \nu$ by Lemma 4.2. Hence, for every $\lambda_1 \in \Lambda$ there exists a point $\lambda_2 \in (\lambda_1, \nu)$ such that $\phi(a,\lambda_2) < \phi(a,\lambda_1) - \pi$. By the Intermediate Value Theorem and since $\psi(\Lambda) \subset [-\pi, 0)$, relation (20) holds for at least one point $\lambda \in (\lambda_1, \lambda_2)$, but not identically on (λ_1, λ_2). Therefore, (λ_1, λ_2) contains at least one eigenvalue as well as some interval which consists of non-eigenvalues.

LEMMA 4.3. *If* $(\mathrm{SL})(\lambda_1) \succ (\mathrm{SL})(\lambda_2)$ *holds for all* $\lambda_1 < \lambda_2$ *in* Λ*, then* $\phi(a, \cdot)$ *is strictly decreasing on* Λ*.*

Proof. For $\lambda_1 < \lambda_2$ in Λ, let $x_0 := \max\{\xi(\lambda_1), \xi(\lambda_2)\}$ (with ξ defined as in the beginning of the main proof). Since $(\mathrm{SL})(\lambda_1) \succ (\mathrm{SL})(\lambda_2)$ and $u(x, \lambda_i) \neq 0$ for all $x \in [x_0, b)$, the Comparison Theorem for Principal Solutions [4, Chapter XI, Corollary 6.5]) implies

$$\frac{v(x_0,\lambda_1)}{u(x_0,\lambda_1)} \leq \frac{v(x_0,\lambda_2)}{u(x_0,\lambda_2)}.$$

Therefore, $\phi(x_0,\lambda_1) \geq \phi(x_0,\lambda_2)$. From Sturm's Comparison Theorem (see, e.g., [14, Theorem 13.1]) it follows that $\phi(a,\lambda_1) > \phi(a,\lambda_2)$. ∎

LEMMA 4.4. *If* $(\mathrm{SL})(\nu)$ *is non-oscillatory on* I *and* $(\mathrm{SL})(\lambda) \succ (\mathrm{SL})(\nu)$ *holds for all* $\lambda \in \Lambda$*, then* $\phi(a, \cdot)$ *is bounded from below by* $-(N(\nu) + 1)\pi$*.*

Proof. As in the proof of Lemma 4.3 (with λ, ν instead of λ_1, λ_2) we obtain $\phi(a,\lambda) > \phi(a,\nu)$ for all $\lambda \in \Lambda$, and the assertion follows from the inequality $\phi(a,\nu) \geq -(N(\nu)+1)\pi$. ∎

Using the above results, we can now prove the assertion of Theorem 4.3. From (M) and Lemma 4.3 it follows that $\phi(a, \cdot)$ is strictly decreasing, while ψ is increasing on Λ. Further, (L) and Lemma 4.4 imply that $\phi(a,\lambda) > -(N(\nu)+1)\pi$ for all $\lambda \in \Lambda$. Together with (19) we conclude that $\{\phi(a,\lambda) : \lambda \in \Lambda\} \subset [-(N(\nu)+1)\pi, -N(\omega)\pi)$, and relation (20) holds for at most $N(\nu) - N(\omega) + 1$ points λ in Λ. If, in addition, (P) is satisfied, then $\phi(a, \cdot)$ is continuous on Λ by Lemma 4.1. In this case, we have $\{\phi(a,\lambda) : \lambda \in \Lambda\} \supset [-N(\nu)\pi, -(N(\omega)+1)\pi)$, and relation (20) holds for at least $N(\nu) - N(\omega) - 1$ points $\lambda \in \Lambda$.

LEMMA 4.5. *Suppose that* $(\mathrm{SL})(\nu)$ *is non-oscillatory and* $(\mathrm{SL})(\lambda) \succ (\mathrm{SL})(\nu)$ *holds for all* $\lambda \in \Lambda$*. If* $\Lambda_0 \subset \Lambda$ *and* $\nu = \sup \Lambda_0$*, then there ex-*

ists a strictly increasing sequence $(\lambda_n)_1^\infty$ *in* Λ_0 *with* $\nu = \lim_{n\to\infty} \lambda_n$ *such that* $(\phi(a,\lambda_n))_1^\infty$ *is strictly decreasing and* $\phi(a,\nu) = \lim_{n\to\infty} \phi(a,\lambda_n)$.

Proof. Since (SL)(ν) is non-oscillatory and (SL)$(\lambda) \succ$ (SL)(ν) for all $\lambda \in \Lambda$, it follows from Sturm's Comparison Theorem (see [14, Theorem 13.1]) that (SL)(λ) is disconjugate on $(\xi(\nu), b)$ for all $\lambda \in \overline{\Lambda}$. We fix some $\eta \in (\xi(\nu), b)$, and without loss of generality we can assume that the principal solutions $u(\,\cdot\,,\lambda)$ of (SL)(λ) at b satisfy $u(\eta,\lambda) = 1$ for all $\lambda \in \overline{\Lambda}$. Now, the Comparison Theorem for Principal Solutions (see [4, Chapter XI, Corollary 6.5]) implies $v(\eta,\lambda) \le v(\eta,\nu)$ for all $\lambda \in \Lambda$. Hence, $\sigma = \sup_{\lambda\in\Lambda_0} v(\eta,\lambda)$ exists, and we have $\sigma \le v(\eta,\nu)$. In particular, there exists a strictly increasing sequence $(\mu_n)_1^\infty$ in Λ_0 with $\nu = \lim_{n\to\infty} \mu_n$ such that $(v(\eta,\mu_n))_1^\infty$ is increasing and $v(\eta,\mu_n) \to \sigma$ as $n \to \infty$. We first claim that $\sigma = v(\eta,\nu)$. Otherwise, if $\sigma < v(\eta,\nu)$ and u_0 is the solution of (SL)(ν) with the initial values

$$u_0(\eta) = 1, \quad v_0(\eta) = \sigma,$$

where $v_0 := p(\,\cdot\,,\nu)u_0'$, then

$$\frac{v_0(\eta)}{u_0(\eta)} < \frac{v(\eta,\nu)}{u(\eta,\nu)}.$$

From [4, Chapter XI, Theorem 6.4 (iii)] it follows that u_0 has exactly one zero in (η, b). Moreover, by the Existence and Uniqueness Theorem, we get $u(\,\cdot\,,\mu_n) \to u_0$ and $v(\,\cdot\,,\mu_n) \to v_0$ locally uniformly on I as $n \to \infty$. Hence, as in the second paragraph of the proof of Lemma 4.2, we can show that $u(\,\cdot\,,\mu_n)$ also has at least one zero in (η, b) for sufficiently large n. However, from the disconjugacy of (SL)(λ) on $(\xi(\nu), b)$ it follows that $u(\,\cdot\,,\mu_n) \ne 0$ on (η, b) for all n (see [9, p. 208]), a contradiction. Therefore, again by the Existence and Uniqueness Theorem, we have

$$u(\,\cdot\,,\mu_n) \to u(\,\cdot\,,\nu), \quad v(\,\cdot\,,\mu_n) \to u(\,\cdot\,,\nu)$$

uniformly on any compact subset $K \subset I$ as $n \to \infty$. Thus, as in the proof of Lemma 4.1 (with ν instead of λ_0) we can show $\lim_{n\to\infty} \phi(a,\mu_n) = \phi(a,\nu)$. Moreover, as (SL)$(\mu_n) \succ$ (SL)(ν) by (L) and

$$\phi(\eta,\mu_n) = \operatorname{Arccot} v(\eta,\mu_n) > \operatorname{Arccot} v(\eta,\nu) = \phi(\eta,\nu),$$

we can apply Sturm's Comparison Theorem which yields $\phi(a,\mu_n) > \phi(a,\nu)$ for all $n \in \mathbb{N}$. Hence, there exists a subsequence $(\lambda_n)_1^\infty$ of $(\mu_n)_1^\infty$ such that $\phi(a,\lambda_n)$ is strictly decreasing and $\phi(a,\lambda_n) \to \phi(a,\nu)$ as $n \to \infty$. ∎

It remains to verify Theorem 4.2. For this purpose, let (SL)(ν) be non-oscillatory on I and suppose that (SL)(λ) $\succ$ (SL)(ν) holds for all $\lambda \in \Lambda$. Further denote by Γ_0 the set of all eigenvalues in the interval Λ. We assume that ν is an accumulation point of eigenvalues. Then $\nu = \sup \Gamma_0$ and, by Lemma 4.5, there exists a sequence of eigenvalues $(\lambda_n)_1^\infty$ with $\nu = \lim_{n\to\infty} \lambda_n$ and $\phi(a,\nu) = \lim_{n\to\infty} \phi(a,\lambda_n)$ such that $(\phi(a,\lambda_n))_1^\infty$ is strictly decreasing. Moreover, Lemma 4.3 implies that $\phi(a,\lambda_n) \geq -(N(\nu)+1)\pi$ for all $n \in \mathbb{N}$. Since relation (20) holds for all λ_n and ψ is continuous on $\overline{\Lambda}$, it follows that $\phi(a,\nu) \equiv \psi(\nu) \bmod \pi$. This means, ν is an eigenvalue of the spectral problem (SL)(λ), (BC)(λ), which proves the second assertion of Theorem 4.2. Further, if (M) or (L) holds, then there exists a subsequence $(\mu_n)_1^\infty$ of $(\lambda_n)_1^\infty$ with the property that $(\psi(\mu_n))_1^\infty$ is increasing and $\psi(\mu_n) \to \psi(\nu)$ as $n \to \infty$. Since $(\phi(a,\mu_n))_1\infty$ is strictly decreasing, relation (20) can hold for at most finitely numbers μ_n, a contradiction. This completes the proof of Theorem 4.2. ▮

Remark 4. 1. The assertion of Theorem 4.1 has been proved in [5] by using the Implicit Function Theorem under the additional hypothesis that the coefficients p and q are continuous on $I \times \Lambda$. Moreover, the first assertion of Theorem 4.2 is equivalent to [5, Theorem 2.1 (ii)] under the stronger assumption that both (P) and (M) are satisfied, while the second assertion of Theorem 4.2 seems to be new in this context. Assuming that p and q are continuous, also the results in [7] on the eigenvalues of singular boundary problems for λ-nonlinear Hamiltonian systems can be applied to the spectral problem (SL)(λ), (BC)(λ). In particular, [7, Theorem 2.1, (ii)] contains a lower bound for the number of eigenvalues in a given compact subinterval of Λ.

THEOREM 4.4 (Continuity of Principal Solutions). *If there exists a continuous function $c : \Lambda \longrightarrow (a,b)$ such that $q(\cdot,\lambda) \geq 0$ holds a.e. in $[c(\lambda), b)$ and either $\int_I \frac{1}{p(t,\lambda)}\,dt = \infty$ or $\int_I q(t,\lambda)\,dt = \infty$ for all $\lambda \in \Lambda$, then the assumptions* (iii) *and* (P) *are satisfied.*

Proof. The assertion follows from [4, Chapter XI, Corollary 6.6] and Lemma A.1 in the Appendix. ▮

5. APPLICATION TO THE DIRAC OPERATOR

In the following we apply the above results to the Dirac operator

$$H = -\mathrm{i}\,\alpha \cdot \nabla + \beta + V(|x|)I$$

in $\mathrm{L}^2(\mathbb{R}^3)^4$ with a spherically symmetric potential $V : (0,\infty) \longrightarrow \mathbb{R}$. The units are chosen such that $\hbar = m = c = 1$, I denotes the 4×4 unit matrix, and

$$\alpha = (\alpha_1, \alpha_2, \alpha_3), \quad \beta = \alpha_0,$$

where α_k are hermitian 4×4 matrices satisfying the commutation relations

$$\alpha_i\alpha_j + \alpha_j\alpha_i = 2\,\delta_{ij} I, \quad i,j \in \{0,\dots,3\}.$$

Further we make the following assumptions on the potential V:

(a) $V \in \mathrm{L}^\infty_{\mathrm{loc}}(0,\infty)$, $\lim_{x\to\infty} V(x) = 0$,

(b) $\limsup_{x\to 0}|xV(x)| < \frac{1}{2}\sqrt{3}$,

(c) $\int_0^1 |V(x) - \frac{\rho}{x}|\,dx < \infty$ with some $\rho \in [0, \frac{1}{2}\sqrt{3})$.

Then, by [12, Theorem 1], the operator H is self-adjoint on the domain $\mathcal{D}(H) = \mathrm{H}^1(\mathbb{R}^3)^4$, and $\sigma_{ess}(H) = (-\infty,-1] \cup [1,\infty)$. Moreover (see [14, Section 1]), there exists an orthogonal decomposition

$$\mathrm{L}^2(\mathbb{R}^3)^4 = \bigoplus_{\kappa\in\mathbb{Z}\setminus\{0\}} \bigoplus_{\ell=1}^{\ell(\kappa)} \mathcal{S}_{\kappa,\ell}$$

which completely reduces H, and the restriction $H \upharpoonright \mathcal{S}_{\kappa,\ell}$ of H to $\mathcal{S}_{\kappa,\ell}$ is unitarily equivalent to a differential operator generated by the differential expression $\tau_{\kappa,\ell} = \tau_\kappa$, $\ell \in \{1,\dots,\ell(\kappa)\}$, given by

$$\tau_\kappa y(x) = Jy'(x) + \begin{pmatrix} -1 + V(x) & \frac{\kappa}{x} \\ \frac{\kappa}{x} & 1 + V(x) \end{pmatrix} y(x), \quad x \in (0,\infty),$$

in $\mathrm{L}^2(0,\infty)^2$, where J denotes the constant matrix

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For a fixed $\kappa \in \mathbb{Z}\setminus\{0\}$ the maximal operator H_κ generated by τ_κ has the domain

$$\mathcal{D}(H_\kappa) = \left\{\, y \in \mathrm{L}^2(0,\infty)^2 \cap \mathrm{AC}(0,\infty)^2 : \tau_\kappa y \in \mathrm{L}^2(0,\infty)^2 \right\},$$

while the corresponding minimal operator $H_{\kappa,0}$ is defined on

$$\mathcal{D}(H_{\kappa,0}) = \{\, y \in \mathcal{D}(H_\kappa) : y \text{ has compact support in } (0,\infty) \,\}$$

(see [14, Section 3]). Below we prove that H_κ is the unique self-adjoint extension of the (densely defined and symmetric) operator $H_{\kappa,0}$ in $\mathrm{L}^2(0,\infty)^2$, and that the essential spectrum of this so-called *radial Dirac operator* H_κ is given by $\sigma_{ess}(H_\kappa) = (-\infty,-1] \cup [1,\infty)$. Moreover, we investigate the problem of eigenvalue accumulation from within the gap $(-1,1)$ in the essential spectrum at the endpoints -1 and 1.

Now, for a fixed $\lambda \in [-1,1]$, the differential equation $(\tau_\kappa - \lambda)y = 0$ is equivalent to the Dirac system

$$Jy'(x) + Q(x,\lambda)y(x) = 0, \quad x \in (0,\infty), \tag{21}$$

where

$$Q(x,\lambda) = \frac{1}{x}\begin{pmatrix} \rho & \kappa \\ \kappa & \rho \end{pmatrix} + \begin{pmatrix} V(x) - \frac{\rho}{x} - 1 - \lambda & 0 \\ 0 & V(x) - \frac{\rho}{x} + 1 - \lambda \end{pmatrix}$$

is of the form (12). In addition, there exists a point $x_0 \in (0,\infty)$ such that $|V(x)| \le \varepsilon < 1$ for all $x \ge x_0$. Since the function $V - \frac{\rho}{x}$ is integrable on $(0,a]$ for any $a \in (x_0,\infty)$, we can apply Theorem 3.1 which yields that (21) has a fundamental system of solutions

$$y_1(x,\lambda) = x^\alpha h_1(x,\lambda), \quad y_2(x,\lambda) = x^{-\alpha} h_2(x,\lambda)$$

on the x-interval $(0,a]$, where $\alpha := \sqrt{\kappa^2 - \rho^2}$ and $h_i : (0,a]\times[-1,1] \longrightarrow \mathbb{R}^2$, $i \in \{1,2\}$, are continuous functions satisfying $\lim_{x\to 0} h_i(x,\lambda) = \eta_i$ with some constant vectors $\eta_i \ne 0$. Moreover, $\alpha > \frac{1}{2}$ implies that a solution y of (21) is square-integrable on $(0,a]$ if and only if y is a constant multiple of $y_1(\,\cdot\,,\lambda)$.

Lemma 5.1. *The differential operator $H_{\kappa,0}$ is essentially self-adjoint, H_κ is the closure of $H_{\kappa,0}$, and $\sigma_{ess}(H_\kappa) = (-\infty,-1] \cup [1,\infty)$.*

Proof. By [14, Theorem 6.8], τ_κ is in the limit point case at ∞. Moreover, $y_2(\,\cdot\,,0)$ is a solution of (21) for $\lambda = 0$ which is not square-integrable on $(0,a]$. Hence, by Weyl's alternative (see, e.g., [14, Theorem 5.6]), τ_κ is also in the limit point case at 0. Now, [14, Theorem 5.8 (i)] implies that the closure $\overline{H_{\kappa,0}}$ of $H_{\kappa,0}$ is the only self-adjoint extension of H_κ, and from [14, Theorem 3.9] it follows that $\overline{H_{\kappa,0}} = H_\kappa$. In order to prove the second assertion, define

$$Q_0 := \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From $\lim_{x\to\infty} V(x) = 0$ it follows that $\lim_{x\to\infty} \frac{1}{x}\int_1^x |Q(x,0) - Q_0|\, dt = 0$. Hence, [14, Theorem 16.6] yields that $\mathbb{R} \setminus (-1,1) \subset \sigma_{ess}(H_\kappa)$ (-1 and 1 being the the eigenvalues of Q_0). Moreover, if A is any self-adjoint realization of τ_k on $[a,\infty)$, then [14, Theorem 16.6] implies that $\sigma_{ess}(A) \cap (-1,1) = \emptyset$. It remains to prove that $\sigma_{ess}(B) \cap (-1,1) = \emptyset$ for any self-adjoint realization of τ_k on $(0,a]$ since in this case $\sigma_{ess}(H_\kappa) = \sigma_{ess}(A) \cup \sigma_{ess}(B)$ according to [14, p. 165]. For this purpose, let B be the self-adjoint realization of τ_k on $(0,a]$ induced by the boundary condition

$$y(a) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \tag{22}$$

with some $\theta \in \mathbb{R}$. Moreover, let $y_0(\,\cdot\,,\lambda)$ be the solution of the differential equation (21) on $(0,a]$ satisfying (22) at a for all $\lambda \in (-1,1)$, and suppose that $\phi(\,\cdot\,,\lambda)$ is the angle function of $y_0(\,\cdot\,,\lambda)$ with the property that $\phi(a,\lambda) = \theta$. From Proposition 3.1 it follows that $\lim_{x\to 0} \phi(x,\lambda)$ exists for every $\lambda \in (-1,1)$, and by [14, Theorem 16.4] we have

$$\dim\left(E(\lambda_1) - E(\lambda_2)\right) \le \lim_{x\to 0} \frac{1}{\pi}\left(\phi(a,\lambda_1) - \phi(a,\lambda_2)\right) < \infty$$

for all $\lambda_1 < \lambda_2$ in $(-1,1)$, where E denotes the spectral resolution of B. Hence, $\sigma_{ess}(B) \cap (-1,1) = \emptyset$. ▮

Next, we consider the question if 1 is an accumulation point of eigenvalues of H_κ from below. For this purpose, we reduce the eigenvalue problem for H_κ to a Sturm-Liouville problem which satisfies the assumptions of the previous section.

We first note that for any $a \in [x_0,\infty)$ a point $\lambda \in (-1,1)$ is an eigenvalue of H_κ if and only if (21), restricted to $[a,\infty)$, has a solution $y \in \mathrm{L}^2[a,\infty)^2$ satisfying the interface condition

$$y(a) = C y_1(a,\lambda) \tag{23}$$

with some $C \in \mathbb{R} \setminus \{0\}$. In addition, we can fix some point $\lambda_0 \in [0,1)$ such that $1 - \lambda + V > 0$ on $[x_0,\infty)$ for all $\lambda \in [\lambda_0, 1)$. Now, if we introduce the transformation

$$y(x) = \begin{pmatrix} x^{\kappa} v(x) \\ x^{-\kappa} u(x) \end{pmatrix}, \quad x \in [a,\infty), \tag{24}$$

then, for every $\lambda \in [\lambda_0, 1)$, the system (21) on the x-interval $[a,\infty)$ is equivalent to the Sturm-Liouville equation

$$(p(x,\lambda)u'(x))' - q(x,\lambda)u(x) = 0, \quad x \in [a,\infty), \tag{25}$$

where

$$p(x,\lambda) = \frac{x^{-2\kappa}}{1+\lambda - V(x)}, \quad q(x,\lambda) = x^{-2\kappa}(1-\lambda+V(x))$$

and $v(x) = p(x,\lambda)u'(x)$. Obviously, the coefficients p and q satisfy the hypotheses (i), (ii) of Section 4 on $[a,\infty) \times [\lambda_0, 1)$. Moreover, there exists a continuous function $c : [\lambda_0, 1) \longrightarrow (a, \infty)$ such that $|V| \le \frac{1}{2}(1-\lambda)$ on $[c(\lambda),\infty)$ for all $\lambda \in [\lambda_0, 1)$. Hence,

$$\begin{aligned} \frac{1}{2}\,x^{2\kappa} \;&\le\; \frac{1}{p(x,\lambda)} \;\le\; 2\,x^{2\kappa}, \\ \frac{1}{2}\,(1-\lambda)\,x^{-2\kappa} \;&\le\; q(x,\lambda) \;\le\; \frac{3}{2}\,(1-\lambda)\,x^{-2\kappa} \end{aligned} \tag{26}$$

for $x \in [c(\lambda),\infty)$, $\lambda \in [\lambda_0, 1)$, and Theorem 4.4 implies that the assumptions (iii) and (P) of Section 4 are also satisfied.

In order to establish the boundary conditions, we write $y_1(\,\cdot\,,\lambda)$ in the form

$$y_1(x,\lambda) = \begin{pmatrix} v_1(x,\lambda) \\ u_1(x,\lambda) \end{pmatrix}, \quad x \in (0,a].$$

Since the function $x^{-\kappa}u_1(\,\cdot\,,\lambda)$ is a nontrivial solution of (25) on $[x_0,a]$, it has no accumulation points of zeros in this compact interval according to the Separation Theorem. Hence, we can assume that $u_1(a,1) \ne 0$ (otherwise, replace a by a point in $[x_0,a]$ with this property). Now, from Theorem 3.2 it follows that $u_1(a,\lambda) \ne 0$ for all $\lambda \in [\mu,1]$ with some constant $\mu \in [\lambda_0,1)$, and $Q(x,\lambda_1) - Q(x,\lambda_2) = (\lambda_2-\lambda_1)I \ge 0$ for $\lambda_1 < \lambda_2$ implies that the mapping

$$\lambda \longmapsto \frac{v_1(a,\lambda)}{u_1(a,\lambda)}$$

is decreasing on $[\mu,1]$. If we define

$$\alpha(\lambda) := a^{-\kappa}v_1(a,\lambda), \quad \beta(\lambda) := -a^{\kappa}u_1(a,\lambda)$$

then the functions α, $\beta : [\mu,1] \longrightarrow \mathbb{R}$ satisfy hypothesis (iv) in Section 4, where $\beta(\lambda) \ne 0$ for $\lambda \in [\mu,1]$, and $\lambda \longmapsto \frac{\alpha(\lambda)}{\beta(\lambda)}$ is increasing on $[\mu,1]$. Finally, by (23) and (24), a point $\lambda \in [\mu,1)$ is an eigenvalue of H_κ if and only if (21) has a nontrivial solution u such that $x^{-\kappa}u$, $x^{\kappa}v$ belong to $\mathrm{L}^2([a,\infty),\mathbb{R})$ and

$$\alpha(\lambda)u(a) + \beta(\lambda)v(a) = 0. \tag{27}$$

Now, from (26) and Lemma A.2 (see Appendix) it follows that a solution u of (21) satisfies $x^{-\kappa}u$, $x^{\kappa}v \in \mathrm{L}^2([a,\infty),\mathbb{R})$ if and only if u is principal at ∞. Hence, the eigenvalues of H_κ in $[\mu,1)$ coincide with the eigenvalues of the λ-nonlinear Sturm-Liouville problem (25), (27). Since, in addition to (i) – (iv) and (P), the monotonicity conditions (L) and (M) are satisfied, we can apply Corollary 4.1 with $\nu = 1$ which yields the following result.

THEOREM 5.1. *Suppose that the potential V satisfies the assumptions* (a) – (c). *Then the eigenvalues of the radial Dirac operator H_κ, $\kappa \in \mathbb{Z}\setminus\{0\}$, in the gap $(-1,1)$ accumulate at 1 if and only if the differential equation*

$$\left(\frac{x^{-2\kappa}u'(x)}{2-V(x)}\right)' - x^{-2\kappa}V(x)u(x) = 0 \tag{28}$$

is oscillatory at ∞.

By a similar reasoning, reducing (21) to a Sturm-Liouville equation for the component v in (24) and replacing λ by $-\lambda$, we obtain an analoguous result on the eigenvalue accumulation at the point -1.

THEOREM 5.2. *Suppose that the potential V satisfies the assumptions* (a) – (c). *Then the eigenvalues of the radial Dirac operator H_κ, $\kappa \in \mathbb{Z}\setminus\{0\}$, in the gap $(-1,1)$ accumulate at -1 if and only if the differential equation*

$$\left(\frac{x^{2\kappa}v'(x)}{2+V(x)}\right)' + x^{2\kappa}V(x)v(x) = 0 \tag{29}$$

is oscillatory at ∞.

Finally, if we apply Sturm's Comparison Theorem to (28), (29) and the differential equation

$$\left(x^{\gamma+1}u'(x)\right)' - c\,x^{\gamma-1}u(x) = 0$$

which is oscillatory at ∞ in case of $c < -\frac{1}{4}\gamma^2$ and non-oscillatory at ∞ in case of $c > -\frac{1}{4}\gamma^2$, we obtain the following criterion:

THEOREM 5.3. *Suppose that the potential V satisfies the assumptions* (a) – (c). *Then the eigenvalues of the radial Dirac operator H_κ, $\kappa \in \mathbb{Z}\setminus\{0\}$, in the interval $(-1,1)$ accumulate at*

$$\begin{cases} 1 & \textit{if} \quad \limsup_{x\to\infty} x^2V(x) \;<\; -\frac{1}{8}(2\kappa+1)^2, \\ -1 & \textit{if} \quad \liminf_{x\to\infty} x^2V(x) \;>\; \frac{1}{8}(2\kappa-1)^2. \end{cases}$$

Further, the eigenvalues of H_κ in the gap $(-1,1)$ do not accumulate at

$$\begin{cases} 1 & \textit{if} \quad \liminf_{x\to\infty} x^2 V(x) \; > \; -\frac{1}{8}(2\kappa+1)^2, \\ -1 & \textit{if} \quad \limsup_{x\to\infty} x^2 V(x) \; < \; \frac{1}{8}(2\kappa-1)^2. \end{cases}$$

Remark 5. 1. The assumptions (a), (b), (c) on the potential V required in Theorem 5.3 are much weaker than the conditions established by Griesemer and Lutgen (see [3, Theorem 3]), where, e.g., V is assumed to be continuously differentiable and bounded at 0. The first assertion of Theorem 5.3 concerning the eigenvalue accumulation of H_κ at 1 was proved in the Appendix of [3] under the additional assumption that $-2 \le V \le 0$ a.e. in $(0,\infty)$.

COROLLARY 5.1. *Suppose that the potential V satisfies the assumptions* (a) – (c). *Then the eigenvalues of the Dirac operator H in $(-1,1)$ accumulate at*

$$\begin{cases} 1, & \textit{if} \quad \limsup_{x\to\infty} x^2 V(x) \; < \; -\frac{1}{8}, \\ -1, & \textit{if} \quad \liminf_{x\to\infty} x^2 V(x) \; > \; \frac{1}{8}. \end{cases}$$

Proof. If $\lambda \in (-1,1)$ is an eigenvalue of H_κ for some $\kappa \in \mathbb{Z} \setminus \{0\}$, then λ is an eigenvalue of H, and the assertion follows from Theorem 5.3. ▮

Remark 5. 2. It is well known (see [13, Theorem 10.37]) that -1 is not an accumulation point of eigenvalues of the Dirac operator H if the potential V is non-positive, i.e., $V(x) \le 0$, $x \in (0,\infty)$. For the radial Dirac operators H_κ, $\kappa \in \mathbb{Z} \setminus \{0\}$, this result follows from Theorem 5.2 since equation (28) is non-oscillatory at ∞ in case of $V \le 0$ (see Lemma A.1 in the Appendix).

APPENDIX

In the following we recall some basic notations and results for Sturm-Liouville equations

$$(p(x)u'(x))' - q(x)u(x) = 0, \quad x \in I, \tag{A.1}$$

on an interval $I = [a,b)$, $-\infty < a < b \le \infty$, where $p > 0$ holds a.e. in I and $\frac{1}{p}$, $q : I \longrightarrow \mathbb{R}$ are locally integrable functions (see [4, Chapter XI, Section 6] or [9, Chapter IV, Section 3]). A function $u : I \longrightarrow \mathbb{R}$ is said to be a *solution* of (A.1) if u and its quasi-derivative $v = p\,u'$ are absolutely continuous on I and $v' = q\,u$ holds a.e. in I. The differential equation (A.1) is called *disconjugate* on a subinterval $J \subset I$ if every nontrivial solution of (A.1) has at most one zero in J; if at least one nontrivial solution has infinitely many zeros in J, then (A.1) is said to be *oscillatory* on J.

In the case that (A.1) is non-oscillatory at b (i.e. non-oscillatory on every subinterval $[c, b)$ of I), then a nontrivial solution u of (A.1) is called *principal* at b if there exists a solution $u_0 \not\equiv 0$ linearly independent of u such that $\frac{u(x)}{u_0(x)} \to 0$ as $x \to b$ (such a principal solution always exists, and it is unique up to a multiplicative constant). A solution $u \not\equiv 0$ of (A.1) is principal at b if and only if

$$\int_c^b \frac{1}{p(t)u(t)^2}\,dt = \infty$$

where the lower limit of integration c is any point beyond the last zero of u.

The following lemma is a generalization of a known result for the case of continuous coefficients p, q (see [4, Chapter XI, Corollary 6.4 and Exercise 6.7]).

LEMMA A.1. *If $q \ge 0$ a.e. in $[c, b)$ for some $c \in I$, then equation (A.1) is disconjugate on $[c, b)$. Moreover, there exists a principal solution u_1 of (A.1) at b satisfying*

$$u_1(x) > 0, \quad p(x)u_1'(x) < 0, \quad x \in [c, b), \tag{A.2}$$

and a non-principal solution u_2 with the property that

$$u_2(x) \ge 1, \quad p(x)u_2'(x) \ge 1, \quad x \in [c, b). \tag{A.3}$$

In addition, if either $\int_I \frac{1}{p(t)}\,dt = \infty$ or $\int_I q(t)\,dt = \infty$ holds, then the solution u_1 is uniquely determined by (A.2) and the initial value $u_1(c)$.

Proof. Let u be the solution of (A.1) which satisfies $u(c) = 1$ and $v(c) = 1$ where $v = p\,u'$. Then, for every $x_0 \in [c, b)$, we have

$$u(x) = u(x_0) + \int_{x_0}^{x} \frac{1}{p(t)} v(t)\,dt, \;\; v(x) = v(x_0) + \int_{x_0}^{x} q(t)u(t)\,dt \tag{A.4}$$

for all $x \in [x_0, b)$. Since $q \ge 0$ holds a.e. in $[c, b)$, (A.4) with $x_0 = c$ implies that $u(x) \ge 1$ and $v(x) \ge 1$ in a right neighbourhood of c. Further, let

$$\gamma := \sup\{\xi \in (c, b) : u(x) \ge 1,\, v(x) \ge 1,\, x \in [c, \xi)\}.$$

Assuming $\gamma < b$, (A.4) with $x_0 = \gamma$ implies that $u(x) \geq 1$ and $v(x) \geq 1$ in a right neighbourhood of γ, a contradiction. Therefore $\gamma = b$, and we obtain $u(x) \geq 1$, $v(x) \geq 1$ for all $x \in [c, b)$. Now, from $u(x) \geq 1 + \int_c^x \frac{1}{p(t)}\, dt$, $x \in [c, b)$, it follows that

$$\begin{aligned}\int_c^x \frac{1}{p(t)u(t)^2}\, dt &\leq \int_c^x \frac{1}{p(t)} \left(1 + \int_c^t \frac{1}{p(s)}\, ds\right)^{-2} dt \\ &= 1 - \left(1 + \int_c^x \frac{1}{p(t)}\, dt\right)^{-1} \leq 1, \quad x \in [c, b).\end{aligned}$$

Hence $\int_c^b \frac{1}{p(t)u(t)^2}\, dt < \infty$, and $u_2 = u$ is a non-principal solution of (A.1) at b which satisfies (A.3). Now, let us define

$$u_1(x) := u(x) \int_x^b \frac{1}{p(t)u(t)^2}\, dt, \quad x \in I.$$

Obviously, $u_1 > 0$ on $[c, b)$. Moreover, the quasi-derivative $v_1 = p\, u_1'$ satisfies

$$v_1(x) = v(x) \int_x^b \frac{1}{p(t)u(t)^2}\, dt - \frac{1}{u(x)}, \quad x \in I,$$

and therefore $v_1'(x) = q(x)u_1(x)$ for $x \in I$. In particular, u_1 and u_2 are linearly independent solutions of (A.1) which have no zeros in $[c, b)$, and thus (A.1) is disconjugate on $[c, b)$. Since v_1 is increasing on $[c, b)$, we get

$$v_1(x) \leq \int_x^b \frac{v(t)}{p(t)u(t)^2}\, dt - \frac{1}{u(x)} = -\lim_{t \to b} \frac{1}{u(t)} \leq 0, \quad x \in [c, b).$$

Moreover, from

$$\int_c^x \frac{1}{p(t)u_1(t)^2}\, dt = \left(\int_x^b \frac{1}{p(t)u(t)^2}\, dt\right)^{-1} - \left(\int_c^b \frac{1}{p(t)u(t)^2}\, dt\right)^{-1}, \ x \in [c, b),$$

it follows that $\int_c^b \frac{1}{p(t)u_1(t)^2}\, dt = \infty$. This means, u_1 is a principal solution of (A.1) at b which has the required properties. In order to prove that u_1 is uniquely determined if $\int_I \frac{1}{p(t)}\, dt = \infty$ or $\int_I q(t)\, dt = \infty$, suppose that $u_0 \neq u_1$ is another solution of (A.1) which satisfies $u_0(c) = u_1(c)$ and the condition (A.2). The disconjugacy of (A.1) on $[c, b)$ implies $u_0(x) \neq u_1(x)$ for all $x \in (c, b)$; otherwise, $u_0 - u_1$ would be a nontrivial solution of (A.1) with at least two zeros in $[c, b)$. Without loss of generality, we can assume that $u_0(x) > u_1(x)$ for all $x \in (c, b)$. If we define $v_0 := p\,u_0'$, then

$$\int_c^x \frac{1}{p(t)}\,(v_0(t) - v_1(t))\;dt = u_0(x) - u_1(x) \geq 0,\;\; x \in [c, b), \tag{A.5}$$

implies $v_0(x) \geq v_1(x)$ for all $x \in [c, b)$, and from the Existence and Uniqueness Theorem it follows that $v_0(c) > v_1(c)$. Moreover,

$$v_0(x) - v_1(x) = v_0(c) - v_1(c) + \int_c^x q(t)\,(u_0(t) - u_1(t))\;dt, \quad x \in [c, b),$$

yields that $v_0(x) - v_1(x) \geq \varepsilon_1$ for all $x \in [c, b)$ with some constant $\varepsilon_1 > 0$, and from (A.5) we deduce $u_0(x) - u_1(x) \geq \varepsilon_2$ for all $x \in [x_0, b)$ with some $x_0 \in (c, b)$ and $\varepsilon_2 > 0$. Since, in addition, $u_1(x) > 0$ for $x \in [c, b)$, we obtain

$$v_0(x) \geq v_0(x_0) + \varepsilon_2 \int_{x_0}^x q(t)\,dt,\;\; u_0(x) \geq \varepsilon_1 \int_{x_0}^x \frac{1}{p(t)}\,dt \tag{A.6}$$

for all $x \in [x_0, b)$. If $\int_I q(t)\,dt = \infty$, then (A.6) implies $v_0(x) \to \infty$ as $x \to b$, in contrast to $v_0(x) \leq 0$ for $x \in [c, b)$. Similarly, $\int_I \frac{1}{p(t)}\,dt = \infty$ and (A.6) imply that $u_0(x) \to \infty$ as $x \to b$, a contradiction to

$$u_0(x) = u_0(c) + \int_c^b \frac{1}{p(t)} v_0(t)\,dt \leq u_0(c)$$

for $x \in [c, b)$. ∎

LEMMA A.2. *Suppose that $q \geq 0$ holds a.e. in $[c, b)$ for some $c \in I$. If u is a principal solution of* (A.1) *and $v := p\,u'$ is its quasi-derivative, then*

$$\int_I \left(\frac{1}{p(t)} v(t)^2 + q(t) u(t)^2\right)\,dt < \infty.$$

Moreover, if $\int_I q(t)\,dt = \infty$, a nontrivial solution u of (A.1) *is principal at b if and only if* $\lim_{x\to b} u(x) = 0$.

Proof. Let u_1 be the principal solution specified in Lemma A.1 and $v_1 := p\,u_1'$. From (A.2) we conclude that u_1 is strictly decreasing on $[c, b)$ and bounded from below, while v_1 is increasing on $[c, b)$ and bounded from above. Hence, $U_1 = \lim_{x\to b} u_1(x) < \infty$ and $V_1 = \lim_{x\to b} v_1(x) < \infty$. Since

$$(u_1 \cdot v_1)' = \frac{1}{p}\,v_1^2 + q\,u_1^2,$$

we obtain

$$\int_I \left(\frac{1}{p(t)} v_1(t)^2 + q(t)u_1(t)^2\right)\, dt = U_1 V_1 - u_1(a)v_1(a) < \infty.$$

Moreover, if $\int_I q(t)\, dt = \infty$, we have $U_1 = 0$; otherwise,

$$v_1(x) \geq v_1(c) + U_1 \int_c^x q(t)(t)\, dt$$

implies $v_1(x) \to \infty$ as $x \to b$, a contradiction. The converse of the second assertion holds since u_1 and the non-principal solution u_2 in Lemma A.1 with the property (A.3) form a fundamental system of (A.1). ∎

REFERENCES

1. C. D. Ahlbrandt, *Principal and antiprincipal solutions of selfadjoint differential systems and their reciprocals*, Rocky Mountain J. Math., **2** (1972), pp. 169–182.
2. M. S. P. Eastham, *The Asymptotic Solution of Linear Differential Systems. Applications of the Levinson Theorem*, London Mathematical Society Monographs (New Series) 4, The Clarendon Press, Oxford University Press, New York, 1989.
3. M. Griesemer and J. P. Lutgen, *Accumulation of discrete eigenvalues of the radial Dirac operator*, J. Funct. Anal., **162** (1999), pp. 120–134.
4. P. Hartman, *Ordinary Differential Equations*, John Wiley & Sons, Inc., New York – London – Sydney, 1964.
5. J. P. Lutgen, *Eigenvalue accumulation for singular Sturm–Liouville problems nonlinear in the spectral parameter*, J. Differential Equations, **159** (1999), pp. 515–542.
6. R. Mennicken, H. Schmid and A. A. Shkalikov, *On the eigenvalue accumulation of Sturm–Liouville problems depending nonlinearly on the spectral parameter*, Math. Nachr., **189** (1998), pp. 157–170.
7. T. W. Prevatt, *Singular boundary value problems for linear Hamiltonian systems depending on a parameter*, J. Differential Equations, **20** (1976), pp. 1–17.
8. W. T. Reid, *A continuity property of principal solutions of linear Hamiltonian differential systems*, Scripta Math., **29** (1973), pp. 337–350.
9. ———, *Sturmian Theory for Ordinary Differential Equations*, Applied Mathematical Sciences, **31**, Springer-Verlag, New York – Berlin, 1980.
10. H. Schmid, *Das Häufungsverhalten der Eigenwerte bei Randeigenwertproblemen zu Störungen regulär-singulärer Differentialgleichungen*, Dissertation, Universität Regensburg, 1998.
11. B. Thaller, *The Dirac Equation*, Texts and Monographs in Physics, Springer-Verlag, Berlin, 1992.
12. V. Vogelsang, *Remark on essential selfadjointness of Dirac operators with Coulomb potentials*, Math. Z., **196** (1987), pp. 517–521.
13. J. Weidmann, *Linear Operators in Hilbert Spaces*, Graduate Texts in Mathematics, **68**, Springer-Verlag, New York – Berlin, 1980.
14. ———, *Spectral Theory of Ordinary Differential Operators*, Lecture Notes in Mathematics, **1258**, Springer-Verlag, Berlin – New York, 1987.