Change of base, Cauchy-completness and reversibility

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Abstract: We investigate the effect of the change of base 2-functors $\mathcal{V}-Cat \to \mathcal{W}-Cat$ induced by two-sided enrichments $\mathcal{V} \to \mathcal{W}$ on Cauchy-complete objects. We restrict our study to the case of locally posetal bases. Reversibility is defined for locally posetal bicategories, two-sided enrichments and, Cauchy-completion. We show that a reversible left adjoint two-sided enrichment $F : \mathcal{V} \to \mathcal{W}$ between locally posetal bicategories induces some adjunction $F_\sim \dashv F^\sim : \mathcal{V}-SkCRcCat \to \mathcal{W}-SkCRcCat$ between sub-categories of skeletal and Cauchy-reversible-complete enrichments. We give two examples of application, sheaves and group actions.

keywords: Enriched categories, two-sided enrichments, change of base, reversibility, Cauchy-completion, sheaves.

1 Introduction

Our motivation for the study of the change of base for enrichments over bicategories originated with the characterisation of sheaves as skeletal and Cauchy-complete enrichments over bicategories [Wal81], together with the fact a continuous map $f : X \to Y$ yields an adjunction between the categories of sheaves $f_* \dashv f^* : Sh(Y) \rightharpoonup Sh(X)$. The problem was first to define a good notion of base morphism $F : \mathcal{V} \to \mathcal{W}$. We had in mind that such a morphism should first induce a 2-adjunction at the level of enrichments, say $F_@ \dashv F^@ : \mathcal{V}-Cat \to \mathcal{W}-Cat$. The secund point is that this adjunction should also "behave well" with respect to the skeletality and Cauchy-completness of objects: there should exist a categorical connection between the sub-2-categories of skeletal and Cauchy-complete

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objects respectively of $\mathcal{V}-Cat$ and $\mathcal{W}-Cat$. The first point of our problem was answered in [KLSS99]. Let us start from the definition of monoidal functor between monoidal categories [Ben63] and enrichments over them [EiKe66], [Law73]. A monoidal functor $F : \mathcal{V} \to \mathcal{W}$ induces a 2-functor $F_@ : \mathcal{V}-Cat \to \mathcal{W}-Cat$. $MonCat$ is equipped with a 2-categorical structure by defining 2-cells in it as monoidal natural transformations ([EiKe66]). The process $(-)_@$ of sending $\mathcal{V}$ to $\mathcal{V}-Cat$ and F to $F_@$ extends to a 2-functor from $MonCat$ to $2-Cat$. Adjunctions in $MonCat$ were characterised in [Kel74]. Moving to the case of enrichments over bicategories, several notions of "morphism" between bicategories where proposed in the literature, but adjoints were not preserved by $(-)_@$. A two-sided enrichment $F : \mathcal{V} \to \mathcal{W}$ between two bicategories was proposed in [KLSS99] as a slight generalisation of monoidal functors and Benabou's lax functors [Ben67]. With these new morphisms one obtained a bicategory $Base$ together with the expected pseudo-functor $(-)_@ : Base \to 2-Cat$. Introducing some 3-cells on $Base$, one eventually gets a tricategory $Caten$ such that the 2-categories of enrichments are representable $\mathcal{V}-Cat \cong Caten(1,\mathcal{V})$ (1 is the unit bicategory). These new morphisms of bicategories could be seen as a two-sided version of the usual enrichments over bicategories, hence they were named "two-sided enrichments". Adjunctions in $Caten$ were characterised: left adjoint two-sided enrichments are exactly the pseudo-functors with "local right adjoints". This notion of local adjointness is slightly more general than the one in [BePo88]. To complete the picture, let us recall the following results. The cartesian product of bicategories extends to a pseudo-functor $Caten \times Caten \to Caten$ that makes $Caten$ into a monoidal tricategory. $Caten$ has a closed structure $Caten(\mathcal{V}\times\mathcal{W},\mathcal{Z}) \cong Caten(\mathcal{V}, Conv(\mathcal{W},\mathcal{Z}))$. The two-sided enrichments are also just enrichments since thus $Caten(\mathcal{V},\mathcal{W}) \cong Conv(\mathcal{V},\mathcal{W})-Cat$.

This paper treats the second point of our original problem: the effect of change of base $F_@$ for a two-sided enrichment F, on skeletal and Cauchy-complete objects. For convenience, since our purpose was to give a satisfying categorical generalisation of the above adjunction between categories of sheaves and also for convenience, we restricted our study to the case when the base bicategories are locally posetal.

First we investigate conditions on two-sided enrichments F such that $F_@$ preserves skeletality, Cauchy-completion, Morita-equivalence, and so on. We introduce the super two-sided enrichments. They generalise the super monoidal functors used to code uniformly continuous maps as some enriched functors in [Sc00]. They are more general than left adjoint two-sided enrichments and have nice properties. It was known from [KLSS99] that a two-sided enrichment F induces a normal lax functor between bicategories of modules $F_\sharp : \mathcal{V}-Mod \to \mathcal{W}-Mod$. For any super F, $F_\sharp : \mathcal{V}-Mod \to \mathcal{W}-Mod$ preserves adjoints and $F_@ : \mathcal{V}-Cat \to \mathcal{W}-Cat$ preserves the Morita-equivalence.

Eventually we need to introduce the notion of reversibility for bicategories, two-

sided-enrichments and Cauchy-completion. Most familiar examples of enrichments over locally posetal bicategories (sheaves and group actions) enjoy a nice property of symmetry that we call reversibility. As pointed out by Kasangian [BeWal82] a Cauchy-completion of a reversible enrichment fails in general to be reversible. This fact forces us to define a completion for reversible enrichments that preserves reversibility. This is the so-called Cauchy-reversible-completion. Thus we get the desired generalisation: an adjoint pair in $Caten$ $F \dashv G : \mathcal{V} \rightharpoonup \mathcal{W}$ where F is reversible induces some adjunction between sub-categories of skeletal and Cauchy-reversible-complete objects $\mathcal{V}-SkCRcCat \rightharpoonup \mathcal{W}-SkCRcCat$. An instance of this theorem is also the classical result in group theory that a group morphism $f : H \to G$ induces an adjunction between group actions $f_* \dashv f^* : H - Set \rightharpoonup G - Set$.

This paper is organised as follows. Section 2 recalls basics of the theory of enrichments over locally posetal bicategories to make the paper self-contained and to allow later comparisons with the reversible case. Section 3 treats the change of base for enrichments and the connection with two-sided enrichments in the case when the base categories are locally posetal. We first recall some elements of the theory of two-sided enrichments developped [KLSS99]: the definition of the tricategory $Caten$, the connection with enrichments, the characterisation of adjoints in $Caten$, and the change of base. Then we present the various new results of our investigation into the effect of the change of base bicategories on skeletality, and Cauchy-completness of enrichments. The super two-sided enrichments are introduced. Section 4 is devoted to the development of the theory in the reversible case. The notion of reversibility is defined for bicategories, (two-sided) enrichments, enriched modules, completion, etc. We retrace all the steps of the theory. Two detailed examples are given, namely sheaves and group actions.

2 Basic enriched category theory

Definition 2.1 *A bicategory $\mathcal{V}$ is* locally posetal *(respectively* locally preordered*), when for any objects a and b, the homset $\mathcal{V}(a,b)$ is a partial order (respectively a preorder).*

Further on in this section, $\mathcal{V}$ will denote a locally posetal bicategory.

Definition 2.2 (Enrichments over a locally posetal bicategory) *An* enrichment *A over $\mathcal{V}$, also called a $\mathcal{V}$-*category*, is a set $Obj(A)$ — the objects of A, with two mappings $(-)^A_+ : Obj(\mathcal{A}) \longrightarrow Obj(\mathcal{V})$ (sometimes just written $(-)_+$) and $A(-,-) : Obj(A) \times Obj(A) \longrightarrow Arrows(\mathcal{V})$ that satisfy:*
- $(enr-i)$ for any objects a,b of A, $A(a,b) : a_+ \longrightarrow b_+$;
- $(enr-ii)$ for any object a of A, $I_{a_+} \leq A(a,a)$;
- $(enr-iii)$ for any objects a,b,c of A, $A(b,c) \circ A(a,b) \leq A(a,c)$.
*Given $\mathcal{V}$-categories A and B, a $\mathcal{V}$-*functor *f from A to B is a map $f : Obj(A) \longrightarrow$*

$Obj(B)$ such that:
- $(fun-i)$ $(-)^A_+ = (-)^B_+ \circ f$;
- $(fun-ii)$ *for any objects* a, b *of* A, $A(a,b) \leq B(fa, fb)$.
Given $\mathcal{V}$*-functors* $f, g : A \to B$ *there is a unique* $\mathcal{V}$-natural transformation *from* f *to* g, *when for any object* a *of* A, $I_{a_+} \leq B(fa, ga)$.

$\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations form a locally preordered 2-category denoted $\mathcal{V} - Cat$.

Further on, we suppose that:
- $\mathcal{V}$ is *biclosed* i.e. for any arrow $f : a \to b$ and any object c of $\mathcal{V}$, the functors $f \circ - : \mathcal{V}(c,a) \to \mathcal{V}(c,b)$ and $- \circ f : \mathcal{V}(c,b) \to \mathcal{V}(c,a)$ are adjoints.
- $\mathcal{V}$ is *locally cocomplete* i.e. each homset $\mathcal{V}(a,b)$ is cocomplete.

Definition 2.3 ($\mathcal{V}$-module) *Given two* $\mathcal{V}$*-categories* A *and* B, *a* $\mathcal{V}$-module φ *from* A *to* B — *denoted* $\varphi : A \multimap B$ — *is a map* $\varphi : Obj(B) \times Obj(A) \to Obj(\mathcal{V})$ *such that:*
- $(mod-i)$ *for all objects* a *of* A, *and* b *of* B, $\varphi(b,a)$ *is an arrow* $b_+ \to a_+$ *in* $\mathcal{V}$.
- $(mod-ii)$ *for all objects* a, a' *of* A, *and* b *of* B, $A(a,a') \circ \varphi(b,a) \leq \varphi(b,a')$;
- $(mod-iii)$ *for all objects* $a \in Obj(A)$, *and* b, b' *of* B, $\varphi(b,a) \circ B(b',b) \leq \varphi(b',a)$.

For any two $\mathcal{V}$*-modules* $\varphi : A \multimap B$ *and* $\psi : B \multimap C$, *their composite* $\psi \bullet \varphi : A \multimap C$ *is defined by* $\psi \bullet \varphi(c,a) = \bigvee_{b \in Obj(B)} \varphi(b,a) \circ \psi(c,b)$. *The set* $\mathcal{V}-Mod(A,B)$ *of* $\mathcal{V}$*-modules from* A *to* B *is partially ordered by* $\varphi \leq \psi \Leftrightarrow \forall (b,a) \in Obj(B) \times Obj(A), \varphi(b,a) \leq \psi(b,a)$.

Fact 2.4 $\mathcal{V}$*-categories and* $\mathcal{V}$*-modules with partial orders defined above, constitute a locally posetal 2-category denoted* $\mathcal{V} - Mod$.

In $\mathcal{V} - Mod$, the identity in A is the module with underlying map $A(-,-) : Obj(A) \times Obj(A) \to Obj(\mathcal{V})$ sending (y,x) to $A(y,x)$. For any $\mathcal{V}$-categories A and B, a $\mathcal{V}$-module $\varphi : A \multimap B$ has right adjoint $\psi : B \multimap A$ if and only if $A(-,-) \leq \psi \bullet \varphi$ and $\varphi \bullet \psi \leq B(-,-)$. For any left adjoint module φ, $\tilde{\varphi}$ will denote its (unique!) right adjoint. Any $\mathcal{V}$-functor $f : A \to B$ corresponds to a pair of adjoint modules $f_\diamond \dashv f^\diamond$, $f_\diamond : A \multimap B$, $f^\diamond : B \multimap A$, as follows: for all objects a of A, and b of B, $f_\diamond(b,a) = B(b,fa)$, $f^\diamond(a,b) = B(fa,b)$. For any object u in $\mathcal{V}$, $\hat{u}$ will denote the enrichment on $\mathcal{V}$, with one object say $*$, with $*_+ = u$ and $\hat{u}(*,*) = I_u$. If B is a $\mathcal{V}$-category, for any object b in B, $b : \hat{b_+} \to B$ will denote the $\mathcal{V}$-functor sending $*$ to b.

Further on $\mathcal{V} - AMod$ will denote the locally posetal 2-category with:
- objects $\mathcal{V}$-categories,
- arrows: *left adjoint* $\mathcal{V}$-modules;

- 2-cells: partial orders inherited from $\mathcal{V}-Mod$.

There is a 2-functor $J_{\mathcal{V}} : \mathcal{V}-Cat \to \mathcal{V}-Mod$ as follows. It is the identity on objects, and is the map $(-)_\diamond$ on arrows sending any $\mathcal{V}$-functor f to the $\mathcal{V}$-module $f_\diamond$. Note that for any $\mathcal{V}$-categories A and B, for any $\mathcal{V}$-functors $f, g : A \to B$, $f \Rightarrow g$ if and only if $f_\diamond \leq g_\diamond$.

Definition 2.5 (Skeletality, Cauchy-completness) *Let B be a $\mathcal{V}$-category. B is called* skeletal *when for any $\mathcal{V}$-category A, the map $(-)_\diamond : \mathcal{V}-Cat(A,B) \to \mathcal{V}-AMod(A,B)$ is injective. B is called* Cauchy-complete *when for any $\mathcal{V}$-category A, the map $(-)_\diamond : \mathcal{V}-Cat(A,B) \to \mathcal{V}-AMod(A,B)$ is surjective.*

Observation 2.6 *A $\mathcal{V}$-category B is* skeletal *when for any of its objects a and b, if $a_+ = b_+$, $I_{a_+} \leq B(a,b)$ and, $I_{a_+} \leq B(b,a)$ then $a = b$.*

Fact 2.7 *A $\mathcal{V}$-category B is* Cauchy-complete *when for any object u of $\mathcal{V}$ and any left adjoint $\mathcal{V}$-module $\varphi : \hat{u} \multimap B$, $\varphi = b_\diamond$ for some object b of B with $b_+ = u$.*

proof: Suppose that $\varphi : A \multimap B$ has right adjoint. Then for any object a of A, $\varphi \bullet a_\diamond : \hat{a_+} \multimap B$ and $a^\diamond \bullet \tilde{\varphi} : B \multimap \hat{a_+}$ are adjoints. For any objects a of A and b of B,

$$\varphi \bullet a_\diamond(b,*) = \bigvee_{a' \in Obj(A)} A(a',a) \circ \varphi(b,a') = \varphi(b,a)$$

and

$$a^\diamond \bullet \tilde{\varphi}(*,b) = \bigvee_{b' \in Obj(B)} \tilde{\varphi}(a',b) \circ A(a,a') = \tilde{\varphi}(a,b).$$

By assumption, for any object a of A, there is a $\mathcal{V}$-functor $f_a : \hat{a_+} \to B$ such that $(f_a)_\diamond = \varphi \bullet a_\diamond$ and $(f_a)^\diamond = a^\diamond \bullet \tilde{\varphi}$. This means that for any objects a of A and b of B, $\varphi(b,a) = B(b, f_a(*))$ and $\tilde{\varphi}(a,b) = B(f_a(*), b)$. Let f denote the map from $Obj(A)$ to $Obj(B)$ sending a to $f_a(*)$. f defines a $\mathcal{V}$-functor from A to B since for any objects a,a' of A,

$$\begin{aligned} B(fa, fa') &= \textstyle\bigvee_{b \in Obj(B)} B(b, fa') \circ B(fa, b) \\ &= \textstyle\bigvee_{b \in Obj(B)} \varphi(b,a') \circ \tilde{\varphi}(a,b) \\ &= \tilde{\varphi} \bullet \varphi(a,a') \\ &\geq A(a,a'). \end{aligned}$$

∎

We define the following locally posetal 2-categories.

- $\mathcal{V}-SkCcCat$, with:
  - objects: skeletal and Cauchy-complete $\mathcal{V}$-categories,
  - arrows and 2-cells: inherited from $\mathcal{V}-Cat$.

- $\mathcal{V} - SkCcAMod, with$:
  - objects: skeletal and Cauchy-complete $\mathcal{V}$-categories,
  - arrows and 2-cells: inherited from $\mathcal{V} - AMod$.

Fact 2.8 *The restriction of $J_{\mathcal{V}}$ on $\mathcal{V} - SkCcCat$ is an isomorphism onto $\mathcal{V} - SkCcAMod$.*

Definition 2.9 (Cauchy-completion) *Let A be a $\mathcal{V}$-category. The* Cauchy-completion *of A is the $\mathcal{V}$-category $\bar{A}$ defined as follows. Its objects are the left-adjoint $\mathcal{V}$-modules of the form $\varphi : \hat{u} \multimap A$, u ranging in $Obj(\mathcal{V})$. For any object φ of $\bar{A}$ with $\varphi : \hat{u} \multimap A$, $\varphi_+ = u$. For convenience we will consider modules in $Obj(\bar{A})$ as well as their adjoints, as maps with domains A. For any objects φ,ψ of $\bar{A}$, $\bar{A}(\varphi, \psi) = (\tilde{\varphi} \bullet \psi)(*, *)$.*

Let us check that the $\bar{A}$ of 2.9 is a well defined $\mathcal{V}$-category. $\bar{A}$ trivially satisfies $(enr - i)$. Let us show that it satifies $(enr - ii)$. Let φ be an object of $\bar{A}$, then

$$\begin{aligned}\bar{A}(\varphi, \varphi) &= \bigvee_{a \in Obj(A)} \varphi(a) \circ \tilde{\varphi}(a) \\ &= \tilde{\varphi} \bullet \varphi(*, *) \\ &\geq I_u.\end{aligned}$$

Let us show that $\bar{A}$ satisfies $(enr - iii)$. Let φ, ψ, γ be objects of $\bar{A}$. Then

$$\begin{aligned}\bar{A}(\gamma, \psi) \circ \bar{A}(\varphi, \gamma) &= (\bigvee_{a \in Obj(A)} \psi(x) \circ \tilde{\gamma}(a)) \circ (\bigvee_{a' \in Obj(A)} \gamma(a') \circ \tilde{\varphi}(a')) \\ &= \bigvee_{a, a' \in Obj(A)} \psi(a) \circ \tilde{\gamma}(a) \circ \gamma(a') \circ \tilde{\varphi}(a') \\ &\leq \bigvee_{a, a' \in Obj(A)} \psi(a) \circ A(a', a) \circ \tilde{\varphi}(a') \\ &= \bigvee_{a \in Obj(A)} \psi(a) \circ \tilde{\varphi}(a) \\ &= \bar{A}(\varphi, \psi).\end{aligned}$$

Let A be a $\mathcal{V}$-category with Cauchy-completion $\bar{A}$. For any objects a of A and φ of $\bar{A}$, $\bar{A}(a_\diamond, \varphi) = \varphi(a)$ and $\bar{A}(\varphi, a_\diamond) = \tilde{\varphi}(a)$. The map $i_A : Obj(A) \to Obj(\bar{A})$, sending a to $a_\diamond$ is a $\mathcal{V}$-functor from A to $\bar{A}$ since for any object a of A, $(a_\diamond)_+ = a_+$, and for any objects a,b of A, $\bar{A}(a_\diamond, b_\diamond) = \bigvee_{c \in Obj(A)} A(c, b) \circ A(a, c) = A(a, b)$.

Remark 2.10 *For any $\mathcal{V}$-category A,*
- A is skeletal if and only if the map $i_A : Obj(A) \to Obj(\bar{A})$ is injective;
- A is Cauchy-complete if and only the map i_A is onto.

Fact 2.11 *For any $\mathcal{V}$-category A, $\bar{A}$ is skeletal.*

proof: Suppose that φ, ψ are some objects of $\bar{A}$ such that $\varphi_+ = \psi_+ = u$ and, (i) $I_u \leq \bar{A}(\varphi, \psi)$ and, (ii) $I_u \leq \bar{A}(\psi, \varphi)$. The inequation (i) is equivalent to $(1_{\hat{u}})_\diamond \leq \tilde{\varphi} \bullet \psi$ which is equivalent to $\varphi \leq \psi$ since $\varphi \dashv \tilde{\varphi}$. Similarly, (ii) is equivalent to $\psi \leq \varphi$. Therefore $\varphi = \psi$. ∎

Lemma 2.12 *Let A be a $\mathcal{V}$-category and, $\Gamma : \hat{u} \multimap \bar{A}$ be some $\mathcal{V}$-module where u is an object of $\mathcal{V}$. For any object φ of $\bar{A}$, $\tilde{\varphi}$ denoting its adjoint, $\Gamma(\varphi) = \bigvee_{a \in Obj(A)} \Gamma(a_\diamond) \circ \tilde{\varphi}(a)$.*

proof: Let a be some object of A,

$$\begin{aligned}\Gamma(a_\diamond) &= \bigvee_{\psi \in Obj(\bar{A})} \Gamma(\psi) \circ \bar{A}(a_\diamond, \psi), \\ &= \bigvee_{\psi \in Obj(\bar{A})} \Gamma(\psi) \circ \psi(a).\end{aligned}$$

Therefore, for any object φ of $\bar{A}$,

$$\begin{aligned}\bigvee_{a \in Obj(A)} \Gamma(a_\diamond) \circ \tilde{\varphi}(a) &= \bigvee_{a \in Obj(A)} (\bigvee_{\psi \in Obj(\bar{A})} \Gamma(\psi) \circ \psi(a)) \circ \tilde{\varphi}(a) \\ &= \bigvee_{a \in Obj(A)} \bigvee_{\psi \in obj(\bar{A})} \Gamma(\psi) \circ \psi(a) \circ \tilde{\varphi}(a) \\ &= \bigvee_{\psi \in Obj(\bar{A})} \Gamma(\psi) \circ \bar{A}(\varphi, \psi) \\ &= \Gamma(\varphi).\end{aligned}$$

∎

Fact 2.13 *For any $\mathcal{V}$-category A, $\bar{A}$ is Cauchy-complete.*

proof: Given a right adjoint module $\Gamma : \hat{u} \multimap \bar{A}$ with u an object of $\mathcal{V}$. We define the maps $\gamma, \gamma' : Obj(A) \to Arrows(\mathcal{V})$ respectively by $\gamma(a) = \Gamma(a_\diamond)$ and $\gamma'(a) = \tilde{\Gamma}(a_\diamond)$. We are going to show that γ is an object of $\bar{A}$ and $\Gamma = \bar{A}(-, \gamma)$. It is immediate that γ and γ' are modules respectively from $\hat{u}$ to A and from A to $\hat{u}$. Let us show that $\gamma \dashv \gamma'$. According to 2.12 (and its dual!),

$$\begin{aligned}\bigvee_{\varphi \in Obj(\bar{A})} \Gamma(\varphi) \circ \tilde{\Gamma}(\varphi) &= \bigvee_{\varphi \in Obj(\bar{A})} \bigvee_{a,b \in Obj(A)} \Gamma(a_\diamond) \circ \tilde{\varphi}(a) \circ \varphi(b) \circ \tilde{\Gamma}(b_\diamond) \\ &\leq \bigvee_{a,b \in Obj(A)} \Gamma(a_\diamond) \circ A(b,a) \circ \tilde{\Gamma}(b_\diamond) \\ &= \bigvee_{a,b \in Obj(A)} \Gamma(a_\diamond) \circ \bar{A}(b_\diamond, a_\diamond) \circ \tilde{\Gamma}(b_\diamond) \\ &= \bigvee_{a \in Obj(A)} \Gamma(a_\diamond) \circ \tilde{\Gamma}(a_\diamond).\end{aligned}$$

Since $\Gamma \dashv \tilde{\Gamma}$,

$$\begin{aligned}I_u &\leq \bigvee_{\varphi \in Obj(\bar{A})} \Gamma(\varphi) \circ \tilde{\Gamma}(\varphi) \\ &= \bigvee_{a \in Obj(A)} \Gamma(a_\diamond) \circ \tilde{\Gamma}(a_\diamond) \\ &= \bigvee_{a \in Obj(A)} \gamma(a) \circ \gamma'(a)\end{aligned}$$

and for any objects a,b of A,

$$\begin{aligned}\gamma \bullet \gamma'(a,b) &= \gamma'(b) \circ \gamma(a) \\ &= \tilde{\Gamma}(b_\diamond) \circ \Gamma(a_\diamond) \\ &= \Gamma \circ \tilde{\Gamma}(a_\diamond, b_\diamond) \\ &\leq \bar{A}(a_\diamond, b_\diamond) \\ &= A(a,b).\end{aligned}$$

Now, since γ is an object of $\bar{A}$, 2.12 shows that $\Gamma = \bar{A}(-,\gamma)$. ∎

Lemma 2.14 *For any $\mathcal{V}$-category A, $i_{A_\diamond} : A \multimap \bar{A}$ and $i_A{}^\diamond : \bar{A} \multimap A$ are inverse modules.*

proof: For any object φ of $\bar{A}$, the maps $\bar{A}(-,\varphi)$ and $\bar{A}(\varphi,-)$ — sending respectively an object ψ of $\bar{A}$ to $\bar{A}(\psi,\varphi)$ and to $\bar{A}(\varphi,\psi)$ — define adjoint modules. Therefore, for any objects a, b of A,

$$\begin{aligned}(i_A{}^\diamond \bullet i_{A\diamond})(x,y) &= \bigvee_{\varphi\in Obj(\bar{A})} \bar{A}(\varphi, b_\diamond) \circ \bar{A}(a_\diamond, \varphi) \\ &= \bar{A}(b_\diamond, a_\diamond) \\ &= A(a,b).\end{aligned}$$

For any objects φ, ψ of $\bar{A}$,

$$\begin{aligned}(i_{A\diamond} \bullet i_A{}^\diamond)(\varphi,\psi) &= \bigvee_{a\in Obj(A)} \bar{A}(a_\diamond, \psi) \circ \bar{A}(\varphi, a_\diamond) \\ &= \bigvee_{a\in Obj(A)} \psi(a) \circ \tilde{\varphi}(a) \\ &= \bar{A}(\varphi,\psi).\end{aligned}$$

∎

According to this,

Fact 2.15 *$\mathcal{V}-AMod$ and $\mathcal{V}-SkCcAMod$ are equivalent. There is a 2-functor $S : \mathcal{V}-AMod \to \mathcal{V}-SkCcCat$ which is an equivalence of 2-categories. S is defined on objects by $S(A) = \bar{A}$ and on arrow by $S(\varphi) = f_\varphi$ where for any left adjoint $\mathcal{V}$-module $\varphi : A \multimap B$, f_φ is the unique $\mathcal{V}$-functor $f : \bar{A} \to \bar{B}$ satisfying $f_\diamond \bullet i_{A\diamond} = i_{B\diamond} \bullet \varphi$. The inclusion 2-functor $\mathcal{V}-SkCcCat \to \mathcal{V}-Cat$ is a right 2-adjoint.*

According to 2.15, skeletal and Cauchy-complete $\mathcal{V}$-categories form a reflective subcategory of the underlying category of $\mathcal{V}-Cat$. The left adjoint of the inclusion sends a $\mathcal{V}$-category A to $\bar{A}$ and the unit takes value $i_A : A \to \bar{A}$ in A.

According to 2.11 and 2.14, the following is coherent

Definition 2.16 (Morita-equivalence) *Two $\mathcal{V}$-categories A and B are* Morita-equivalent *when one the following equivalent assertions is satisfied:*
(i) Their Cauchy-completions are isomorphic in $\mathcal{V}-Cat$;
(ii) They are isomorphic in $\mathcal{V}-Mod$.

3 Change of base

The study of the change of base for enrichments over bicategories yielded the general concept of two-sided enrichments (see [KLSS99] for the general treatment). We will

specify the theory in the particular and simpler case when the base bicategories for enrichments are locally posetal (3.1-3.4). We keep on using the original terminology of [KLSS99]. Section 3.5 presents a study of the effect on the change of base bicategories on skeletality and Cauchy-completness of enrichments. Super two-sided enrichments are introduced.

3.1 Two-sided enrichments

Caten is the tricategory as follows. Its objects are locally posetal bicategories. An arrow $A : \mathcal{V} \to \mathcal{W}$ in *Caten* is a *two-sided enrichment*. It consists in:
- a span in *Set*

$$\begin{array}{ccc} & Obj(A) & \\ {\scriptstyle (-)^A_-}\swarrow & & \searrow{\scriptstyle (-)^A_+} \\ Obj(\mathcal{V}) & & Obj(\mathcal{W}) \end{array}$$

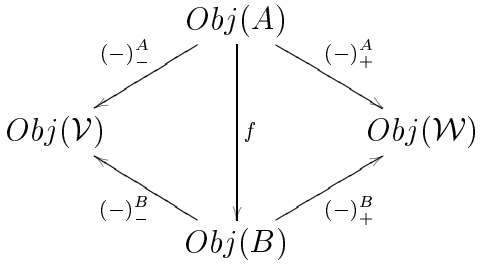
where $Obj(A)$ is the set of "objects" of A,
- a collection of functors $A_{a,b} : \mathcal{V}(a_-, b_-) \to \mathcal{W}(a_+, b_+)$, (a, b) ranging in $Obj(A)^2$, that satisfy the coherence conditions (c),(u) below.
(c) For any objects a,b,c of A, $A_{a,b}(-) \otimes A_{b,c}(?) \leq A_{a,c}(- \otimes ?)$, i.e there exists a 2-cell in *Po*, the locally posetal 2-category of partial orders, monotonous maps with pointwise ordering, as follows

$$\begin{array}{ccc} \mathcal{V}(a_-, b_-) \times \mathcal{V}(b_-, c_-) & \xrightarrow{\circ^{\mathcal{V}}} & \mathcal{V}(a_-, c_-) \\ \downarrow{\scriptstyle A_{a,b} \times A_{b,c}} & \swarrow & \downarrow{\scriptstyle A_{a,c}} \\ \mathcal{W}(a_+, b_+) \times \mathcal{W}(b_+, c_+) & \xrightarrow[\circ^{\mathcal{W}}]{} & \mathcal{W}(a_+, c_+) \end{array}$$

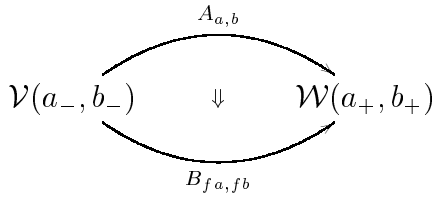
(u) For any object a of A, $I^{\mathcal{W}}_{a_+} \leq A_{a,a}(I^{\mathcal{V}}_{a_-})$ which corresponds to the 2-cell in *Po*

$$\begin{array}{ccc} 1 & \xrightarrow{I^{\mathcal{V}}_{a_-}} & \mathcal{V}(a_-, a_-) \\ \| & \swarrow & \downarrow{\scriptstyle A_{a,a}} \\ 1 & \xrightarrow[I^{\mathcal{W}}_{a_+}]{} & \mathcal{W}(a_+, a_+) \end{array}$$

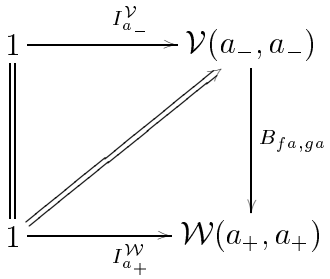
For any two-sided enrichments $\mathcal{V}$ and $\mathcal{W}$, the bicategory $Caten(\mathcal{V}, \mathcal{W})$ is locally posetal, as follows.
A 2-cell $f : A \to B : \mathcal{V} \to \mathcal{W}$ in *Caten* is called a *functor*, it consists in a map $f : Obj(A) \to Obj(B)$ such that:
- $(-)^A_- = (-)^B_- \circ f$;
- $(-)^A_+ = (-)^B_+ \circ f$;



- for any objects a,b of A, $A_{a,b} \leq B_{fa,fb} : \mathcal{V}(a_-, b_-) \to \mathcal{W}(a_+, b_+)$, i.e. there is a 2-cell in Po



2-cells in Caten are ordered as follows. $f \leq g : A \to B : \mathcal{V} \to \mathcal{W}$ when for any object a of A, $I^{\mathcal{V}}_{a_+} \leq B(fa, ga)(I^{\mathcal{W}}_{a_-}) : a_+ \to a_+$, which corresponds to the 2-cell in Po



The (vertical) composition of the 2-cells $f : A \to B, g : B \to C : \mathcal{V} \to \mathcal{W}$ is the composite of the maps on objects f and g. The identity 1_A of $A : \mathcal{V} \to \mathcal{V}$ has underlying map the identity map on $Obj(A)$.

Let $A : \mathcal{U} \to \mathcal{V}$ and $B : \mathcal{V} \to \mathcal{W}$ be two-sided enrichments. Their composite $B \circ A : \mathcal{U} \to \mathcal{W}$ has for span the composition of the spans of A and B: its objects are the pairs (a, b) in $Obj(A) \times Obj(B)$ such that $a_+ = b_-$ and the maps $(-)^{B\circ A}_-$ and $(-)^{B\circ A}_+$ are given by $(a, b)_- = a_-$ and $(a, b)_+ = b_+$. The functors $B \circ A_{(a,b),(a',b')}$ are the composites

$$\mathcal{U}(a_-, a'_-) \xrightarrow{A_{(a,a')}} \mathcal{V}((a_+, a'_-) = \mathcal{V}(b_-, b'_-) \xrightarrow{B_{(b,b')}} \mathcal{W}((b_+, b'_-) \; .$$

The horizontal composition of the 2-cells $f : A \to B : \mathcal{U} \to \mathcal{V}$ and $g : C \to D : \mathcal{V} \to \mathcal{W}$ is the map sending $(a, c) \in Obj(C \circ A)$ to (fa, gc).

3.2 A closed structure on Caten

For any (locally posetal) bicategory $\mathcal{V}$, the 2-category $\mathcal{V}-Cat$ is representable

$$\mathcal{V}-Cat \cong Caten(1,\mathcal{V}).$$

The cartesian product of locally posetal bicategories extends straightforwardly to a pseudofunctor $Caten \times Caten \to Caten$ that makes $Caten$ into a monoidal tricategory. $Caten$ has a biclosed structure:

$$Caten(\mathcal{U}\times\mathcal{V},\mathcal{W}) \cong Caten(\mathcal{U}, Conv(\mathcal{V},\mathcal{W})).$$

where $Conv(\mathcal{V},\mathcal{W})$ is defined as follows:

Definition 3.1 (The bicategory Conv(V,W)) *Given two locally posetal bicategories $\mathcal{V}$, $\mathcal{W}$ the locally posetal bicategory $Conv(\mathcal{V},\mathcal{W})$ is defined as follows. Its objects are ordered pairs $(v,w) \in Ob(\mathcal{V}) \times Ob(\mathcal{W})$. Its arrows $(v,w) \to (v',w')$ are functors $\mathcal{V}(v,v') \to \mathcal{W}(w,w')$, (i.e. monotonous maps). The partial order on arrows is pointwise: For any $F,G : (v,w) \to (v',w')$, $F \leq G$ when for any $h : v \to v'$ in $\mathcal{V}$, $F(h) \leq G(h) : w \to w'$ in $\mathcal{W}$. For any $F : (v,w) \to (v',w')$ and $G : (v',w') \to (v'',w'')$, the (horizontal) composite $G\bar{\circ}F : (v,w) \to (v'',w'')$ is the left Kan extension of*

$$\mathcal{V}(v,v') \times \mathcal{V}(v',v'') \xrightarrow{F\times G} \mathcal{W}(w,w') \times \mathcal{W}(w',w'') \xrightarrow{\circ^{\mathcal{W}}} \mathcal{W}(w,w'')$$

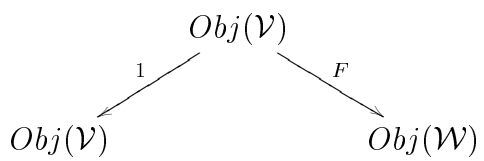
along $\mathcal{V}(v,v') \times \mathcal{V}(v',v'') \xrightarrow{\circ^{\mathcal{V}}} \mathcal{U}(v,v'')$, where $\circ^{\mathcal{V}}$, $\circ^{\mathcal{W}}$ are the horizontal compositions respectively of $\mathcal{V}$ and $\mathcal{W}$. This means that for any arrow h in $\mathcal{V}(v,v'')$, $F\bar{\circ}G(h) = \bigvee\{G(g)\circ F(f) \mid g\circ f \leq h\}$. The identity in (u,v) is the left Kan extension of $I^{\mathcal{W}}_w : 1 \to \mathcal{W}(w,w)$ along $1^{\mathcal{V}}_v : 1 \to \mathcal{V}(v,v)$, thus for any arrow f in $\mathcal{V}(v,v)$, $1_{(v,w)}(f) = \mathcal{V}(v,v)(I_v,f)\cdot 1_v$.

There is a further connection between enrichments over bicategories and two-sided enrichments: two-sided enrichments are also just enrichments since

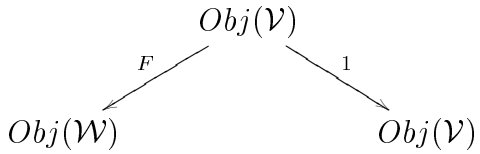
$$Caten(\mathcal{V},\mathcal{W}) \cong Caten(1, Conv(\mathcal{V},\mathcal{W})) \cong Conv(\mathcal{V},\mathcal{W})-Cat.$$

3.3 Adjoints in Caten

Adjoints $F \dashv G : \mathcal{V} \to \mathcal{W}$ in $Caten$ are (up to isorphism) as follows. $Obj(F) = Obj(G) = Obj(\mathcal{V})$, the span of F is a left adjoint in the bicategory of spans on Set



the span of G is its right adjoint



and the functors $F_{a,b} : \mathcal{V}(a,b) \to \mathcal{W}(Fa, Fb)$ have right adjoints $G_{a,b} : \mathcal{W}(Fa, Fb) \to \mathcal{V}(a,b)$. Indeed the left adjoint F may be seen as a 2-functor $\mathcal{V} \to \mathcal{W}$ with "local (right) adjoints". In this case, the above family of right adjoints satisfy some coherence properties. It can be shown that for any lax functor $F : \mathcal{V} \to \mathcal{W}$ such that for any objects a, b in $\mathcal{V}$, the restriction $F : \mathcal{V}[a,b] \to \mathcal{W}[Fa, Fb]$ admits a right adjoint $G_{a,b} : \mathcal{W}[Fa, Fb] \to \mathcal{V}[a,b]$, these adjoints satisfy the coherence conditions, on composition:

(1) for all objects a,b,c of $\mathcal{V}$, for all $\beta : Fa \to Fb$ and $\gamma : Fb \to Fc$,
$$G_{b,c}\gamma \circ G_{a,b}\beta \leq G_{a,c}(\gamma \circ \beta);$$

and on identities:

(2) for all objects a of A, $I_a \leq G_{a,a} I_{Fa}$.

if and only if F is a 2-functor.

3.4 The change of base for enrichments

Using the representability for $\mathcal{V}$-categories and the fact that $Caten$ is a tricategory, one has the change of base theorem for enrichments over bicategories. The composition on the left by any two-sided enrichment $F : \mathcal{V} \to \mathcal{W}$, yields a 2-functor $F \circ - : Caten[1, \mathcal{V}] \to Caten[1, \mathcal{W}]$, and by the representability result, it corresponds to a 2-functor $F_@ : \mathcal{V}\text{-}Cat \to \mathcal{W}\text{-}Cat$. In the same way, with any 2-cells $\sigma : F \to G$ of $Caten$ yields a 2-natural transformation $\sigma_@ : F_@ \to G_@ : \mathcal{V} - Cat \to \mathcal{W} - Cat$. Indeed,

Theorem 3.2 *There is a pseudo-functor $(-)_@$ from $Caten$ to $2 - Cat$ (the 2-category of 2-categories, 2-functors and 2-natural transformations). It sends a bicategory $\mathcal{V}$ to the 2-category $\mathcal{V} - Cat$, its actions on two-sided enrichments and 2-cells are the ones described above.*

The above $F_@ : \mathcal{V} - Cat \to \mathcal{W} - Cat$ is now described in terms of enrichments. For any $\mathcal{V}$-category A, $F_@A$ is given by:
- $Obj(F_@A) = \{(a,x) \mid a \in Obj(\mathcal{V}), x \in Obj(F), a_+ = x_-\}$;
- $(a,x)_+^{F_@A} = x_+$;
- $F_@A((a,x),(b,y)) = F_{x,y}(A(a,b))$.

For any $\mathcal{V}$-functor $f : A \to B$, $F_@(f)$ is the $\mathcal{W}$-functor with underlying map

$(a, x) \mapsto (fa, x)$.

Note that if F is moreover a 2-functor, then for any $\mathcal{V}$-category A, $F_@A$ is given by:
- $Obj(F_@A) = Obj(A)$;
- $a_+^{F_@A} = Fa_+$;
- $(F_@A)(x, y) = FA(x, y)$;
and for any $\mathcal{V}$-functor $f : A \to B$, $F_@(f)$ is the $\mathcal{W} - functor$ with the same underlying map as f.

From an adjoint pair $F \dashv G$ in $Caten$, one gets the adjoint pair $F_@ \dashv G_@$ in $2 - Cat$. Precisely,

Theorem 3.3 *If the 2-functor between locally posetal 2-categories $F : \mathcal{V} \to \mathcal{W}$ admits local adjoints $G_{u,v}$, (u, v) ranging in $Obj(\mathcal{V})^2$ then the 2-functor $F_@$ admits a right 2-adjoint, denoted $F^@$ and defined on objects as follows.*
For any object B of $\mathcal{W} - Cat$, $F^@B$ is given by:
- $Obj(F^@B) = \{(b, v) \mid b \in Obj(B), v \in Obj(\mathcal{V}), b_+ = Fv\}$;
- $(b, v)_+ = v$;
- $F^@B((b, v), (b', v')) = G_{v,v'}(B(b, b'))$.

In the case above $F^@$ is just $G_@$, the collections of local adjoints $G_{u,v}$ defining the right adjoint G to F in $Caten$.

3.5 Change of base, skeletality and Cauchy-completion

Further on we consider a two-sided enrichment $F : \mathcal{V} \to \mathcal{W}$.

Lemma 3.4 *For any $\mathcal{W}$-category B, if B is skeletal then also is $F^@B$.*

proof: Suppose that $(b, v), (b', v) \in Obj(F^@B)$ are such that $F^@B((b, v), (b', v)) \geq I_v$, i.e. $G_{v,v}B(b, b') \geq I_v$. Since $F_{v,v} \dashv G_{v,v}$, $B(b, b') \geq FI_v = I_{Fv}$. By skeletality of B, $b = b'$. ∎

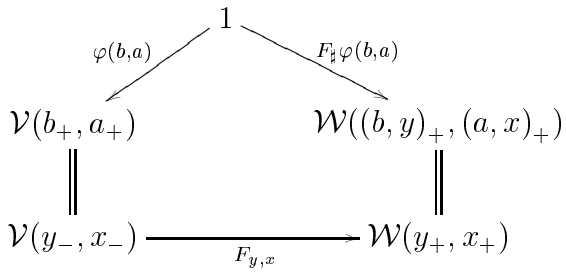
Further on we suppose moreover that $\mathcal{V}$ and $\mathcal{V}$ are locally cocomplete and biclosed.

Fact 3.5 *There is a lax normal [1] functor $F_\sharp : \mathcal{V} - Mod \to \mathcal{W} - Mod$ that extends $F_@$ in the sense the diagram above commutes:*

$$\begin{array}{ccc} \mathcal{V} - Cat & \xrightarrow{F_@} & \mathcal{W} - Cat \\ {\scriptstyle J_\mathcal{V}}\downarrow & & \downarrow{\scriptstyle J_\mathcal{W}} \\ \mathcal{V} - Mod & \xrightarrow[F_\sharp]{} & \mathcal{W} - Mod \end{array}$$

[1] A lax functor is *normal* when it strictly preserves identities.

For any $\mathcal{V}$-module $\varphi : A \multimap B$, $F_\sharp(\varphi) : F_@A \multimap F_@B$ is defined by: for all objects (a,x) of $F_@A$ and (b,y) of $F_@B$, $F_\sharp(\varphi)((b,y),(a,x)) = F_{y,x}(\varphi(b,a))$. This is depicted by the diagram in Cat below:



proof: Let $\varphi : A \multimap B$. For any objects (a,x), (a',x') of $F_@A$ and (b,y) of $F_@B$,

$$\begin{aligned} F_@A((a,x),(a',x')) \circ F_\sharp\varphi((b,y),(a,x)) &= F_{x,x'}A(a,a') \circ F_{y,x}\varphi(b,a) \\ &\leq F_{y,x'}(A(a,a') \circ \varphi(b,a)) \\ &\leq F_{y,x'}(\varphi(b,a')) \\ &= F_\sharp\varphi((y,b),(a',x')). \end{aligned}$$

Similarly, for any objects (a,x) of $F_@A$ and (b,y), (b',y') of $F_@B$, $F_\sharp\varphi((b,y),(a,x)) \circ F_@B((b',y'),(b,y)) \leq F_\sharp\varphi((b',y'),(a,x))$.

Now given $\mathcal{V}$-module, $\varphi : A \multimap B$ and $\psi : B \multimap C$, for any objects (a,x) of $F_@A$ and (c,z) of $F_@C$,

$$\begin{aligned} F_\sharp((\psi \bullet \varphi)((c,z),(a,x))) &= F_{z,x}(\psi \bullet \varphi)(c,a), \\ &= F_{z,x}(\textstyle\bigvee_{b \in Obj(B)} \varphi(b,a) \circ \psi(c,b)), \\ &\geq \textstyle\bigvee_{b \in Obj(B)} F_{z,x}(\varphi(b,a) \circ \psi(c,b)) \\ &\geq \textstyle\bigvee_{(b,y) \in Obj(F_@B)} F_{y,x}\varphi(b,a) \circ F_{z,y}\psi(c,b) \\ &= (F_\sharp\psi \bullet F_\sharp\varphi)((c,z),(a,x)). \end{aligned}$$

By definition $F_\sharp(f_\diamond) = (F_@f)_\diamond$ for any $\mathcal{V}$-functor f, thus for any $\mathcal{V}$-category A, $F_\sharp(1_{A\diamond}) = (F_@1_A)_\diamond = (1_{F_@A})_\diamond$. ∎

Lemma 3.6 *Let $f : A \to B$ be a $\mathcal{V}-$functor. Then:*
(i) $F_\sharp(f_\diamond) \dashv F_\sharp(f^\diamond)$ in $\mathcal{W}-$Mod;
(ii) If $f_\diamond$ is an isomorphism in $\mathcal{V}-$Mod then $F_\sharp(f^\diamond) \bullet F_\sharp(f_\diamond) = 1$.

proof: (*i*): Results from the facts that for any $\mathcal{V}$-functor f, $F_\sharp(f_\diamond) = (F_@(f))_\diamond$ and also $F_\sharp(f^\diamond) = (F_@(f))^\diamond$.
(*ii*): Suppose $f_\diamond \bullet f^\diamond = 1$ and $f^\diamond \bullet f_\diamond = 1$. Composing the second quality by $F_\sharp$ which is lax and normal, one gets $F_\sharp(f^\diamond) \bullet F_\sharp(f_\diamond) \leq F_\sharp(f^\diamond \bullet f_\diamond) = F_\sharp(1) \leq 1$. Thus $F_\sharp(f^\diamond) \bullet F_\sharp(f_\diamond) = 1$ since $F_\sharp(f_\diamond) \dashv F_\sharp(f^\diamond)$. ∎

These later facts motivated the somewhat technical definition below, as we were looking for sufficient conditions on a two-sided enrichment F in order $F_@$ to preserve the Morita-equivalence.

Definition 3.7 (Super two-sided enrichments) *A two-sided enrichment $G : \mathcal{V} \to \mathcal{W}$ is* super *when for any object x of G, and any family of pairs of arrows*

$$x_- \xrightarrow{f_i} a_i \xrightarrow{g_i} x_-$$

in $\mathcal{V}$, i ranging in $\mathcal{I}$, if $I_{x_-} \leq \bigvee_{i\in\mathcal{I}} g_i \circ f_i$ then

$$I_{x_+} \leq \bigvee_{\{y\in Obj(G)|\exists i\in\mathcal{I}, a_i = y_-\}} G_{y,x}(g_i) \circ G_{x,y}(f_i).$$

It is straightforward to check that the composition of super two-sided-enrichments is super. Trivially identities are super.

Lemma 3.8 *If F is super then $1 \leq F_\sharp(i_{A_\diamond}) \bullet F_\sharp(i_A{}^\diamond) : F_@A \to F_@A$ in $\mathcal{W} - Mod$.*

proof: Consider some objects φ, ψ of $\bar{A}$, and x, y of F, with $\varphi : \hat{(x_-)} \multimap A$, $\psi : \hat{(y_-)} \multimap A$. Since $I^{\mathcal{V}}_{x_-} \leq \bigvee_{a\in Obj(A)} \varphi(a) \circ \tilde{\varphi}(a)$,

$$\begin{aligned}
(F_\sharp(i_{A_\diamond}) \bullet F_\sharp(i_A{}^\diamond))((\varphi, x), (\psi, y)) &= \bigvee_{(a,z)\in Obj(F_@A)} (F_\sharp(i_A{}^\diamond)((a, z), (\psi, y)) \circ \\
&\qquad F_\sharp(i_{A_\diamond})((\varphi, x), (a, z))) \\
&= \bigvee_{(a,z)\in Obj(F_@A)} F_{z,y}\psi(a) \circ F_{x,z}\tilde{\varphi}(a) \\
&\geq \bigvee_{(a,z)\in Obj(F_@A)} (F_{z,y}(\bar{A}(\varphi, \psi) \circ \varphi(a)) \circ F_{x,z}\tilde{\varphi}(a)) \\
&\geq \bigvee_{(a,z)\in Obj(F_@A)} ((F_{x,y}\bar{A}(\varphi, \psi) \circ F_{z,x}\varphi(a)) \circ F_{x,z}\tilde{\varphi}(a)) \\
&= F_{x,y}\bar{A}(\varphi, \psi) \circ \bigvee_{(a,z)\in Obj(F_@A)} (F_{z,x}\varphi(a) \circ F_{x,z}\tilde{\varphi}(a)) \\
&\geq F_{x,y}\bar{A}(\varphi, \psi) \circ I_{x_+} \\
&\geq F_{x,y}\bar{A}(\varphi, \psi).
\end{aligned}$$

∎

According to this and 3.6,

Lemma 3.9 *If F is super then $F_\sharp(i_{A_\diamond})$ is an isomorphism, in this case $F_\sharp$ preserves the Morita-equivalence, i.e. if the $\mathcal{V}$-categories A and B are Morita equivalent then also are $F_@A$ and $F_@B$.*

One has also

Lemma 3.10 *If F is super then $F_\sharp : \mathcal{V} - Mod \to \mathcal{W} - Mod$ preserves adjoints.*

proof: Suppose that F is super. Consider a left adjoint $\varphi : A \multimap B$ in $\mathcal{V} - Mod$. First let us show that $1 \leq F_\sharp(\tilde{\varphi}) \bullet F_\sharp(\varphi)$. Note that for any object (a, x) of $F_@A$, Since $I^{\mathcal{V}}_{a_+} \leq \bigvee_{b\in Obj(B)} (\varphi(b, a) \circ \tilde{\varphi}(a, b))$ one has $I^{\mathcal{W}}_{x_+} \leq \bigvee_{(b,z)\in Obj(F_@B)} F_{z,x}\varphi(b, a) \circ F_{x,z}\tilde{\varphi}(a, b)$. Thus for any objects $(a, x), (a', y)$ of $F_@A$.

$$\begin{aligned}
F_{@}A((a,x),(a',y)) &= F_{x,y}A(a,a') \\
&\leq F_{x,y}A(a,a') \circ (\bigvee_{(b,z)\in Obj(F_{@}B)} F_{z,x}\varphi(b,a) \circ F_{x,z}\tilde{\varphi}(a,b)) \\
&= \bigvee_{(b,z)\in Obj(F_{@}B)} ((F_{x,y}A(a,a') \circ F_{z,x}\varphi(b,a)) \circ F_{x,z}\tilde{\varphi}(a,b)) \\
&\leq \bigvee_{(b,z)\in Obj(F_{@}B)} (F_{z,y}\varphi(b,a')) \circ F_{x,z}\tilde{\varphi}(a,b)) \\
&= (F_{\sharp}\tilde{\varphi} \bullet F_{\sharp}\varphi)((a,x),(a',y)).
\end{aligned}$$

Now since $F_{\sharp}$ is lax and normal and $\varphi \bullet \tilde{\varphi} \leq 1$, $F_{\sharp}(\varphi) \bullet F_{\sharp}(\tilde{\varphi}) \leq F_{\sharp}(\varphi \bullet \tilde{\varphi}) \leq F_{\sharp}(1)$ $= 1$. ∎

Further on, we suppose that F has right adjoint G in $Caten$, i.e. $F : \mathcal{V} \to \mathcal{W}$ is a 2-functor with local adjoints $G_{a,b}$, (a,b) ranging in $Obj(\mathcal{V})^2$.

Fact 3.11 *$F_{\sharp} : \mathcal{V} - Mod \to \mathcal{W} - Mod$ is a 2-functor.*

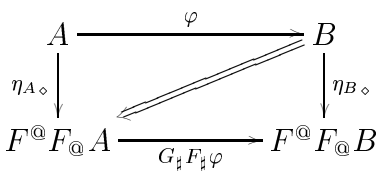
proof: We already know that $F_{\sharp}$ is lax and normal. It remains to show that $F_{\sharp}$ preserves composition. Let $\varphi : A \multimap B$ and $\psi : B \multimap C$ be $\mathcal{V}$-modules. Since F has local right adjoints, it "preserves local least upper bounds", therefore for any objects a of A and c of C,

$$\begin{aligned}
F_{\sharp}(\psi \bullet \varphi)(c,a) &= F(\bigvee_{b\in Obj(B)} \varphi(b,a) \circ \psi(c,b)), \\
&= \bigvee_{b\in Obj(B)} F(\varphi(b,a) \circ \psi(c,b)) \\
&= \bigvee_{b\in Obj(B)} F\varphi(b,a) \circ F\psi(c,b) \\
&= (F_{\sharp}\psi \bullet F_{\sharp}\varphi)(c,a).
\end{aligned}$$

∎

We shall finish with technical lemmas which relevance will appear in the next section treating reversibility.

Lemma 3.12 *Let η be the unit of the adjunction $F_{@} \dashv F^{@}$. For any left adjoint module $\varphi : A \multimap B$ there is a 2-cell in $\mathcal{V} - Mod$:*



proof: Let $\varphi : A \multimap B$ be a right adjoint in $\mathcal{V} - Mod$. Consider some objects a of A and $(v,b) \in F^{@}F_{@}B$. Then

$$\begin{aligned}
(\eta_{B\diamond} \bullet \varphi)((v,b),a) &= \bigvee_{b'\in Obj(B)} \varphi(b',a) \circ G_{v,b'_{+}}FB(b,b') \\
&\leq \bigvee_{b'\in Obj(B)} G_{b'_{+},a_{+}}F\varphi(b',a) \circ G_{v,b'_{+}}FB(b,b') \\
&\leq G_{v,a_{+}}F\varphi(b,a);
\end{aligned}$$

Also,

$$
\begin{aligned}
(G_\sharp F_\sharp \varphi \bullet \eta_{A_\diamond})((v,b),a) &= \bigvee_{(a',v')\in Obj(F^{@}F_{@}A)} G_{v',a_+} FA(a',a) \circ G_{v,v'} F\varphi(b,a') \\
&\leq \bigvee_{(a',v')\in Obj(F^{@}F_{@}A)} G_{v,a_+} F(A(a',a) \circ \varphi(b,a')) \\
&\leq G_{v,a_+} F\varphi(b,a)
\end{aligned}
$$

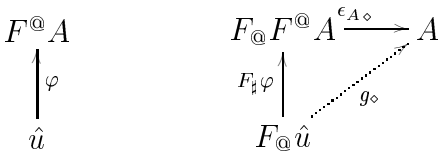
But since $(a,a_+) \in Obj(F^{@}F_{@}A)$ and

$$
\begin{aligned}
G_{v,a_+} F\varphi(b,a) &\leq G_{a_+,a_+} FI_{a_+} \circ G_{v,a_+} F\varphi(b,a) \\
&\leq G_{a_+,a_+} FA(a,a) \circ G_{v,a_+} F\varphi(b,a) \\
&\leq (G_\sharp F_\sharp \varphi \bullet \eta_{A_\diamond})((v,b),a)
\end{aligned}
$$

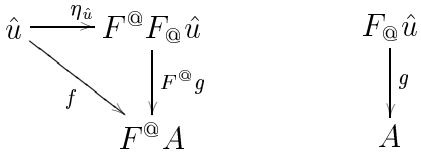
∎

Lemma 3.13 *Let A be a Cauchy-complete $\mathcal{W}$-category. For any object u of $\mathcal{V}$ and any left adjoint $\mathcal{V}$-module $\varphi : \hat{u} \multimap F^{@}A$, there is a $\mathcal{V}$-functor $f : \hat{u} \to F^{@}A$ with $f_\diamond \geq \varphi$.*

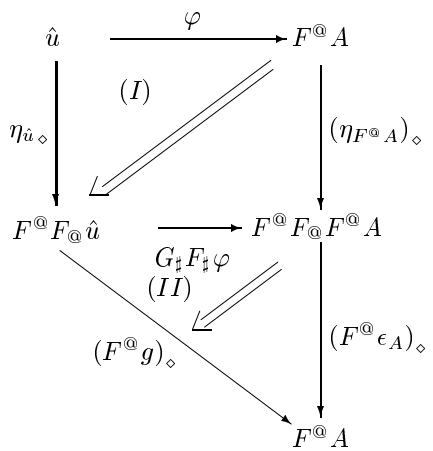
proof: Let u be some object of $\mathcal{V}$ and $\varphi : \hat{u} \multimap F^{@}A$ be a left adjoint $\mathcal{V}$-module. According to Fact 3.11, $F_\sharp \varphi : F_{@}\hat{a} \multimap F_{@}F^{@}A$ is some left adjoint $\mathcal{W}$-module, therefore also is $\epsilon_{A_\diamond} \bullet F_\sharp \varphi : F_{@}\hat{u} \multimap A$, ϵ denoting the co-unit of the adjunction $F_{@} \dashv F^{@}$ (see diagram below).



Since A is Cauchy-complete there is some $\mathcal{V}$-functor $g : F_{@}\hat{u} \to A$ such that $g_\diamond = \epsilon_{A_\diamond} \bullet F_\sharp \varphi$. η denoting the unit of the adjunction $F_{@} \dashv F^{@}$, let us see that the $\mathcal{W}$-functor $f = F^{@}g \circ \eta_{\hat{u}} : \hat{u} \to F^{@}A$ — see diagram below — satisfies $f_\diamond \geq \varphi$.



The arrow results from the pasting of diagrams in $\mathcal{V} - Mod$:



2-cell (I) is given by 3.12, and 2-cells (II) is given by the laxity of $G_\sharp$. ∎

4 Reversibility

In this section we will restrict the theory of two-sided enrichments over locally posetal bicategories to the symmetric case. We will define the notion of reversibility for bicategories, two-sided enrichments, modules, and completion. Definitions will be suggested by the examples and will allow to rebuild the complete theory in this special case. Eventually we study the effect of the change of bases on the completness and skeletality of enrichments in the reversible context. The main result is theorem 4.31.

4.1 Reversible enrichments

Definition 4.1 *A 2-category $\mathcal{V}$ is* reversible *when there are isomorphisms of categories $s_{u,v} : \mathcal{V}(u,v) \cong \mathcal{V}(v,u)$ for objects u and v of $\mathcal{V}$, and these isomorphisms satisfy the conditions:*
- $rev-(i)$: $s_{u,v} \circ s_{v,u} = 1$, for any objects u and v;
- $rev-(ii)$: for any objects u,v,w the diagram (in Cat) below commutes:

$$\begin{array}{ccc}
\mathcal{V}(v,w) \times \mathcal{V}(u,v) & \xrightarrow{\circ_{u,v,w}} & \mathcal{V}(u,w) \\
\cong \downarrow & & \\
\mathcal{V}(u,v) \times \mathcal{V}(v,w) & & \downarrow s_{u,w} \\
{\scriptstyle s_{u,v} \times s_{v,w}} \downarrow & & \\
\mathcal{V}(v,u) \times \mathcal{V}(w,v) & \xrightarrow[\circ_{w,v,u}]{} & \mathcal{V}(w,u)
\end{array}$$

where $\circ$ is the horizontal composition in $\mathcal{V}$;
- $rev-(iii)$: for any object v the diagram below commutes

$$\begin{array}{ccc}
\mathbf{1} & \xrightarrow{u_v} & \mathcal{V}(v,v) \\
\| & & \downarrow s_{v,v} \\
\mathbf{1} & \xrightarrow[u_v]{} & \mathcal{V}(v,v)
\end{array}$$

Further on $\mathcal{V}$ is a reversible locally posetal bicategory.

Definition 4.2 (reversible enrichments) *A $\mathcal{V}$-category A is* reversible *when for any objects a, b of A, $A(b,a) = s_{a_+,b_+}(A(a,b))$.*

Let A and B be $\mathcal{V}$-categories. If B is reversible then the preorder $\mathcal{V}-Cat(A,B)$, denoted here $\Rightarrow$, is an equivalence since for any $\mathcal{V}$-functors $f, g : A \to B$,

$f \Rightarrow g$ if and only if for all object a of A, $I_{a_+} \leq B(fa, ga)$,
- , $s(I_{a_+}) \leq s(B(fa, ga))$
- , $I_{a_+} \leq B(ga, fa)$
- , $g \Rightarrow f$.

Observation 4.3 *If B above is moreover skeletal then $\Rightarrow$ is the identity on $\mathcal{V}-Cat(A,B)$.*

Further on we assume that the bicategory $\mathcal{V}$ is biclosed and locally cocomplete.

Fact 4.4 *Let A and B be reversible $\mathcal{V}$-categories.*
(1) *If $\varphi : A \multimap B$ is a $\mathcal{V}$-module then the map defined on $Obj(A) \times Obj(B)$ sending (a,b) to the arrow $s_{b_+,a_+}(\varphi(b,a))$ defines a $\mathcal{V}$-module $\varphi^s : B \multimap A$.*
(2) *If φ is left adjoint then φ^s is right adjoint with $(\tilde{\varphi})^s \dashv \varphi^s$.*

proof: Assertion (1) is straightforward. Let us see (2) : Suppose that φ is left adjoint. From $1 \leq \tilde{\varphi} \bullet \varphi$, i.e. for any objects a, a' of A,

$$A(a,a') \leq \bigvee_{b \in Obj(B)} \varphi(b,a') \circ \tilde{\varphi}(a,b)$$

one gets by applying $s_{(a_+,a'_+)}$ on both sides of the inequality that for any a, a',

$$A(a',a) \leq \bigvee_{b \in Obj(A)} (\tilde{\varphi})^s(b,a) \circ \varphi^s(a',b),$$

i.e. $1 \leq \varphi^s \bullet (\tilde{\varphi})^s$. Similarly one gets from $(\tilde{\varphi})^s \bullet \varphi^s \leq 1$ that $\varphi \bullet \tilde{\varphi} \leq 1$. ∎

There are reversible enrichments with non-reversible Cauchy-completions. A very simple example (due to S. Kasangian) may be found in [BeWal82]. This motivates the further notion of Cauchy-reversible-completion.

Definition 4.5 (Reversible modules) *A $\mathcal{V}$-module φ between reversible $\mathcal{V}$-categories is* reversible *when it has right adjoint φ^s.*

Remark 4.6 *If f is a $\mathcal{V}$-functor between reversible $\mathcal{V}$-categories then the $\mathcal{V}$-module $f_\diamond$ is reversible.*

Definition 4.7 (Cauchy-reversible-completness) *A reversible $\mathcal{V}$-category B is* Cauchy-reversible-complete *when one the following two equivalent assertions holds:*
(i) For any reversible $\mathcal{V}$-module $\varphi : A \multimap B$, $\varphi = f_\diamond$ for some $\mathcal{V}$-functor $f : A \to B$;
(ii) For any object u of $\mathcal{V}$ and any reversible $\mathcal{V}$-module $\varphi : \hat{u} \multimap B$, $\varphi = b_\diamond$ for some object b of B with $b_+ = u$.

Let us prove the equivalence of assertions (i) and (ii) of the previous definition.
$(i) \Rightarrow (ii)$: immediate.
$(ii) \Rightarrow (i)$ Note that for some reversible module $\varphi : A \multimap\!\!\!\!\rightarrow B$, for any object a of A, $\varphi \bullet a_\diamond \dashv a^\diamond \bullet \varphi^s = (\varphi \bullet a_\diamond)^s$. Then one concludes analogously as for 2.7.

Clearly, any reversible Cauchy-complete $\mathcal{V}$-category is also a Cauchy-reversible-complete one.

Definition 4.8 (Cauchy-reversible-completion) *Let A be a reversible $\mathcal{V}$-category with Cauchy-completion $\bar{A}$. A^s, the* Cauchy-reversible-completion *of A is the reversible $\mathcal{V}$-category where $Obj(A^s) \subseteq Obj(\bar{A})$, is the set of reversible $\mathcal{V}$-modules of the form $\varphi : \hat{u} \multimap\!\!\!\!\rightarrow A$, u ranging in $Obj(\mathcal{V})$. The maps $A^s(-,-)$ and $(-)_+$ of A^s are the restrictions of the maps $\bar{A}(-,-)$ and $(-)_+$ of $\bar{A}$.*

Similarly to 2.9, one checks that the data above defines a $\mathcal{V}$-category A^s. A^s is moreover reversible as shown below. Let $\varphi : \hat{u} \multimap\!\!\!\!\rightarrow A$, $\psi : \hat{v} \multimap\!\!\!\!\rightarrow A$ be reversible adjoints, then

$$\begin{aligned}\bar{A}(\varphi,\psi) &= \textstyle\bigvee_{a\in Obj(A)} \psi(a) \circ \varphi^s(a) \\ &= \textstyle\bigvee_{a\in Obj(A)} s_{v,u}(\varphi(a) \circ \psi^s(a)) \\ &= s_{v,u}(\textstyle\bigvee_{a\in Obj(A)} \varphi(a) \circ \psi^s(a)) \\ &= s_{v,u}(\bar{A}(\psi,\varphi)).\end{aligned}$$

Let A, $\bar{A}$ and, A^s be as in 4.7. The map $i_A : Obj(A) \to Obj(\bar{A})$ takes values in $Obj(A^s)$. Thus it defines a $\mathcal{V}$-functor from A to A^s further on denoted j_A.

An immediate consequence of 2.11 is

Fact 4.9 *For any reversible $\mathcal{V}$-category A, A^s is skeletal.*

Adapting the proof of 2.12 one shows

Lemma 4.10 *Let A be a reversible $\mathcal{V}$-category and, $\Gamma : \hat{u} \multimap\!\!\!\!\rightarrow A^s$ be some $\mathcal{V}$-module where u is an object of $\mathcal{V}$. For any object φ of A^s, $\Gamma(\varphi) = \bigvee_{a\in Obj(A)} \Gamma(a_\diamond) \circ \varphi^s(a)$.*

Fact 4.11 *For any reversible $\mathcal{V}$-category A, A^s is Cauchy-reversible-complete.*

proof: Merely an adaptation of the proof of 2.13. Given $\Gamma : \hat{u} \multimap\!\!\!\!\rightarrow A^s$ a reversible $\mathcal{V}$-module. One checks that the maps $\gamma : Obj(A) \to Arrows(\mathcal{V})$ given by for any $a \in Obj(A)$,

$$\gamma(a) = \Gamma(a_\diamond)$$

defines a reversible module from $\hat{u}$ to A with adjoint γ' given by for any object a of A,

$$\gamma'(a) = \Gamma^s(a_\diamond).$$

Then according to 4.10, $\Gamma = A^s(-,\gamma)$. ∎

Lemma 4.12 *For any reversible $\mathcal{V}$-category A, $j_{A_\diamond} : A \multimap A^s$ and $j_A{}^\diamond : A^s \multimap A$ are inverse reversible modules.*

proof: The proof that $(j_A)_\diamond$ and $(j_A)^\diamond$ are inverse modules is the one for 2.14. Since A^s is reversible, $j_A{}^\diamond = (j_{A_\diamond})^s$. ∎

We define the following 2-categories.

- $\mathcal{V} - RCat$ with:
  - objects: reversible $\mathcal{V}$-categories,
  - arrows and 2-cells: inherited from $\mathcal{V} - Cat$.

- $\mathcal{V} - SkCRcCat$ with:
  - objects: skeletal and Cauchy-reversible-complete $\mathcal{V}$-categories,
  - arrows and 2-cells: inherited from $\mathcal{V} - Cat$.

- $\mathcal{V} - RMod$ with:
  - objects: reversible $\mathcal{V}$-categories,
  - arrows: reversible $\mathcal{V}$-modules,
  - 2-cells: inherited from $\mathcal{V} - Mod$.

According to 4.12,

Proposition 4.13 *$\mathcal{V} - SkCRcCat$ and $\mathcal{V} - RMod$ are 2-equivalent. There is a 2-functor $S' : \mathcal{V} - RMod \to \mathcal{V} - SkCRcCat$ which is an equivalence of categories. S' is defined on objects by $S'(A) = A^s$ and on arrows by $S'(\varphi) = f_\varphi$ where for any reversible $\mathcal{V}$-module $\varphi : A \multimap B$, $f_\varphi : A^s \to B^s$ is the unique $\mathcal{V}$-functor f satisfying $(f_\varphi)_\diamond \bullet j_A{}^\diamond = j_{B_\diamond} \bullet \varphi$. The inclusion 2-functor $\mathcal{V} - SkCsycCat \to \mathcal{V} - RCat$ has a left 2-adjoint.*

According to 4.3, $\mathcal{V} - SkCRcCat$ and $\mathcal{V} - RMod$ are locally discrete, i.e. the only 2-cells are identies. Therefore we shall sometimes consider them just as categories. Skeletal and Cauchy-reversible-complete $\mathcal{V}$-categories form a reflective subcategory of the underlying category of $\mathcal{V} - RCat$. The left adjoint of the inclusion sends a reversible $\mathcal{V}$-category A to A^s and, the unit takes value $j_A : A \to A^s$ in A.

We may define,

Definition 4.14 (Morita-reversible-equivalence) *Two reversible $\mathcal{V}$-categories A and B are* Morita-reversible-equivalent *when one the following equivalent assertions is satisfied:*
(i) Their Cauchy-reversible-completions are isomorphic in $\mathcal{V} - Cat$;
(ii) They are isomorphic in $\mathcal{V} - RMod$.

Note that if two reversible $\mathcal{V}$-categories are Morita-reversible-equivalent then they are Morita-equivalent.

4.2 Examples

Further on we give two examples of reversible enrichments on locally posetal bicategories. Note that for the second example, the reversibility isomorphisms are not identities.

Example 4.15 (Sheaves) *[Wal81]*

To any locale L corresponds a locally posetal 2-category $\mathcal{C}_L$ given by:
- objects are the $u \in L$;
- arrows from u to v are the $w \in L$ with $w \leq u \wedge v$;
- the partial order on $\mathcal{C}_L(u,v)$ is that of L;
- the composition of arrows is the intersection;
- the unit in u is u.

Such a $\mathcal{C}_L$ is locally cocomplete and biclosed. It is also reversible where the isomorphisms $s_{u,v}$ are the identity maps $\mathcal{C}_L(u,v) \to \mathcal{C}_L(v,u)$.

If L is a locale, $Sh(L)$ will denote the category of sheaves on L. In [Wal81], the following result is proved.

Theorem 4.16 *Given a locale L, $Sh(L)$ is equivalent to the category of reversible skeletal Cauchy-complete $\mathcal{C}_L-$enrichments.*

The equivalence J above is defined as follows. For any sheaf F on L, $J(F)$ is the enrichment over $\mathcal{C}_L$ where:
- $Obj(J(F))$ is the set of partial sections of F;
- For any section s, s_+ is its domain $dom(s)$;
- For any sections s, s', $J(F)(s,s') = \bigvee\{u \leq dom(s) \wedge dom(s') \mid s\lfloor u = s'\lfloor u\} - s\lfloor u$ denotes the restriction of s to $u \leq dom(s)$. If $h : F \to G$ is a morphism of $Sh(L)$, $J(h)$ is the map that sends any partial section $s \in Fu$ to $h(s) \in Gu$.

It happens in this case that Cauchy-reversible-complete enrichments are Cauchy-complete. This is due to

Fact 4.17 *For any locale L, any left adjoint $\mathcal{C}_L$-module is reversible.*

proof: Let $u \in L$ and $\varphi : \hat{u} \multimap A$ be some left adjoint module with $\varphi \dashv \tilde{\varphi}$. Then the map ψ given by: $\psi(s) = \varphi(s) \wedge \tilde{\varphi}(s)$ defines a module $\psi : \hat{u} \multimap A$ with itself as adjoint. Since $\psi \leq \varphi$ and $\psi \leq \tilde{\varphi}$, $\varphi = \psi = \tilde{\varphi}$. ∎
So we may reformulate 4.16

Theorem 4.18 *Given a locale L, $Sh(L)$ is equivalent to the category of skeletal Cauchy-reversible-complete $\mathcal{C}_L-$enrichments.*

Example 4.19 (Group actions)

Let (G, X, α) denote the action α of the group G on the set X. We note for such an action:
- $< g, x >_\alpha$ (or simply $< g, x >$), the action of $g \in G$ on $x \in X$,
- "$\cdot$", the composition law of G,
- u, the unit of G.
(G, X, α) may be also written (X, α).

Given a group G, $G - Set$ denotes the category with objects actions of G and with arrows action-preserving maps i.e. $f : (Z, \beta) \to (X, \alpha)$ is an arrow of $G - Set$ when the underlying map $f : Z \to X$ makes the following diagram commute:

$$\begin{array}{ccc} G \times Z & \xrightarrow{\beta} & Z \\ {\scriptstyle 1\times f}\downarrow & & \downarrow{\scriptstyle f} \\ G \times X & \xrightarrow[\alpha]{} & X \end{array}$$

If α is an action of G on X, a locally posetal 2-category $\mathcal{C}_{(G,X,\alpha)}$ is defined as follows:
- Objects are the $x \in X$;
- Arrows from x to x' are the subsets of $\{g \in G \mid < g, x >= x'\}$;
- the partial order on $\mathcal{C}_{(G,X,\alpha)}(x, x')$ is the inclusion;
- the composition $g \circ f$ of $f : x \to x'$ and $g : x' \to x''$ is $\{\tau \cdot \sigma \mid \sigma \in f \ \ and \ \ \tau \in g\}$;
- the unit at x is $\{u\}$.
Clearly $\mathcal{C}_{(G,X,\alpha)}$ is locally cocomplete and biclosed. $\mathcal{C}_{(G,X,\alpha)}$ is reversible with isomorphisms s: $s_{x,x'}(g) = \{\sigma^{-1} \mid \sigma \in g\}$, for any $g : x \to x'$.

Lemma 4.20 *Let (G, X, α) be a group action and A be a reversible enrichment on $\mathcal{C}_{(G,X,\alpha)}$.*

(1) *For any $z_0 \in Obj(A)$ and any $\sigma_0 \in G$, the map φ given by*

$$(*) \quad \forall z \in Obj(A), \varphi(z) = \{\sigma_0\} \circ A(z, z_0)$$

defines a left adjoint module $\varphi : \hat{x} \multimap A$ where $x =< \sigma_0, (z_0)_+ >$, with right adjoint $\tilde{\varphi}$ given by

$$\forall z \in Obj(A), \tilde{\varphi}(z) = A(z_0, z) \circ \{{\sigma_0}^{-1}\}.$$

Conversely for any $x \in X$ and any left adjoint module $\varphi : \hat{x} \multimap A$ there is some $z_0 \in Obj(A)$ and some $\sigma_0 \in G$ such that the map φ is given by formulae $(*)$ above.

(2) *If $\varphi, \psi : \hat{x} \multimap A$ are left adjoint modules given by:*

$$\forall z \in Obj(A), \varphi(z) = \{\sigma_0\} \circ A(z, z_0)$$

and

$$\forall z \in Obj(A), \psi(z) = \{\tau_0\} \circ A(z, {z'}_0)$$

where $\sigma_0, \tau_0 \in G$ *and* $z_0, z_0' \in Obj(A)$, *then* $\varphi = \psi$ *if and only if* $(\sigma_0, z_0) \sim (\tau_0, {z'}_0)$, $\sim$ *being the equivalence relation on* $G \times X$*:*

$$\forall \sigma, \sigma' \in G, \forall z, z' \in Obj(A), \ \ (\sigma, z) \sim (\sigma', z') \Leftrightarrow (\exists \gamma \in A(z, z'), \sigma = \sigma' \circ \gamma).$$

proof: (1) : Let $z_0 \in Obj(A)$, $\sigma_0 \in G$, and φ be the map on $Obj(A)$ defined by $\varphi(z) = \{\sigma_0\} \circ A(z, z_0)$. For short, let $x =< \sigma_0, (z_0)_+ >$. φ defines indeed a module $\hat{x} \multimap A$. φ satisfies $(mod - i)$ since for any $z \in Obj(A)$, $\varphi(z)$ is an object of $\mathcal{C}_{(G,X,\alpha)}(z_+, x)$. φ satisfies $(mod - ii)$ since for any $z \in Obj(A)$, $\{u\} \circ \varphi(z) \subseteq \varphi(z)$. φ satisfies $(mod - iii)$, since for any $z, z' \in Obj(A)$,

$$\begin{aligned} \varphi(z) \circ A(z', z) &= \{\sigma_0\} \circ A(z, z_0) \circ A(z', z) \\ &\subseteq \{\sigma_0\} \circ A(z', z_0) \\ &= \varphi(z') \end{aligned}$$

Analogously one shows that the map sending any $z \in Obj(A)$ to $A(z_0, z) \circ \{{\sigma_0}^{-1}\}$ defines a module $\tilde{\varphi} : A \multimap \hat{x}$. $\tilde{\varphi}$ is right-adjoint to φ as shown below.
Since $u \in \bigcup_{z \in Obj(A)} A(z, z_0) \circ A(z_0, z)$,

$$\begin{aligned} \{u\} &\subseteq \bigcup_{z \in Obj(A)} \{\sigma_0\} \circ A(z, z_0) \circ A(z_0, z) \circ \{{\sigma_0}^{-1}\} \\ &= \bigcup_{z \in Obj(A)} \varphi(z) \circ \tilde{\varphi}(z). \end{aligned}$$

For any $z, z' \in Obj(A)$,

$$\begin{aligned} \tilde{\varphi}(z) \circ \varphi(z') &= A(z_0, z) \circ \{{\sigma_0}^{-1}\} \circ \{\sigma_0\} \circ A(z', z_0) \\ &\subseteq A(z', z). \end{aligned}$$

Now consider some $x \in X$, and some left adjoint module $\varphi : \hat{x} \multimap A$ with right adjoint $\tilde{\varphi}$. Then $\{u\} \subseteq \bigcup_{z \in Obj(A)} \varphi(z) \circ \tilde{\varphi}(z)$. This implies that there are some $z_0 \in Obj(A)$ and some $\sigma_0 \in \varphi(z_0)$ with ${\sigma_0}^{-1} \in \tilde{\varphi}(z_0)$. Let $\psi : \hat{x} \multimap A$ be the left adjoint module defined by $\forall z \in Obj(A), \psi(z) = \{\sigma_0\} \circ A(z, z_0)$. Then for any $z \in Obj(A)$,

$$\begin{aligned} \psi(z) &= \{\sigma_0\} \circ A(z, z_0) \\ &\subseteq \varphi(z_0) \circ A(z, z_0) \\ &\subseteq \varphi(z). \end{aligned}$$

Thus $\psi \leq \varphi$. Analogously for any $z \in Obj(A)$, $A(z_0, z) \circ \{{\sigma_0}^{-1}\} \subseteq \tilde{\varphi}(z)$ i.e. $\tilde{\psi} \leq \tilde{\varphi}$. Therefore $\psi = \varphi$.

(2) : First let us check that $\sim$ is an equivalence relation. It is reflexive since for any object z of A, $u \in A(z, z)$. It is symmetric since if $(\sigma, z) \sim (\sigma', z')$ then there is a $\gamma \in A(z, z')$ such that $\sigma = \sigma' \circ \gamma$, then, $\gamma^{-1} \in A(z', z)$ and $\sigma' = \sigma \circ \gamma^{-1}$. To check

transitivity, suppose $(\sigma, z) \sim (\sigma', z')$ and $(\sigma', z') \sim (\sigma'', z'')$. Then there are some $\gamma \in A(z, z')$ and some $\gamma' \in A(z', z'')$ such that $\sigma = \sigma' \circ \gamma$ and $\sigma' = \sigma'' \circ \gamma'$. Thus $\gamma' \circ \gamma \in A(z', z'') \circ A(z, z') \subseteq A(z, z'')$ and $\sigma = \sigma'' \circ \gamma' \circ \gamma$.

Consider further on $\varphi, \psi : \hat{x} \multimap A$, $\sigma_0, \sigma_0' \in G$ and, $z_0, z_0' \in Obj(A)$ such that $\forall z \in Obj(A)$, $\varphi(z) = \{\sigma_0\} \circ A(z, z_0)$ and $\psi(z) = \{\sigma_0'\} \circ A(z, z_0')$. First, suppose $(\sigma_0, z_0) \sim (\sigma_0', z_0')$ i.e., there is some $\gamma \in A(z_0, z_0')$ such that $\sigma_0 = \sigma_0' \circ \gamma$. Then for any $z \in Obj(A)$,

$$\begin{aligned}
\varphi(z) &= \{\sigma_0\} \circ A(z, z_0) \\
&= \{\sigma_0'\} \circ \{\gamma\} \circ A(z, z_0) \\
&= \{\sigma_0'\} \circ A(z_0, z_0') \circ A(z, z_0) \\
&\subseteq \{\sigma_0'\} \circ A(z, z_0') \\
&= \psi(z),
\end{aligned}$$

then $\varphi \leq \psi$. Since $\sigma_0' = \sigma_0 \circ \gamma^{-1}$ and then $\gamma^{-1} \in A(z_0', z_0)$, one obtains analogously $\psi \leq \varphi$.

Now suppose $\varphi = \psi$. Then for any $z \in Obj(A)$, $\{\sigma_0\} \circ A(z, z_0) = \{\sigma_0'\} \circ A(z, z_0')$. For $z = z_0$, we get $\{\sigma_0\} \subseteq \{\sigma_0'\} \circ A(z_0, z_0')$, i.e. there is $\gamma \in A(z_0, z_0')$, such that $\sigma_0 = \sigma_0' \circ \gamma$. ∎

Corollary 4.21 *Let (G, X, α) be a group action and A be a reversible enrichment on $\mathcal{C}_{(G,X,\alpha)}$. Any left adjoint module $\varphi : \hat{x} \multimap A$ has right adjoint φ^s.*

Proposition 4.22 *Let (G, X, α) be a group action. The category of skeletal and Cauchy-reversible-complete enrichments on $\mathcal{C}_{(G,X,\alpha)}$ is equivalent to the category $G - Set \downarrow (X, \alpha)$ of objects over (X, α). Precisely the equivalence $J : G - Set \downarrow (X, \alpha) \cong \mathcal{C}_{(G,X,\alpha)} - SkCcCat$ is given by:*
- For any $f : (Z, \beta) \to (X, \alpha)$ in $G - Set$, $J(f)$ is the enrichment with set of objects Z, and such that for any $z \in Z$, $z_+ = f(z)$, and for any $z, z' \in Z$, $J(f)(z, z')$ is the set $\{\sigma \in G \mid < \sigma, z >_\beta = z'\}$;
- For any arrow $< f_1, h, f_2 >$ of $G - Set \downarrow (X, \alpha)$ with $f_1 : (Z_1, \beta_1) \to (X, \alpha)$ and $f_2 : (Z_2, \beta_2) \to (X, \alpha)$, $J(h)$ is the $\mathcal{C}_{(G,X,\alpha)}$-functor from $J(f_1)$ to $J(f_2)$ defined by the map $h : Z_1 \to Z_2$.
J has adjoint J' defined on objects as follows. For any skeletal Cauchy-reversible-complete enrichment A on $\mathcal{C}_{(G,X,\alpha)}$, $J'(A) = (G, Obj(A), \beta)$ where for any $\sigma \in G$ and any $z \in Obj(A)$, $< \sigma, z >_\beta$ is the unique z' such that $\sigma \in A(z, z')$.

proof: Let us check that given $f : (Z, \beta) \to (X, \alpha)$, $J(f)$ is a skeletal Cauchy-reversible-complete $\mathcal{C}_{(G,X,\alpha)}$-enrichment.
- It satisfies $(enr - i)$: Since f is a morphism of $G - Set$, for any $z \in Z$ and any $\sigma \in G$, $< \sigma, f(z) >_\alpha = f(< \sigma, z >_\beta)$. Therefore for any $z, z' \in Z$,

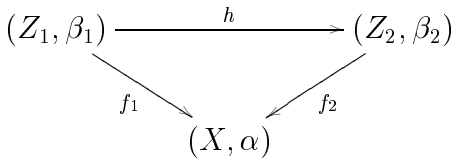
$$J(f)(z,z') = \{\sigma \in G \mid <\sigma, z>_\beta = z'\}$$
$$\subseteq \{\sigma \in G \mid <\sigma, f(z)>_\alpha = f(z')\}$$

i.e. $J(f)(z,z')$ is an object of $\mathcal{C}_{(G,X,\alpha)}(z_+, {z'}_+)$.
- $J(f)$ satisfies $(enr-ii)$ since for any $z \in Z$, $\{u\} \subseteq J(f)(z,z)$.
- It trivially satisfies $(enr-iii)$.
- $J(f)$ is trivially reversible.
- $J(f)$ is skeletal since for any $z, z' \in G$, $u \in J(f)(z,z') \Rightarrow z = z'$.
- Let us show that it is Cauchy-complete. Consider a module $\varphi : \hat{x} \multimap J(f)$ with right adjoint $\tilde{\varphi}$. According to Lemma 4.20 there are some $z_0 \in Z$ and some $\sigma_0 \in \varphi(z_0)$ such that $\varphi(z) = \{\sigma_0\} \circ J(f)(z, z_0)$. Let $z_1 = <\sigma_0, z_0>_\beta$. Then $f(z_1) =< \sigma_0, f(z_0) >= x$. We are going to show that $\varphi = {z_1}_\diamond$. ${z_1}_\diamond \leq \varphi$ since for any $z \in Z$,

$$J(f)(z, z_1) = \{\sigma_0\} \circ J(f)(z, z_0)$$
$$\subseteq \varphi(z_0) \circ J(f)(z, z_0)$$
$$\subseteq \varphi(z).$$

Analogously ${z_1}^\diamond \leq \tilde{\varphi}$ since for any $z \in Z$, $J(f)(z_1, z) = J(f)(z_0, z) \circ \{{\sigma_0}^{-1}\}$. Thus ${z_1}_\diamond = \varphi$.

Let us check that given some arrow $< f_1, h, f_2 >$ of $G-Set \downarrow (X,\alpha)$, $J(h)$ is a well defined $\mathcal{C}_{(G,X,\alpha)}$-functor. Suppose $f_1 : (Z_1, \beta_1) \to (X,\alpha)$ and $f_2 : (Z_2, \beta_2) \to (X,\alpha)$. h is an arrow of $G-Set$ that makes the following diagram in $G-Set$ commute:



Since h is an arrow of $G-Set$, for any $z, z' \in Z_1$, $J(f_1)(z,z') \subseteq J(f_2)(hz, hz')$ and the map $h : Z_1 \to Z_2$ defines a $\mathcal{C}_{(G,X,\alpha)}$-functor from $J(f_1)$ to $J(f_2)$.

Now it is straightforward to check that J as defined above is a functor from $G-Set \downarrow (X,\alpha)$ to $\mathcal{C}_{(G,X,\alpha)}-Cat$.

Let us check that given some skeletal reversible Cauchy-complete enrichment A on $\mathcal{C}_{(G,X,\alpha)}$, $J'(A) = (Obj(A), \beta)$ is a well defined object of $G-Set \downarrow (X,\alpha)$.

First we need to prove the property $(*)$:
Given $x \in X$, $z_0 \in Obj(A)$ and $\sigma_0 \in G$ such that $<\sigma_0, {(z_0)}_+ >= x$, there is a unique $z_1 \in Obj(A)$ with ${(z_1)}_+ = x$ and $\{\sigma_0\} \subseteq A(z_0, z_1)$.

Let x, z_0, and σ_0 be as above and $\varphi : \hat{x} \multimap A$ be the left adjoint module defined by $\varphi(z) = \{\sigma_0\} \circ A(z, z_0)$. Due to the Cauchy-completness of A, there is

a $z_1 \in Obj(A)$ such that $\varphi = z_{1\diamond}$. Such a z_1 satisfies therefore $(z_1)_+ = x$ and $\{\sigma_0\} \subseteq A(z_0, z_1)$. Due to skeletality, such a z_1 is unique.

According to $(*)$, $\beta : G \times Obj(A) \to Obj(A)$ is a well defined map. β defines indeed an action, of G on $Obj(A)$, as shown below (for short we omit the subscript β).
- For any $z \in Obj(A)$, $< u, z >= z$ since $u \in A(z, z)$.
- For any $z \in Obj(A)$ and any $g_1, g_2 \in G$, $\{g_1\} \subseteq A(z, < g_1, z >)$ and $\{g_2\} \subseteq A(< g_1, z >, < g_2, < g_1, z >>)$. Therefore

$$\begin{aligned} g_2 \cdot g_1 &\in A(< g_1, z >, < g_2, < g_1, z >> \circ A(z, < g_1, z >) \\ &\subseteq A(z, < g_2, < g_1, z >>), \end{aligned}$$

proving $< g_2 \cdot g_1, z >=< g_2, < g_1, z >>$.

Since by definition for any $z, z' \in Obj(A)$, $A(z, z') = \{\sigma \in G \mid < \sigma, z >_\beta = z'\}$, it is clear that $(-)_+$ defines an arrow of $G - Set$ from $(Obj(A), \beta)$ above to (X, α).

In order to prove that J is an equivalence of category with adjoint J' we must prove $J' \circ J \cong 1$ and $J \circ J' \cong 1$. It is immediate to see that the two natural isomorphims above are identities. ∎

We found convenient to present now the following result that we shall use later (4.34).

Lemma 4.23 *If A is a reversible enrichment on $\mathcal{C}_{(G,X,\alpha)}$ with (reversible) Cauchy-completion $\bar{A}$, then $J(\bar{A})$ is isomorphic to the arrow of $G - Set$*

$$f : ((G \times Obj(A))/\sim, \delta) \to (X, \alpha)$$

where:
- $\sim$ is the equivalence on $G \times Obj(A)$ defined in Lemma 4.20-(2);
- G acts on $(G \times Obj(A))/\sim$ as follows:

$$< \tau, [(\sigma, z)] >_\delta = [(\tau \circ \sigma, z)];$$

- for any $z \in Obj(A)$ and any $\sigma \in G$,

$$f([(\sigma, z)]) = < \sigma, z_+ >_\alpha$$

proof: First, let us show that the maps δ and f are well defined. Suppose $(\sigma, z) \sim (\sigma', z')$ for some $\sigma, \sigma' \in G$ and some $z, z' \in Obj(A)$. By definition there is a $\gamma \in A(z, z')$ such that $\sigma = \sigma' \circ \gamma$. For any $\tau \in G$, $\tau \circ \sigma = \tau \circ \sigma' \circ \gamma$, therefore $(\tau \circ \sigma, z) \sim (\tau \circ \sigma', z')$. Since $\gamma \in A(z, z')$, $< \gamma, z_+ >_\alpha = (z')_+$ and $< \sigma, z_+ >_\alpha = < \sigma' \circ \gamma, z_+ >_\alpha = < \sigma', (z')_+ >_\alpha$. δ trivially defines an action of G on $(G \times Obj(A))/\sim$.

Further on, let $\bar{A}$ denote the Cauchy-completion of A. According to Lemma 4.20-(2), the map $h : Obj(\bar{A}) \to (G \times X)/\sim$ sending any φ to $[(\sigma_0, z_0)]$ such that $\forall z, \varphi(z) = \{\sigma_0\} \circ A(z, z_0)$ is well defined and is a bijection.

According to 4.22, in order to conclude, it is enough to show that h defines an isomorphism of $\mathcal{C}_{(G,X,\alpha)} - Cat$ from $\bar{A}$ to $J'(f) = ((G \times Obj(A))/\sim, f, d)$. $f \circ h = (-)_+$ since for any $\varphi : \hat{x} \multimap A = h([(\sigma_0, z_0)])$, $\varphi_+ = x = <\sigma_0, (z_0)_+>_\alpha = f([(\sigma_0, z_0)])$. For any $\sigma, \sigma' \in G$ and any $z, z' \in Obj(A)$,

$$\begin{aligned}
A([\sigma, z], [\sigma', z']) &= \{\tau \in G \mid <\tau, [\sigma, z]>_\delta = [\sigma', z']\} \\
&= \{\tau \in G \mid [\tau \circ \sigma, z] = [\sigma', z']\} \\
&= \{\tau \in G \mid \exists \gamma \in A(z, z'), \tau \circ \sigma = \sigma' \circ \gamma\} \\
&= \{\sigma'\} \circ A(z, z') \circ \{\sigma^{-1}\} \\
&= \{\sigma'\} \circ \bigvee_{y \in Obj(A)} (A(y, z') \circ A(z, y)) \circ \{\sigma^{-1}\} \\
&= \bar{A}(h([\sigma, z]), h([\sigma', z'])).
\end{aligned}$$

∎

4.3 Change of base, reversibility and completions

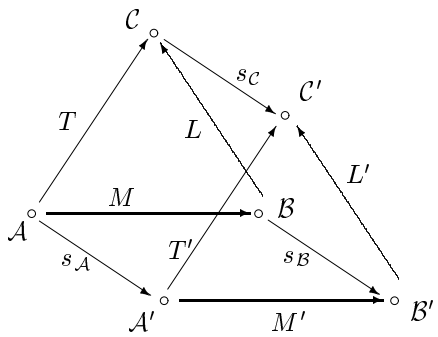
If $\mathcal{U}$ and $\mathcal{V}$ are reversible then $Conv(\mathcal{U}, \mathcal{V})$ may be provided with a reversible structure. The isomorphism $s_{(u,v),(u',v')} : Conv(\mathcal{U}, \mathcal{V})((u, v), (u', v')) \cong Conv(\mathcal{U}, \mathcal{V})((u', v'), (u, v))$ sends any functor $F : (u, v) \to (u', v')$ to the functor $F_s : (u', v') \to (u, v)$ such that

$$\begin{array}{ccc}
\mathcal{U}(u, u') & \xrightarrow{F} & \mathcal{V}(v, v') \\
{\scriptstyle s^{\mathcal{U}}_{u',u}}\uparrow & & \downarrow{\scriptstyle s^{\mathcal{V}}_{v,v'}} \\
\mathcal{U}(u', u) & \xrightarrow[F^s]{} & \mathcal{V}(v', v)
\end{array}$$

and any $\sigma : F \to G : (u, v) \to (u', v')$ to $\sigma^s : F^s \to G^s = s^{\mathcal{V}}_{v,v'} \circ \sigma \circ s^{\mathcal{U}}_{u',u}$.

The $s_{(u,v),(u',v')}$ are trivially idempotent. To see that $Conv(\mathcal{U}, \mathcal{V})$ satisfies $rev-(ii)$ and $rev-(iii)$, we will need the following lemma which proof is left to the reader.

Lemma 4.24 *Given functors $M : \mathcal{A} \to \mathcal{B}$, $T : \mathcal{A} \to \mathcal{C}$, $M' : \mathcal{A}' \to \mathcal{B}'$, $T' : \mathcal{A}' \to \mathcal{C}'$ and isomorphisms $s_{\mathcal{A},\mathcal{A}'} : \mathcal{A} \to \mathcal{A}'$, $s_{\mathcal{B},\mathcal{B}'} : \mathcal{B} \to \mathcal{B}'$ and, $s_{\mathcal{C},\mathcal{C}'} : \mathcal{C} \to \mathcal{C}'$ such that:*
- the left Kan extension L of T along M and the left Kan extension L' of T' along M' exist,
- $s_{\mathcal{B},\mathcal{B}'} \circ M = M' \circ s_{\mathcal{A},\mathcal{A}'}$,
- $s_{\mathcal{C},\mathcal{C}'} \circ T = T' \circ s_{\mathcal{A},\mathcal{A}'}$,
then $L' \circ s_{\mathcal{B},\mathcal{B}'} \cong s_{\mathcal{C},\mathcal{C}'} \circ L$.



That $Conv(\mathcal{U},\mathcal{V})$ satisfies $rev-(ii)$ is equivalent to the fact that for any functors $F:\mathcal{U}(u,u')\to\mathcal{V}(v,v')$ and $G:\mathcal{U}(u',u'')\to\mathcal{V}(v',v'')$ the diagram below (in Cat) commutes

$$\begin{array}{ccc}\mathcal{U}(u,u'') & \xrightarrow{G\bar{\circ}F} & \mathcal{V}(v,v'') \\ {\scriptstyle s^{\mathcal{U}}_{u,u''}}\downarrow & & \downarrow{\scriptstyle s^{\mathcal{V}}_{v,v''}} \\ \mathcal{U}(u'',u) & \xrightarrow[F^s\bar{\circ}G^s]{} & \mathcal{V}(v'',v)\end{array}$$

To see that it holds, just apply 4.24 with:

- $M=\ \mathcal{U}(u,u')\times\mathcal{U}(u',u'')\xrightarrow{\circ^{\mathcal{U}}}\mathcal{U}(u,u'')$,
- $T=\ \mathcal{U}(u,u')\times\mathcal{U}(u',u'')\xrightarrow{F\times G}\mathcal{V}(v,v')\times\mathcal{V}(v',v'')\xrightarrow{\circ^{\mathcal{V}}}\mathcal{V}(v,v'')$,
- $M'=\ \mathcal{U}(u'',u')\times\mathcal{U}(u',u)\xrightarrow{\circ^{\mathcal{V}}}\mathcal{U}(u'',u)$,
- $T'=\ \mathcal{U}(u'',u')\times\mathcal{U}(u',u)\xrightarrow{G^s\times F^s}\mathcal{V}(v'',v')\times\mathcal{V}(v',v)\xrightarrow{\circ^{\mathcal{V}}}\mathcal{V}(v'',v)$,

and isomorphisms:

- $\mathcal{U}(u,u')\times\mathcal{U}(u',u'')\xrightarrow{\cong}\mathcal{U}(u',u'')\times\mathcal{U}(u,u')\xrightarrow{s_{u',u''}\times s_{u,u'}}\mathcal{U}(u'',u')\times\mathcal{U}(u',u)$,
- $\mathcal{U}(u,u'')\xrightarrow{s_{u,u''}}\mathcal{U}(u'',u)$,
- $\mathcal{V}(v,v'')\xrightarrow{s_{v,v''}}\mathcal{V}(v'',v)$.

That $Conv(\mathcal{U},\mathcal{V})$ satisfies $rev-(iii)$ is equivalent to the fact that for any objects u of $\mathcal{U}$ and v of $\mathcal{V}$ the following diagram commutes.

$$\begin{array}{ccc}\mathcal{U}(u,u) & \xrightarrow{1_{(u,v)}} & \mathcal{V}(v,v)\ . \\ {\scriptstyle s^{\mathcal{U}}_{u,u}}\downarrow & & \downarrow{\scriptstyle s^{\mathcal{V}}_{v,v}} \\ \mathcal{U}(u,u) & \xrightarrow[1_{(u,v)}]{} & \mathcal{V}(v,v)\end{array}$$

This results from applying 4.24 with:

- $M=\ 1\xrightarrow{1_u}\mathcal{U}(u,u)$,
- $T=\ 1\xrightarrow{1_v}\mathcal{V}(v,v)$,

- $M' = 1 \xrightarrow{1_u} \mathcal{U}(u,u)$,

- $T' = 1 \xrightarrow{1_v} \mathcal{V}(v,v)$,

and isomorphisms $1 : 1 \to 1$, $s_{u,u}$ and, $s_{v,v}$.

Definition 4.25 *A two-sided enrichment $F : \mathcal{V} \to \mathcal{W}$ between reversible locally posetal bicategories is* reversible *when for any objects x, y of F the following diagram commutes*

$$\begin{array}{ccc} \mathcal{V}(x_-, y_-) & \xrightarrow{F_{x,y}} & \mathcal{W}(x_+, y_+) \\ {\scriptstyle s^{\mathcal{V}}_{x_-,y_-}}\downarrow & & \downarrow{\scriptstyle s^{\mathcal{W}}_{x^+,y^+}} \\ \mathcal{V}(y_-, x_-) & \xrightarrow[F_{y,x}]{} & \mathcal{W}(y_+, x_+) \end{array}$$

By definition for reversible locally posetal bicategories $\mathcal{U}$ and $\mathcal{V}$, the isomorphism

$$Caten(\mathcal{U}, \mathcal{V}) \cong Conv(\mathcal{U}, \mathcal{V}) - Cat$$

sends exactly reversible two-sided enrichments to reversible enrichments. Conversely, if $\mathcal{V}$ is reversible the composite isomorphism

$$\mathcal{V} - Cat \cong Conv(1, \mathcal{V}) - Cat \cong Caten(1, \mathcal{V})$$

sends bijectively reversible enrichments onto reversible two-sided-enrichments. It is immediate that reversible enrichments form a subcategory of $Caten$. Thus for any reversible two-sided enrichment $F : \mathcal{V} \to \mathcal{W}$, the corresponding change of base $F_@ : \mathcal{V} \to \mathcal{W}$ preserves the reversibility of objects.

Further on $\mathcal{V}$ and $\mathcal{W}$ will denote two locally posetal reversible bicategories.

Using 3.10, it is straightforward to check that

Observation 4.26 *Given a reversible super two-sided enrichment $F : \mathcal{V} \to \mathcal{W}$, the lax functor $F_\sharp : \mathcal{V} - Mod \to \mathcal{W} - Mod$ defined in 3.11 preserves the reversibility of arrows: for any reversible $\mathcal{V}$-module φ, $F_\sharp(\varphi)$ is reversible.*

According to this and 4.12

Fact 4.27 *For any super reversible two-sided enrichment $F : \mathcal{V} \to \mathcal{W}$, $F_@$ preserves the Morita-reversible-equivalence: if the reversible $\mathcal{V}$-categories A and B are Morita-reversible-equivalent then also are $F_@A$ and $F_@B$.*

Further on, we consider some adjunction $F \dashv G : \mathcal{V} \to \mathcal{W}$ in $Caten$, say F is a 2-functor with local right adjoints $G_{a,b} : \mathcal{W}(Fa, Fb) \to \mathcal{V}(a, b)$.

Lemma 4.28 *F is reversible if and only if G is.*

To prove this, we need

Lemma 4.29 *Given objects a, b of $\mathcal{V}$ If $s : \mathcal{V}(a,b) \to \mathcal{V}(b,a)$ and $s' : \mathcal{W}(Fa,Fb) \to \mathcal{W}(Fb,Fa)$ are functors then*

$$F_{b,a} \circ s \leq s' \circ F_{a,b} \Leftrightarrow s \circ G_{a,b} \leq G_{b,a} \circ s'.$$

proof: ($\Rightarrow$) Consider some arrow of $\mathcal{W}$, $\alpha : Fa \to Fb$. Since $F_{a,b} \dashv G_{a,b}$, $F_{a,b}G_{a,b}(\alpha) \leq \alpha$. Therefore $s'F_{a,b}G_{a,b}(\alpha) \leq s'(\alpha)$. By assumption, $F_{b,a} \circ s \leq s' \circ F_{a,b}$ thus $F_{b,a}sG_{a,b}(\alpha) \leq s'F_{a,b}G_{a,b}(\alpha) \leq s'(\alpha)$. Then, since $F_{b,a} \dashv G_{Fb,Fa}$, $sG_{a,b}(\alpha) \leq G_{b,a}s'(\alpha)$ (see the diagram in $\mathcal{W}(Fb,Fa)$ below where subscripts a,b are omited).

$$\begin{array}{ccc} s'(\alpha) & \longleftarrow & FGs'(\alpha) \\ \uparrow & & \vdots \\ s'FG(\alpha) & \longleftarrow & FsG(\alpha) \end{array}$$

($\Leftarrow$) Consider some arrow of $\mathcal{V}$, $\alpha : a \to b$. Since $F_{a,b} \dashv G_{a,b}$, $\alpha \leq G_{a,b}F_{a,b}(\alpha)$. Therefore $s(\alpha) \leq sG_{a,b}F_{a,b}(\alpha)$. By assumption $s \circ G_{a,b} \leq G_{b,a} \circ s'$, thus $sG_{a,b}F_{a,b}(\alpha) \leq G_{b,a}s'F_{a,b}(\alpha)$. Then since $F_{b,a} \dashv G_{b,a}$, $F_{b,a}s(\alpha) \leq s'F_{a,b}(\alpha)$ (see the diagram in $\mathcal{V}(b,a)$ below).

$$\begin{array}{ccc} s(\alpha) & \longrightarrow & GFs(\alpha) \\ \downarrow & & \vdots \\ sGF(\alpha) & \longrightarrow & Gs'F(\alpha) \end{array}$$

∎

proof of 4.28: As consequence of 4.29, one has that for any objects a, b of $\mathcal{V}$, the following diagram (in Cat) commutes

$$\begin{array}{ccc} \mathcal{W}(Fa,Fb) & \xrightarrow{G_{a,b}} & \mathcal{V}(a,b) \\ {\scriptstyle s^{\mathcal{W}}_{Fa,Fb}}\downarrow & & \downarrow{\scriptstyle s^{\mathcal{V}}_{a,b}} \\ \mathcal{W}(Fb,Fa) & \xrightarrow[G_{b,a}]{} & \mathcal{V}(b,a) \end{array}$$

if and only the diagram

$$\begin{array}{ccc} \mathcal{V}(a,b) & \xrightarrow{F_{a,b}} & \mathcal{W}(Fa,Fb) \\ {\scriptstyle s^{\mathcal{V}}_{a,b}}\downarrow & & \downarrow{\scriptstyle s^{\mathcal{W}}_{Fa,Fb}} \\ \mathcal{V}(b,a) & \xrightarrow[F_{b,a}]{} & \mathcal{W}(Fb,Fa) \end{array}$$

commutes. ∎

Further on, we suppose that the left adjoint F is reversible.

Lemma 4.30 *If A is a Cauchy-reversible-complete $\mathcal{W}$-category then $F^{@}A$ is Cauchy-reversible-complete.*

proof: First we can show that for any object v of $\mathcal{V}$ and any reversible $\mathcal{V}$-module $\varphi : \hat{v} \multimap F^{@}A$, there is a $\mathcal{V}$-functor $f : \hat{v} \to F^{@}A$ with $f_{\diamond} \geq \varphi$. The proof of this merely an adaptation of the proof of 3.13. Let v be an object of $\mathcal{V}$ and $\varphi : \hat{v} \multimap F^{@}A$ be a reversible $\mathcal{V}$-module. According to 4.26, $F_{\sharp}\varphi : F_{@}\hat{v} \multimap F_{@}F^{@}A$ is reversible. Then also is $\epsilon_{A\diamond} \bullet F_{@}\varphi : F_{@}\hat{v} \multimap A$. Due to the Cauchy-reversible-completness of A, there is a $\mathcal{V}$-functor $g : F_{@}\hat{v} \to A$ such that $g_{\diamond} = \epsilon_{A\diamond} \bullet F_{\sharp}\varphi$. Then one concludes as for 3.13. Indeed for the f determined above $f_{\diamond} = \varphi$, since the category $\mathcal{V} - RMod(\hat{v}, F^{@}A)$ is discrete. ∎

Since $F_{@}$ and $F^{@}$ preserves the reversibility of objects and $F^{@}$ preserves the skeletality and the Cauchy-reversible-completness of objects (3.4, 4.30), one deduces from 3.3 the existence for any skeletal and Cauchy-reversible-complete $\mathcal{V}$-category A of a natural isomorphism in B:

$$\mathcal{V} - SkCRcCat(A, F^{@}B) \cong_{B} \mathcal{W} - SkCRcCat((F_{@}A)^{s}, B).$$

Let us sum up

Theorem 4.31 *For any reversible left adjoint $F : \mathcal{V} \to \mathcal{W}$ in $Caten$, the functor $F^{\sim} : \mathcal{W} - SkCRcCat \to \mathcal{W} - SkCRcCat$*

$$\begin{cases} B \mapsto F^{@}B, \\ f : B \to B' \ \mapsto \ F^{@}f : F^{@}B \to F^{@}B' \end{cases}$$

has a left adjoint $F_{\sim}$ defined on objects by $F^{\sim}A = (F_{@}A)^{s}$.

4.4 Examples (continued)

Now we give two applications of theorem 4.31.

Example 4.32 (Geometric morphisms)

Let $f : L \to H$ be a continuous map of locales, with corresponding frame morphims $f^{-} : H \to L$. It corresponds to a 2-functor $F : \mathcal{C}_{H} \to \mathcal{C}_{L}$ that preserves reversibility. F sends any object u to $f^{-}u$; $F_{u,v}$ sends any morphisms $w : u \to v$ (i.e. $w \leq u \wedge v$) to $f^{-}w : f^{-}u \to f^{-}v$. $F_{u,v}$ has a right adjoint $G_{u,v}$, defined for any $w \leq f^{-}u \wedge f^{-}v$ by

$$G_{u,v}(w) = \bigvee \{w' \leq u \wedge v \mid f^{-}w' \leq w\}.$$

Therefore and according to 4.18, we may apply 4.31 to retrieve the following well known result.

Theorem 4.33 *Any continuous map of locales $f : L \to H$ yields an adjunction $f_* \dashv f^* : Sh(H) \to Sh(L)$, where f_* and f^* are respectively the* inverse *and* direct *image functors. f^* is the following. Let f^- denote the frame morphism corresponding to f. For any sheaf $A : L^{op} \to Set$, and any open v of H,*

$$(f^*A)(v) = A(f^-(v)).$$

For any sheaves $A, B : L^{op} \to Set$ and any natural transformation $h : A \to B$, f^*h is the natural transformation defined in each open v of H by

$$(f^*h)_v = h_{f^-v} : A(f^-(v)) \to B(f^-(v)).$$

To see this, it is enough to check that given such a $f : L \to H$, if $F_\sim$ denotes the left adjoint of 4.31 for the F defined above, then f^* is $F_\sim$ up to the equivalence of 4.18. Which is immediate.

Example 4.34 (Group actions)

Recall that any group morphism $\varphi : G \to H$ induces an adjoint pair $\varphi_* \dashv \varphi^* : G - Set \rightharpoonup H - Set$. φ_* is named the *extension* functor) and φ^*, the *restriction* functor. Actually φ^* has also a right adjoint named the *induction* functor.

For any H-set (Y, β), $\varphi^*((Y, \beta))$ is the set Y with action

$$\begin{cases} G \times Y \to Y, \\ (\sigma, y) \mapsto < \varphi(\sigma), y >_\beta. \end{cases}$$

An arrow from (Y_1, β_1) to (Y_2, β_2) of $H - Set$ with underlying map $p : Y_1 \to Y_2$ has image by φ^*, the arrow of $G - Set$ (from $\varphi^*((Y_1, \alpha_1))$ to $\varphi^*((Y_2, \alpha_2))$) with underlying map p.

For any G-set (X, α), its image by φ_* is the *contracted product* $H \wedge^G (X, \alpha)$ defined as follows. Its underlying set is

$$(H \times X)/\sim$$

with $\sim$ the equivalence defined by

$$\forall \tau, \tau' \in H, \ \ \forall x, x' \in X, \ \ (\tau, x) \sim (\tau', x') \Leftrightarrow \exists \sigma \in G, < \sigma, x >_\alpha = x', \tau = \tau' \circ \varphi(\sigma).$$

Our purpose is to show the later adjunction may be retrieved as an instance of 4.31. Precisely, considering a category $Grp - Set$ of group actions, described below, we will show that any arrow of $Grp - Set$, $(\varphi, f) : (G, X, \alpha) \to (H, Y, \beta)$ yields an adjunction $(\varphi, f)^\sim \dashv (\varphi, f)_\sim$ with $(\varphi, f)^\sim : G - Set \downarrow (G, X, \alpha) \to H - Set \downarrow (H, Y, \beta)$. The previous pair $\varphi_* \dashv \varphi^*$ corresponding then to the adjunct pair $(\varphi, 1_{\{*\}})^\sim \dashv (\varphi, 1_{\{*\}})_\sim$ with $(\varphi, 1_{\{*\}})^\sim : (G, \{*\}, !_G) \to (H, \{*\}, !_H)$, $\{*\}$ denoting the one-point

set, $1_{\{*\}}$, the identity map on this set, and $!_G$ and $!_H$, the respective unique actions of G and H on this set.

The category $Grp-Set$ of group actions is as follows. Objects are group actions. Arrows from (G, X, α) to (H, Y, β) are the pairs (φ, f) where $\varphi : G \to H$ is a group morphism, $f : X \to Y$ is a map and, the following diagram commutes:

$$\begin{array}{ccc} G \times X & \xrightarrow{\alpha} & X \\ {\scriptstyle \varphi \times f}\downarrow & & \downarrow{\scriptstyle f} \\ H \times Y & \xrightarrow[\beta]{} & Y \end{array}$$

Consider an arrow $(\varphi, f) : (G, X, \alpha) \to (H, Y, \beta)$. It corresponds to a 2-functor $F : \mathcal{C}_{(G,X,\alpha)} \to \mathcal{C}_{(H,Y,\beta)}$ that preserves reversibility. F sends any object x to $f(x)$ and, $F_{x,x'}$ sends any arrow $L : x \to x'$ — i.e. $L \subseteq \{\sigma \in G \mid < \sigma, x >_\alpha = x'\}$ to the arrow

$$\varphi(L) = \{\varphi(\sigma) \mid \sigma \in L\} : f(x) \to f(x').$$

F has local right adjoints: for any $x, x' \in X$, $G_{x,x'}$, the right adjoint to $F_{x,x'}$, is defined for any $L \subseteq \{\tau \in H \mid < \tau, f(x) >_\beta = f(x')\}$ by

$$G_{x,x'}(L) = \varphi^{-1}(L) \cap \{\sigma \in G \mid < \sigma, x >_\alpha = x'\}.$$

Therefore we may apply 4.31 together with 4.22 and 4.23 to obtain the following result.

Theorem 4.35 *Any arrow $(\varphi, f) : (G, X, \alpha) \to (H, Y, \beta)$ of $Grp - Set$ yields some adjunction $(\varphi, f)^\sim \dashv (\varphi, f)_\sim$, with $(\varphi, f)^\sim : G - Set \downarrow (X, \alpha) \to H - Set \downarrow (Y, \beta)$, given below.*
- If $h : (Z, \gamma) \to (Y, \beta)$ is an arrow of $H - Set$, its image by $(\varphi, f)_\sim$ is the arrow of $G - Set$, $\pi^2 : (Z \times_{h,f} X, \delta) \to (X, \alpha)$, with

$$Z \times_{h,f} X = \{(z, x) \mid h(z) = f(x)\},$$

for any $(z, x) \in Z \times_{h,f} X$,

$$\pi^2((z, x)) = x,$$

and for any $\sigma \in G$ and any $(z, x) \in Z \times_{h,f} X$,

$$< \sigma, (z, x) >_\delta = (< \varphi(\sigma), z >_\gamma, < \sigma, x >_\alpha).$$

- $(\varphi, f)_\sim$ is defined on arrows as follows. Considering the following commuting diagram in $H - Set$:

$$\begin{array}{ccc} (Z_1,\gamma_1) & \xrightarrow{\quad p \quad} & (Z_2,\gamma_2)\ . \\ & {\scriptstyle h_1}\searrow \quad \swarrow{\scriptstyle h_2} & \\ & (Y,\beta) & \end{array}$$

the image by $(\varphi, f)_\sim$ *of the arrow* $< h_1, p, h_2 >$ *of* $H - Set \downarrow (Y, \beta)$ *has underlying arrow in* $G - Set$

$$p' : (Z_1 \times_{(h_1,f)} X, \delta_1) \to (Z_2 \times_{(h_2,f)} X, \delta_2)$$

where for any $(z, x) \in Z_1 \times_{(h_1,f)} X$,

$$p'((z, x)) = (p(z), x)$$

— see the commuting diagram in $G - Set$ *below where for* $i = 1, 2$, $(\varphi, f)_\sim(h_i) = \pi^2 : (Z_i \times_{(h_i,f)} X, \delta_1) \to (X, \alpha)$.

$$\begin{array}{ccc} (Z_1 \times_{(h_1,f)} X, \delta_1) & \xrightarrow{\quad p' \quad} & (Z_2 \times_{(h_2,f)} X, \delta_2) \\ & {\scriptstyle \pi^2}\searrow \quad \swarrow{\scriptstyle \pi^2} & \\ & (X, \alpha) & \end{array}$$

- $(\varphi, f)^\sim$ *is defined on objects of* $G - Set \downarrow (X, \alpha)$ *as follows. For any arrow* $g : (Z, \gamma) \to (X, \alpha)$ *of* $G - Set$, $(\varphi, f)^\sim(g)$ *is the arrow*

$$h : ((H \times Z)/\sim, \delta) \to (Y, \beta)$$

of $H - Set$ *where* $\sim$ *is the equivalence on* $H \times Z$ *defined by*

$$\forall \tau, \tau' \in H,\ \ \forall z, z' \in Z,\ \ (\ (\tau, z) \sim (\tau', z') \Leftrightarrow \exists \sigma \in G, < \sigma, z >= z', \tau = \tau' \circ \varphi(\sigma)\ )$$

for any $\tau, \kappa \in H$ *and* $z \in Z$,

$$< \kappa, [(\tau, z)] >_\delta = [(\kappa \circ \tau, z)]$$

for any $\tau \in H$ *and* $z \in Z$,

$$h([(\tau, z)]) =< \tau, \varphi \circ g(z) >_\beta\ .$$

References

[Ben63] J. Bénabou,
Catégories avec multiplication,
C. R. Acad. Sci. Paris 256, 63, 1887-1890.

[Ben67] J. Bénabou,
Introduction to bicategories,
Lecture Notes in Math. 47 (Springer-Verlag), 67, 1-77.

[BePo88] R.Betti, A.J. Power,
On local adjointness of distributive categories, Boll. Un. Mat. Ital. (7) 2-B, 931-947, (88)

[BeWal82] R. Betti, R.F.C. Walters,
The symmetry of the Cauchy-completion of a category,
LNM 962, 82, 7-12.

[EiKe66] S. Eilenberg, G.M Kelly,
Closed Categories,
Proceedings of the conference on Categorical Algebra at La Jolla, Springer, 66, 421-562.

[Kel74] G.M. Kelly,
Doctrinal adjunction,
Lecture Notes in Math. 420 Springer-Verlag, Berlin, 74, 257-280

[Kel82] G.M. Kelly,
Basic Concepts of Enriched Category Theory,
London Mathematical Society Lecture Notes Series 64 - Cambridge University Press 82.

[KLSS99] G.M. Kelly, A. Labella, V.Schmitt, R.Street,
Categories enriched on two sides,
Communication at Category theory 99 - Coimbra 99. To appear in Journal of Pure and Applied Algebra.

[Law73] F.W Lawvere,
Metric spaces, generalized logic, and closed categories,
Rend. Sem. Mat. Fis. di Milano 43, 73, 135-166

[Sc00] V. Schmitt,
Applying enriched categories to quasi-uniform spaces,
submitted paper.

[Wal81] R.F.C. Walters,
Sheaves and Cauchy-complete categories,
3e Colloque sur les catégories, Amiens, juillet 80, Cah. Topo. Geom. Diff., vol XXII-3, 81