Convergence of Euclidean Radial Basis Approximation on Spheres

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Abstract

In this paper we investigate the convergence of radial basis interpolation when all of the data lie on a sphere. Here, we use strictly conditionally positive definite radial functions defined in the ambient Euclidean space. These are strictly conditionally positive definite of the same order when restricted to the sphere. The analysis of v. Golitschek and Light [9] can then be used to give an error estimate for radial approximation in Euclidean space when the data is restricted to the sphere.

1 Introduction

Let S^d be the unit sphere in $\mathbb{R}^{d+1}$, Π_n denote the polynomials of degree n in $\mathbb{R}^{d+1}$ and P_n denote the polynomials on S^d. We will think of P_n as the restrictions of polynomials from Π_n to the sphere. Let juxtaposition xx_i be used to denote the usual inner product of vectors in $\mathbb{R}^{d+1}$.

A univariate function $\phi : \mathbb{R}_+ \to \mathbb{R}$ is said to be *strictly conditionally positive definite of order* m on $\mathbb{R}^{d+1}$ if, for all $N \in \mathbb{N}$, and points

$x_1, x_2, \ldots, x_N \in \mathbb{R}^{d+1}$, whenever

$$\sum_{i=1}^{N} c_i \pi(x_i) = 0$$

for all $\pi \in \Pi_{m-1}$,

$$\sum_{i,j=1}^{N} c_i c_j \phi(\|x_i - x_j\|) > 0.$$

Similarly, $\psi : [-1,1] \to \mathbb{R}$ is said to be strictly conditionally positive definite of order m on S^d if, for all $N \in \mathbb{N}$, and points $x_1, x_2, \ldots, x_N \in S^d$, whenever

$$\sum_{i=1}^{N} c_i p(x_i) = 0$$

for all $p \in P_{m-1}$,

$$\sum_{i,j=1}^{N} c_i c_j \psi(x_i x_j) > 0.$$

Let us now fix our points $x_1, x_2, \ldots, x_N \in S^d$. We assume that we have a function $f \in \mathbf{C}(S^d)$ for which we know the values $f_i = f(x_i)$, $i = 1, \ldots, N$, and we wish to reconstruct the function f using interpolation. The usual procedure in this context is to use a strictly conditionally positive definite function ψ of some order, m say, on the sphere, and construct an approximation

$$r(x) = \sum_{i=1}^{N} \alpha_i \psi(x x_i) + p(x), \quad x \in S^d, \tag{1}$$

where $p \in P_{m-1}$. We then compute the coefficients $\alpha_1, \ldots, \alpha_N$ and the polynomial p with the interpolation conditions

$$r(x_j) = f(x_j), \quad j = 1, \ldots, N,$$

and

$$\sum_{i=1}^{N} \alpha_i q(x_i) = 0, \text{ for all } q \in P_{m-1}.$$

Building on the analysis presented in Levesley et. al. [2], von Golitschek and Light [9] give error estimates of the form

$$|f(x) - r(x)| \leq C\|f\|_\psi h^k, \qquad x \in s^d,$$

where h is a measure of the spacing of the points on the sphere, and k depends on the smoothness of the function ψ. Here, $\|f\|_\psi$ is the so called native space seminorm of f, this native space and hence the target function f, depending intimately on the original choice of ψ.

In this paper we use this error estimate to compute the convergence rate of the following interpolation strategy. Let $\phi : \mathbb{R} \to \mathbb{R}$ be strictly conditionally positive definite of order m in Euclidean space, and approximate $f \in \mathbf{C}(S^d)$ using

$$s(x) = \sum_{i=1}^{N} \alpha_i \phi(\|x - x_i\|) + \pi(x), \tag{2}$$

where $\|\cdot\|$ is the Euclidean distance and $\pi \in \Pi_{m-1}$. We compute the coefficients α_i, $i = 1, \ldots, N$, and the polynomial π with the interpolation conditions

$$s(x_j) = f(x_j), \quad j = 1, \ldots, N,$$

and

$$\sum_{i=1}^{N} \alpha_i \nu(x_i) = 0, \text{ for all } \nu \in \Pi_{m-1}.$$

In Section 2 we show how each conditionally positive definite function in Euclidean space gives rise to a conditionally positive definite function on the sphere. We will show how to write the interpolation problem in the Euclidean space so that the corresponding interpolation problem on the sphere fits the framework in which the results of [9] may be applied. In Section 4 we give decay rates for Gegenbauer coefficients of the restrictions of the polyharmonic splines $\phi(t) = t^\beta$ or $\phi(t) = t^\beta \log t$ in order to determine the appropriate native spaces. We then apply the results of [9] to give convergence rates for interpolation of functions defined on the sphere using polyharmonic spline approximation in the ambient space.

2 Recasting the interpolation problem on the sphere

Each conditionally positive definite function ϕ in Euclidean space gives rise to a conditionally positive definite function ψ on the sphere in the following way. Since, for $x, x_i \in S^d$,

$$\begin{aligned} \|x - x_i\|^2 &= \|x\|^2 + \|x_i\|^2 - 2xx_i \\ &= 2 - 2xx_i, \end{aligned}$$

we see that, for $x, x_i \in S^d$,

$$\psi(xx_i) = \phi((2 - 2xx_i)^{1/2}) = \phi(\|x - x_i\|). \tag{3}$$

Now, let

$$\sum_{i=1}^{N} c_i p(x_i) = 0$$

for all $p \in P_{m-1}$. Then, since $P_n = \Pi_n|_{S^d}$, and $x_i \in S^d$, $i = 1, \ldots, N$,

$$\sum_{i=1}^{N} c_i \pi(x_i) = 0$$

for all $\pi \in \Pi_{m-1}$. As ϕ is strictly conditionally positive definite of order m,

$$\sum_{i,j=1}^{N} c_i c_j \phi(\|x_i - x_j\|) > 0,$$

and so by (3)

$$\sum_{i,j=1}^{N} c_i c_j \psi(x_i x_j) > 0.$$

Thus we see that ψ is strictly conditionally positive definite of order m on the sphere.

We sum these findings up in the following proposition:

Proposition 1 *Let ϕ be strictly conditionally positive definite of order m on $\mathbb{R}^{d+1}$. Then, $\psi(t) = \phi((2-2t)^{1/2})$ is strictly conditionally positive definite of order m on S^d.*

3 Basic theory and convergence rates

Let

$$[f, g] = \int_{S^d} fg\,d\mu$$

denote the usual inner product of functions defined on the sphere. Here $d\mu$ is used to denote the surface measure on the sphere. Let $H_n =$

$P_n \bigcap P_{n-1}^{\perp}$ be the space of spherical harmonics polynomials of degree n, where orthogonality is with respect to the above inner product. Suppose that $\{Y_n^1, \ldots, Y_n^{d_n}\}$ is an orthonormal basis for H_n. Then every element of $L_1(S^d)$ has an associated Fourier series expansion

$$f \sim \sum_{n=0}^{\infty} \sum_{j=1}^{d_n} a_n^j(f) Y_n^j,$$

where

$$a_n^j(f) = \int_{S^d} f Y_n^j d\mu.$$

Let

$$\psi = \sum_{k=0}^{\infty} \psi_k P_k^{\lambda},$$

where P_k^{λ} are the Gegenbauer polynomials, orthogonal with respect to the weight $(1-t^2)^{\lambda-1/2}$; see Szegö [8] for details. Since ψ is conditionally positive definite of order m we know that $\psi_k \geq 0$ for all $k \geq m$ (see [4]) . We will assume that $\psi_k > 0$ for all such k (this assumption will hold for all restrictions of polyharmonic splines to the sphere) so that ψ is strictly conditionally positive definite of order m. Then, the native space induced by ψ is

$$H_{\psi} = \left\{ f \in L_2(S^d) : \|f\|_{\psi}^2 = \sum_{k=m}^{\infty} \psi_k^{-1} \sum_{j=1}^{d_k} (a_k^j(f))^2 < \infty \right\}.$$

We may assume, without loss of generality, that $\psi_k = 0$ for $0 \leq k < m$. For suppose that this were not the case. Let us consider the new basis function

$$\overline{\psi} = \psi - \sum_{k=0}^{m-1} \psi_k P_k^{\lambda}.$$

Then the interpolation problem for this function would be to find a function of the form

$$\overline{r}(x) = \sum_{i=1}^{N} \overline{\alpha}_i \overline{\psi}(x x_i) + \overline{p}(x), \quad x \in S^d,$$

where $\overline{p} \in P_{m-1}$. We then compute the coefficients $\overline{\alpha}_1, \ldots, \overline{\alpha}_N$ and the polynomial $\overline{p}$ with the interpolation conditions

$$\overline{r}(x_j) = f(x_j), \quad j = 1, \ldots, N,$$

and

$$\sum_{i=1}^{N} \overline{\alpha}_i q(x_i) = 0, \text{ for all } q \in P_{m-1}.$$

By the last set of conditions,

$$\sum_{i=1}^{N} \overline{\alpha}_i \overline{\psi}(xx_i) = \sum_{i=1}^{N} \overline{\alpha}_i \psi(xx_i),$$

so that the conditions satisfied by $\overline{\alpha}_1, \ldots, \overline{\alpha}_N$ and $\overline{p}$ are the same as those satisfied by $\alpha_1, \ldots, \alpha_N$ and p. Since the solution of the above interpolation equations is unique $\overline{r} = r$.

We require that the first m Gegenbauer coefficients of ψ be zero to use Corollary 3.3 in [9], though this result is not stated in the most convenient form for our purposes. A slight alteration in their proof in the style of Theorem 4.4 of [3] gives us

Theorem 1 *For each $h \in (0,1)$, let $X_h = \{x_1, \ldots, x_{N(h)}\} \subset S^d$ be such that*

$$\max_{x \in S^d} \min_{y \in X_h} \cos^{-1}(xy) < h.$$

Given $f \in H_\psi$ let r_f be the interpolant to f as given by (1) and the following interpolation conditions. Then the are numbers C and $h_0 > 0$ such that

$$\begin{aligned} \|f - r_f\|_{\infty, S^d} &\leq C \sqrt{\inf_{q \in \Pi^1_{m-1}} \max_{t \in [1-h^2, 1]} |\psi(t) - q(t)|} \|f\|_\psi \\ &= C \sqrt{\inf_{q \in \Pi^1_{m-1}} \max_{t \in [0,h]} |\phi(t) - q(t)|} \|f\|_\psi \end{aligned}$$

for all $h < h_0$, where Π^1_{m-1} denotes the univariate polynomials of degree $m-1$.

4 Applications

In this section we will apply the results of the previous section when $\psi(t) = (1-t)^\beta$ for $\beta > 0$ or $\phi(t) = (1-t)^\beta \log(1-t)$ for $\beta \in \mathbb{N}$. The corresponding functions in Euclidean space, $t^{2\beta}$ and $t^{2\beta} \log t$, are strictly conditionally negative definite of order $[\beta]+1$ ($[\cdot]$ denotes the integer part) if $[\beta]$ is even, and are strictly conditionally positive definite of order $|\beta|+1$ if $[\beta]$ is odd; see Micchelli [5].

In [6] the Gegenbauer coefficients for ψ above are computed. We give these results in the following proposition.

Proposition 2 *Let $\lambda \geq -1/2$, and $k \in I\!N \cup \{0\}$.*

i) For $\beta \geq 0$,

$$\begin{aligned} a_k^{\beta,\lambda} &:= \int_{-1}^{1} (1-t)^{\beta}(1-t^2)^{\lambda-1/2} P_k^{\lambda}(t)\, dt \\ &= (-1)^k 2^{\beta+2\lambda+2k} \frac{\Gamma(\beta+\lambda+1)\Gamma(\lambda+k+\frac{1}{2})\Gamma(2\lambda+k)}{k!\,\Gamma(2\lambda+2k)\Gamma(\beta+2\lambda+k+1)} \\ &\qquad \times \frac{\Gamma(\beta+1)\Gamma(\lambda+k)}{\Gamma(\beta-k+1)\Gamma(\lambda)} \end{aligned}$$

ii) If $\beta \in I\!N$, $k \leq \beta$,

$$\begin{aligned} b_k^{\beta,\lambda} &:= \int_{-1}^{1} (1-t)^{\beta} \log(1-t)(1-t^2)^{\lambda-1/2} P_k^{\lambda}(t)\, dt \\ &= (-1)^k\, 2^{\beta+2\lambda+2k}\, \frac{\Gamma(\beta+\lambda+\frac{1}{2})\, \Gamma(\lambda+k+\frac{1}{2})\, \Gamma(2\,\lambda+k)}{k!\,\Gamma(2\,\lambda+2\,k)\, \Gamma(\beta+2\,\lambda+k+1)} \\ &\qquad \times \frac{\Gamma(\beta+1)\,\Gamma(\lambda+k)}{\Gamma(\beta-k+1)\Gamma(\lambda)} \\ &\qquad \times (\log 2 + \Psi(\beta+\lambda+1) + \Psi(\beta+1) - \Psi(\beta+2\lambda+k+1) \\ &\qquad\qquad -\psi(\beta-k+1)), \end{aligned}$$

where $\Psi = \Gamma'/\Gamma$ is the digamma function; see [1, 6.3].
If $k > \beta$,

$$\begin{aligned} b_k^{\beta,\lambda} &= (-1)^{\beta+1}\, 2^{\beta+2\lambda+2k}\, \frac{\Gamma(\beta+\lambda+\frac{1}{2})\, \Gamma(\lambda+k+\frac{1}{2})}{k!\,\Gamma(2\,\lambda+2\,k)} \\ &\qquad \times \frac{\Gamma(2\,\lambda+k)\Gamma(\beta+1)\,\Gamma(\lambda+k)(k-\beta)!}{\Gamma(\beta+2\,\lambda+k+1)\,\Gamma(\lambda)}. \end{aligned}$$

The Γ function is positive for positive argument. If $x \in (-2l+1, -2l+2)$ for some $l \in \mathbb{N}$ then $\Gamma(x) < 0$ whilst if $x \in (-2l, -2l+1)$ for some $l \in \mathbb{N}$ then $\Gamma(x) > 0$. By examining the factor $(-1)^k\Gamma(\beta-k+1)$ in i) above we see that, for $l \in \mathbb{N}$,

$$\begin{aligned} a_k^{\beta,\lambda} < 0, \quad k > \beta, \quad \beta \in (2(l-1), 2l-1), \\ a_k^{\beta,\lambda} > 0, \quad k > \beta, \quad \beta \in (2l-1, 2l). \end{aligned}$$

Similarly,

$$\begin{aligned} b_k^{\beta,\lambda} < 0, \quad k > \beta, \quad \beta \text{ even}, \\ b_k^{\beta,\lambda} > 0, \quad k > \beta, \quad \beta \text{ odd}. \end{aligned}$$

Thus, $(-1)^{[\beta]+1}\psi$ is strictly conditionally positive definite of order $[\beta]+1$ on the sphere, which, in conjunction with the remarks at the start of this section, confirms the conclusions of Proposition 1.

Using the asymptotic estimate

$$\Gamma(az+b) \sim \sqrt{2\pi}e^{-az}(az)^{az+b+1/2}$$

(see [1, 6.1.39]) we see that

$$\begin{aligned} a_k^{\beta,\lambda} & \\ \sim\ & 2^{2k}\frac{e^{-k}k^{\lambda+k}e^{-k}k^{2\lambda+k-1/2}e^{-k}k^{\lambda+k-1/2}}{e^{-k}k^{k+1/2}e^{-2k}(2k)^{2\lambda+2k-1/2}e^{-k}k^{\beta+2\lambda+k+1/2}e^{k}k^{\beta-k+1/2}} \\ \sim\ & k^{-2(\beta+1)}. \end{aligned}$$

Using the same asymptotic expression we see that

$$b_k^{\beta,\lambda} \quad \sim \quad k^{-2(\beta+1)}.$$

Hence, in either case, the Hilbert space of functions we approximate is

$$H_\psi = \left\{ f : \sum_{k=m}^{\infty} k^{2(\beta+1)} \sum_{j=1}^{d_k} (a_k^j(f))^2 < \infty \right\}.$$

It is known (see e.g. Schaback [7]) that, for $\phi(t)=t^{2\beta}$, $\beta>0$, or $\phi(t)=t^{2\beta}\log t$ with $\beta\in\mathbb{N}$, and $m=[\beta]+1$,

$$\inf_{q\in\Pi^1_{m-1}} \max_{t\in[0,h]} |\phi(t)-q(t)| \le Ch^\beta,$$

for some constant C independent of h.

Combining this with Theorem 1 we have the following result for polyharmonic spline interpolation on the sphere:

Theorem 2 *Let $\phi(t)=t^{2\beta}$, $\beta>0$, or $\phi(t)=t^{2\beta}\log t$ for $\beta\in\mathbb{N}$. Let $m=[\beta]+1$. For each $h\in(0,1)$, let $X_h=\{x_1,\ldots,x_{N(h)}\}\subset S^d$ be such that*

$$\max_{x\in S^d}\min_{y\in X_h} \cos^{-1}(xy) < h.$$

Given $f\in H_\psi$ let s_f be the interpolant to f as given by (2) and the following interpolation conditions. Then the are numbers C and $h_0>0$ such that

$$\|f-s_f\|_{\infty,S^d} \quad \le \quad Ch^\beta\|f\|_\psi.$$

Remarks 1 *It is reassuring that we recover the same order of convergence as for interpolation using, for example thin plate splines ($\phi(t) = t^2 \log t$) in the plane, where we expect to get $\mathcal{O}(h)$ convergence for h-separated data.*

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