A note on local polynomial approximation on compact manifolds

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Abstract

In [1] we give a local error estimate for approximation of smooth functions by spherical harmonics. For a k–times differentiable function f we show that there is a spherical polynomial p such that $|f(x) - p(x)| \le C\rho^k$, for $x \in D_\rho$, where D_ρ is a disk of radius ρ on the sphere. We generalise this result to all compact manifolds in a very elementary way, retaining the essence of the argument used on the sphere.

Discussion and Result

Let M be a compact d–dimensional C^∞-manifold embedded in $\mathbb{R}^{d+k}$ for some $k > 0$. Let the distance, $d(x,y)$, between $x, y \in M$ be measured as the minimum arc length from x to y. Fix $x \in M$ and, for $\rho > 0$, let $D_\rho(x) = \{y \in M : d(x,y) \le \rho\}$. Let $T_x = \mathbb{R}^d$ be the tangent space to M at x considered as a d-dimensional hyperplane in $\mathbb{R}^{d+k}$. Let $N_\rho = \{y \in \mathbb{R}^d : \|y\| \le \rho\}$. Since M is an embedded C^∞ manifold the orthogonal projection, π_x, from $\mathbb{R}^{d+k}$ to T_x has a derivative at x which is the identity map when restricted to T_x. Hence by the inverse function theorem there exists $\rho_0 > 0$ and a C^∞ map $\phi : N_{\rho_0} \to M$ which is the inverse of the orthogonal projection restricted to a closed neighborhood of x in M. Since $\phi : N_{\rho_0} \subset \mathbb{R}^d \to M$ is an invertible C^∞ map on a compact neighborhood, whose inverse is orthogonal projection restricted to M, there exists a constant C_1 such that

$$\begin{aligned} C_1 d(\phi(y),\phi(z)) \le \|y - z\| &\le ||\phi(y) - \phi(z)|| \\ &\le d(\phi(y),\phi(z)), \quad y, z \in N_{\rho_0}. \quad (1) \end{aligned}$$

The polynomials of degree n on M, denoted P_n, are the restrictions of the polynomials of degree n in $\mathbb{R}^{d+k}$ to M.

Fix $f \in C^k(M)$. By (1) for $\rho \leq \rho_0$, $f \circ \phi \in C^k(\phi^{-1}D_\rho)$. We can now perform a multivariate Taylor expansion,

$$f \circ \phi(y) = \sum_{|\alpha|<k} c_\alpha y^\alpha + R_k(f,y), \tag{2}$$

where the remainder

$$|R_k(f,y)| \leq C_k(f) \max_{t \in D_\rho}(||\phi^{-1}(t)||)^k. \tag{3}$$

Let $t = \phi(y)$ and define $p_{k-1} \in P_{k-1}$ by

$$p_{k-1}(t) \quad = \quad \sum_{|\alpha|<k} c_\alpha y^\alpha,$$

a polynomial since ϕ is the inverse of orthogonal projection. Then we see, using Equations (2) and (3) , that, for $\rho \leq \rho_0$,

$$\max_{y \in D_\rho} |f(t) - p_{k-1}(t)| \leq C_k(f) \max_{t \in D_\rho}(||\phi^{-1}(t)||)^k.$$

Since

$$\max_{t \in D_\rho} ||\phi^{-1}(t)|| \leq \rho,$$

we arrive, using the compactness of the manifold, at the following result:

Theorem 1 *Let $f \in C^k(M)$. Then there exists a positive ρ_0 and a constant $C_k(f)$ such that for each $x \in M$ there exists a degree $k-1$ polynomial p_{k-1} such that, for every $0 < \rho \leq \rho_0$,*

$$\max_{t \in D_\eta} |f(t) - p_{k-1}(t)| \leq C_k(f)\rho^k,$$

where the constant $C_k(f)$ does not depend on ρ.

Remarks 1 *Such a local polynomial approximation theorem is often referred to as a Whitney–type theorem.*

References

[1] J. Levesley and D.L. Ragozin, Local approximation on manifolds using radial functions and polynomials, in *Curve and Surface Fitting: Saint Malo 1999*, A. Cohen, C. Rabut and L. Schumaker(eds.), Vanderbilt University Press, Nashville, 2000, 291–300.