Regions of linearity, Lusztig cones and canonical basis elements for the quantized enveloping algebra of type A_4 [1]

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To Professor Helmut Wielandt on his 90th birthday.

Abstract

Let U_q be the quantum group associated to a Lie algebra $\mathbf{g}$ of rank n. The negative part U^- of U has a canonical basis $\mathbf{B}$ with favourable properties (see Kashiwara [6] and Lusztig [11, §14.4.6]). The approaches of Lusztig and Kashiwara lead to a set of alternative parametrizations of the canonical basis, one for each reduced expression for the longest word in the Weyl group of $\mathbf{g}$. We show that if $\mathbf{g}$ is of type A_4 there are close relationships between the Lusztig cones, canonical basis elements and the regions of linearity of reparametrization functions arising from the above parametrizations. A graph can be defined on the set of simplicial regions of linearity with respect to adjacency, and we further show that this graph is isomorphic to the graph with vertices given by the reduced expressions of the longest word of the Weyl group modulo commutation and edges given by long braid relations.

Keywords: Quantum group, Lie algebra, Canonical basis, Tight monomials, Weyl group, Piecewise-linear functions.

1 Introduction

Let $U = U_q(\mathbf{g})$ be the quantum group associated to a semisimple Lie algebra $\mathbf{g}$ of rank n. The negative part U^- of U has a canonical basis $\mathbf{B}$ with favourable properties (see Kashiwara [6] and Lusztig [11, §14.4.6]). For example, via action on highest weight vectors it gives rise to bases for all the finite-dimensional irreducible highest weight U-modules.

Let W be the Weyl group of $\mathbf{g}$, with Coxeter generators $s_1, s_2, \ldots, s_n$, and let w_0 be the element of maximal length in W. Let $\mathbf{i}$ be a reduced expression for w_0, i.e. $w_0 = s_{i_1}s_{i_2}\cdots s_{i_k}$ is reduced. Lusztig obtains a parametrization of the canonical basis $\mathbf{B}$ for each such reduced expression $\mathbf{i}$, via a correspondence between a basis of PBW-type associated to $\mathbf{i}$ and the canonical basis. This gives a bijection

$$\phi_{\mathbf{i}} : \mathbf{B} \to \mathbb{N}^k,$$

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where $\mathbb{N} = \{0, 1, 2, \dots\}$.

Kashiwara, in his approach to the canonical basis (which he calls the global crystal basis), defines certain root operators $\widetilde{F}_i$ on the canonical basis (see [6, §3.5]) which lead to a parametrization of the canonical basis for each reduced expression $\mathbf{i}$ by a certain subset $Y_{\mathbf{i}}$ of $\mathbb{N}^k$. This gives a bijection $\psi_{\mathbf{i}} : \mathbf{B} \to Y_{\mathbf{i}}$. The subset $Y_{\mathbf{i}}$ is called the string cone.

In Lusztig's theory, an important role is played by two specific reduced expressions $\mathbf{j}$ and $\mathbf{j}'$, which in type A_4 with Dynkin diagram as in Figure 1 are $\mathbf{j} = (1, 3, 2, 4, 1, 3, 2, 4, 1, 3)$ and $\mathbf{j}' = (2, 4, 1, 3, 2, 4, 1, 3, 2, 4)$.

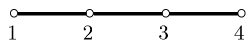


Figure 1: Dynkin diagram of type A_4

The function $R^{\mathbf{j}'}_{\mathbf{j}} = \phi_{\mathbf{j}'}\phi_{\mathbf{j}}^{-1} : \mathbb{N}^k \to \mathbb{N}^k$ was shown by Lusztig to be piecewise-linear, and the regions of linearity of this function were shown to be relevant to understanding the behaviour of the canonical basis.

The function $S^{\mathbf{j}}_{\mathbf{i}} = \phi_{\mathbf{j}}\psi_{\mathbf{i}}^{-1} : Y_{\mathbf{i}} \to \mathbb{N}^k$ is useful in relating Kashiwara's and Lusztig's parametrizations of $\mathbf{B}$. These re-parametrization functions have recently been studied using an approach involving totally positive varieties, in the preprint [4] of Berenstein and Zelevinsky (although not in terms of regions of linearity).

There is no simple way to express the elements of $\mathbf{B}$ in terms of the natural generators $F_1, F_2, \dots, F_n$ of U^-. This has been done only in types A_1, A_2, A_3 and B_2 (see [8, §3.4], [12, §13], [20] and [22]) and appears to become arbitrarily complicated in general. However it seems that there is an interesting subset of $\mathbf{B}$ whose elements are expressible as monomials in $F_1, F_2, \dots, F_n$.

A monomial $F_{i_1}^{(a_1)} F_{i_2}^{(a_2)} \cdots F_{i_k}^{(a_k)}$, where $F_i^{(a)} = F_i^a/[a]!$, is said to be tight if it belongs to $\mathbf{B}$. Lusztig [12] described a method which in low rank cases leads to the construction of tight monomials. He defined, for each reduced expression $\mathbf{i}$ of w_0, a certain cone $C_{\mathbf{i}}$ in $\mathbb{N}^k$ which we shall call the Lusztig cone, and showed that, for types A_1, A_2, A_3, the above monomial is tight for all $\mathbf{a} \in C_{\mathbf{i}}$. In [14] the second author showed that this is also true in type A_4. Let

$$M_{\mathbf{i}} = \{F_{i_1}^{(a_1)} F_{i_2}^{(a_2)} \cdots F_{i_k}^{(a_k)} \ : \ \mathbf{a} \in C_{\mathbf{i}}\}$$

be the set of monomials obtained from elements of $C_{\mathbf{i}}$.

There appears to be an intriguing relationship between the Lusztig cones, the regions of linearity of Lusztig's function $R^{\mathbf{j}'}_{\mathbf{j}}$, and the tight monomials in the canonical basis. This relationship is not fully understood in general, but the aim of this paper is to describe this relationship in the type A_4 case. Types A_1, A_2, A_3 were considered by Lusztig, but the A_4 case is considerably more complicated, while at the same time being amenable to explicit calculation.

In type A_4 we can show the following. For each $\mathbf{a} \in C_{\mathbf{i}}$ we have

$$\widetilde{F}_{i_1}^{a_1} \widetilde{F}_{i_2}^{a_2} \cdots \widetilde{F}_{i_k}^{a_k} \cdot 1 \equiv F_{i_1}^{(a_1)} F_{i_2}^{(a_2)} \cdots F_{i_k}^{(a_k)} \mod v\mathcal{L}',$$

where $\mathcal{L}'$ is the $\mathcal{A}$-lattice spanned by $\mathbf{B}$ and $\mathcal{A}$ is the subring of $\mathbb{Q}(v)$ of functions regular at $v = 0$. Since it has been shown (independently by the second author [15] and Premat [17]) that $C_{\mathbf{i}} \subseteq Y_{\mathbf{i}}$, it follows that $M_{\mathbf{i}} \subseteq \mathbf{B}$ and that, under the Kashiwara parametrization of $\mathbf{B}$, we have

$$\psi_{\mathbf{i}}(M_{\mathbf{i}}) = C_{\mathbf{i}}.$$

We can also show that, under Lusztig's parametrization of $\mathbf{B}$, we have

$$\phi_{\mathbf{j}}(M_{\mathbf{i}}) = X_{\mathbf{i}},$$

where $X_{\mathbf{i}}$ is one of the regions of linearity of the piecewise-linear function $R_{\mathbf{j}}^{\mathbf{j}'}$ and that the transition function

$$S_{\mathbf{i}}^{\mathbf{j}} : C_{\mathbf{i}} \to X_{\mathbf{i}}$$

is linear for all $\mathbf{i}$.

The set of all regions of linearity of $R_{\mathbf{j}}^{\mathbf{j}'}$ in type A_4 was determined by the first author using the ideas outlined in [5]. These regions are described in the present paper. Each such region is defined by a certain set of inequalities. It turns out that the regions $X_{\mathbf{i}}$ are all defined by the minimal possible number of inequalities, and that they give the set of all regions defined by this minimal number of inequalities. We call these the simplicial regions of $R_{\mathbf{j}}^{\mathbf{j}'}$. We thus have a parametrization of the simplicial regions in terms of the reduced expressions $\mathbf{i}$ of w_0 modulo commutation. A graph can be defined on the set of simplicial regions with respect to adjacency, and we further show that this graph is isomorphic to the graph with vertices given by the reduced expressions $\mathbf{i}$ of w_0 modulo commutation and edges given by long braid relations.

Thus in type A_4 there are close relationships between the Lusztig cones $C_{\mathbf{i}}$, the tight monomials $M_{\mathbf{i}}$, and the regions of linearity $X_{\mathbf{i}}$. Examples of N. H. Xi [21] and M. Reineke [18] show, however, that these relationships cannot be expected to hold in the same way in type A_n for arbitrary n.

2 Parametrizations of the canonical basis

Let $\mathbf{g}$ be the simple Lie algebra over $\mathbb{C}$ of type A_n and U be the quantized enveloping algebra of $\mathbf{g}$. Then U is a $\mathbb{Q}(v)$-algebra generated by the elements E_i, F_i, K_μ, $i \in \{1, 2, \ldots, n\}$, $\mu \in Q$, the root lattice of $\mathbf{g}$. Let U^+ be the subalgebra generated by the E_i and U^- the subalgebra generated by the F_i.

Let W be the Weyl group of $\mathbf{g}$. It has a unique element w_0 of maximal length. For each reduced expression $\mathbf{i}$ for w_0 there are two parametrizations of the canonical basis $\mathbf{B}$ for U^-. The first arises from Lusztig's approach to the canonical basis [11, §14.4.6], and the second arises from Kashiwara's approach [6].

Lusztig's Approach
There is an $\mathbb{Q}$-algebra automorphism of U which takes each E_i to F_i, F_i to E_i, K_μ to $K_{-\mu}$ and v to v^{-1}. We use this automorphism to transfer Lusztig's definition of the canonical basis in [8, §3] to U^-.

Let T_i, $i = 1, 2, \ldots, n$, be the automorphism of U as in [10, §1.3] given by:

$$T_i(E_j) = \begin{cases} -F_jK_j, & \text{if } i = j, \\ E_j, & \text{if } |i-j| > 1 \\ -E_iE_j + v^{-1}E_jE_i & \text{if } |i-j| = 1 \end{cases}$$

$$T_i(F_j) = \begin{cases} -K_j^{-1}E_j, & \text{if } i = j, \\ F_j, & \text{if } |i-j| > 1 \\ -F_jF_i + vF_iF_j & \text{if } |i-j| = 1 \end{cases}$$

$$T_i(K_\mu) = K_{\mu - \langle \mu, \alpha_i \rangle h_i}, \text{ for } \mu \in Q,$$

where α_i are the simple roots and h_i the simple coroots of $\mathbf{g}$.

For each i, let r_i be the automorphism of U which fixes E_j and F_j for $j = i$ or $|i-j| > 1$ and fixes K_μ for all μ, and which takes E_j to $-E_j$ and F_j to $-F_j$ if $|i-j| = 1$. Let $T''_{i,-1} = T_i r_i$ be the automorphism of U as in [11, §37.1.3]. Let $\mathbf{c} \in \mathbb{N}^k$, where $k = \ell(w_0)$, and $\mathbf{i}$ be a reduced expression for w_0. Let

$$F_{\mathbf{i}}^{\mathbf{c}} := F_{i_1}^{(c_1)} T''_{i_1,-1}(F_{i_2}^{(c_2)}) \cdots T''_{i_1,-1} T''_{i_2,-1} \cdots T''_{i_{k-1},-1}(F_{i_k}^{(c_k)}).$$

Define $B_{\mathbf{i}} = \{F_{\mathbf{i}}^{\mathbf{c}} : \mathbf{c} \in \mathbb{N}^k\}$. Then $B_{\mathbf{i}}$ is the basis of PBW-type corresponding to the reduced expression $\mathbf{i}$. Let $^-$ be the $\mathbb{Q}$-algebra automorphism from U to U taking E_i to E_i, F_i to F_i, and K_μ to $K_{-\mu}$, for each $i \in [1, n]$ and $\mu \in Q$, and v to v^{-1}. Lusztig proves the following result in [8, §§2.3, 3.2].

Theorem 2.1 *(Lusztig)*
The $Z[v]$-span $\mathcal{L}$ of $B_{\mathbf{i}}$ is independent of $\mathbf{i}$. Let $\pi : \mathcal{L} \to \mathcal{L}/v\mathcal{L}$ be the natural projection. The image $\pi(B_{\mathbf{i}})$ is also independent of $\mathbf{i}$; we denote it by B. The restriction of π to $\mathcal{L} \cap \overline{\mathcal{L}}$ is an isomorphism of $\mathbb{Z}$-modules $\pi_1 : \mathcal{L} \cap \overline{\mathcal{L}} \to \mathcal{L}/v\mathcal{L}$. Also $\mathbf{B} = \pi_1^{-1}(B)$ is a $\mathbb{Q}(v)$-basis of U^-, which is the canonical basis of U^-.

Lusztig's theorem provides us with a parametrization of $\mathbf{B}$, dependent on $\mathbf{i}$. If $b \in \mathbf{B}$, we write $\phi_{\mathbf{i}}(b) = \mathbf{c}$, where $\mathbf{c} \in \mathbb{N}^k$ satisfies $b \equiv F_{\mathbf{i}}^{\mathbf{c}} \mod v\mathcal{L}$. Note that $\phi_{\mathbf{i}}$ is a bijection.

Lusztig considers in [8] two particular reduced expressions for w_0. Let the nodes in the Dynkin diagram of A_n be labelled as in Figure 2.

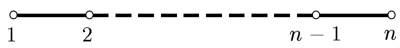


Figure 2: Dynkin diagram of type A_n

Let $\mathbf{j} = 135\cdots246\cdots135\cdots246\cdots$, and let $\mathbf{j}' = 246\cdots135\cdots246\cdots135\cdots$, where both expressions have length $k = \ell(w_0)$ — they are both reduced expressions for w_0. We have bijections $\phi_{\mathbf{j}} : \mathbf{B} \to \mathbb{N}^k$ and $\phi_{\mathbf{j}'} : \mathbf{B} \to \mathbb{N}^k$.

Lusztig defines in [8, §2.6] a function $R_{\mathbf{j}}^{\mathbf{j}'} = \phi_{\mathbf{j}'}\phi_{\mathbf{j}}^{-1} : \mathbb{N}^k \to \mathbb{N}^k$. This function was shown by Lusztig to be piecewise linear and its regions of linearity were shown to have significance for the canonical basis, in the sense that elements b of the canonical basis with $\phi_{\mathbf{j}}(b)$ in the same region of linearity of $R_{\mathbf{j}}^{\mathbf{j}'}$ often have similar form.

Kashiwara's approach

Let $\widetilde{E}_i$ and $\widetilde{F}_i$ be the Kashiwara operators on U^- as defined in [6, §3.5]. Let $\mathcal{A} \subseteq \mathbb{Q}(v)$ be the subring of elements regular at $v = 0$, and let $\mathcal{L}'$ be the $\mathcal{A}$-lattice spanned by arbitrary products $\widetilde{F}_{j_1}\widetilde{F}_{j_2}\cdots\widetilde{F}_{j_m}\cdot 1$ in U^-. We denote the set of all such elements by S. The following results were proved by Kashiwara in [6].

Theorem 2.2 *(Kashiwara)*
(i) Let $\pi' : \mathcal{L}' \to \mathcal{L}'/v\mathcal{L}'$ be the natural projection, and let $B' = \pi'(S)$. Then B' is a $\mathbb{Q}$-basis of $\mathcal{L}'/v\mathcal{L}'$ (the crystal basis).
(ii) Furthermore, $\widetilde{E}_i$ and $\widetilde{F}_i$ each preserve $\mathcal{L}'$ and thus act on $\mathcal{L}'/v\mathcal{L}'$. They satisfy $\widetilde{E}_i(B') \subseteq B' \cup \{0\}$ and $\widetilde{F}_i(B') \subseteq B'$. Also for $b, b' \in B'$ we have $\widetilde{F}_i b = b'$, if and only if $\widetilde{E}_i b' = b$.
(iii) For each $b \in B'$, there is a unique element $\widetilde{b} \in \mathcal{L}' \cap \overline{\mathcal{L}'}$ such that $\pi'(\widetilde{b}) = b$. The set of elements $\{\widetilde{b} : b \in B'\}$ forms a basis of U^-, the global crystal basis *of U^-.*

It was shown by Lusztig [9, 2.3] that the global crystal basis of Kashiwara coincides with the canonical basis of U^-.

There is a parametrization of $\mathbf{B}$ arising from Kashiwara's approach, again dependent on a reduced expression $\mathbf{i}$ for w_0. Let $\mathbf{i} = (i_1, i_2, \ldots, i_k)$ and $b \in B$. Let a_1 be maximal such that $\widetilde{E}_{i_1}^{a_1} b \not\equiv 0 \mod v\mathcal{L}'$; let a_2 be maximal such that $\widetilde{E}_{i_2}^{a_2}\widetilde{E}_{i_1}^{a_1} b \not\equiv 0 \mod v\mathcal{L}'$, and so on, so that a_k is maximal such that $\widetilde{E}_{i_k}^{a_k}\widetilde{E}_{i_{k-1}}^{a_{k-1}}\cdots\widetilde{E}_{i_2}^{a_2}\widetilde{E}_{i_1}^{a_1} b \not\equiv 0 \mod v\mathcal{L}'$. Let $\mathbf{a} = (a_1, a_2, \ldots, a_k)$. We write $\psi_{\mathbf{i}}(b) = \mathbf{a}$. This is the crystal string of b — see [3, §2] and the end of Section 2 in [16]; see also [7]. It is known that $\psi_{\mathbf{i}}(b)$ uniquely determines $b \in \mathbf{B}$ (see [16, §2.5]). We have $b \equiv \widetilde{F}_{i_1}^{a_1}\widetilde{F}_{i_2}^{a_2}\cdots\widetilde{F}_{i_k}^{a_k}\cdot 1 \mod v\mathcal{L}'$. The image of $\psi_{\mathbf{i}}$ is a cone which first appears in [3]. We shall call this the *string cone* $Y_{\mathbf{i}} = \psi_{\mathbf{i}}(\mathbf{B})$.

We next define a function which compares Kashiwara's approach with Lusztig's approach. Consider the maps

$$Y_{\mathbf{i}} \xrightarrow[\psi_{\mathbf{i}}^{-1}]{} B \xrightarrow[\phi_{\mathbf{j}}]{} \mathbb{N}^k$$

where $\mathbf{j} = 135\cdots246\cdots135\cdots246\cdots$. We define $S_{\mathbf{i}}^{\mathbf{j}} = \phi_{\mathbf{j}}\psi_{\mathbf{i}}^{-1} : Y_{\mathbf{i}} \to \mathbb{N}^k$, a reparametrization function. We shall show that this function has some interesting properties in the case in which $\mathbf{g}$ has type A_4.

3 The Lusztig cones and their spanning vectors

Lusztig [12] introduced certain regions which, in low rank, give rise to canonical basis elements of a particularly simple form. We consider reduced expressions $\mathbf{i} = (i_1, i_2, \dots, i_k)$ for w_0. We shall identify this k-tuple with the reduced expression $s_{i_1} s_{i_2} \cdots s_{i_k}$. Given two such reduced expressions we say that $\mathbf{i} \sim \mathbf{i}'$ if there is a sequence of commutations (of the form $s_i s_j = s_j s_i$ with $|i-j| > 1$) which, when applied to $\mathbf{i}$, give $\mathbf{i}'$. This is an equivalence relation on the set of reduced expressions for w_0, and the equivalence classes are called commutation classes.

The *Lusztig cone*, $C_{\mathbf{i}}$, corresponding to a reduced expression $\mathbf{i}$ for w_0 is defined to be the set of points $\mathbf{a} \in \mathbb{N}^k$ satisfying the following inequalities:
(*) For every pair $s, s' \in [1, k]$ with $s < s'$, $i_s = i_{s'} = i$ and $i_p \neq i$ whenever $s < p < s'$, we have

$$(\sum_p a_p) - a_s - a_{s'} \geq 0,$$

where the sum is over all p with $s < p < s'$ such that i_p is joined to i in the Dynkin diagram.

It was shown by Lusztig [12] that if $\mathbf{a} \in C_{\mathbf{i}}$ then the monomial $F_{i_1}^{(a_1)} F_{i_2}^{(a_2)} \cdots F_{i_k}^{(a_k)}$ lies in the canonical basis $\mathbf{B}$, provided $n = 1, 2, 3$. The second author [14] showed that this remains true if $n = 4$. The Lusztig cones have been studied in the second author's preprints [13] and [15] in type A for every reduced expression $\mathbf{i}$ for the longest word, and have also been studied by Bedard in [1] for arbitrary finite (simply-laced) type for reduced expressions compatible with a quiver whose underlying graph is the Dynkin diagram. Bedard describes these vectors using the Auslander-Reiten quiver of the quiver and homological algebra, showing they are closely connected to the representation theory of the quiver.

The reduced expression $\mathbf{i}$ defines an ordering on the set Φ^+ of positive roots of the root system associated to W. We write $\alpha^j = s_{i_1} s_{i_2} \cdots s_{i_{j-1}}(\alpha_{i_j})$ for $j = 1, 2, \dots, k$. Then $\Phi^+ = \{\alpha^1, \alpha^2, \dots \alpha^k\}$. For $\mathbf{a} = (a_1, a_2, \dots, a_k) \in \mathbb{Z}^k$, write $a_{\alpha^j} = a_j$. If $\alpha = \alpha_{ij} := \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}$ with $i < j$, we also write a_{ij} for $a_{\alpha_{ij}}$.

We can regard $C_{\mathbf{i}}$ as a subset of $\mathbb{Z}^k$ defined by the inequalities in (*) above, together with the n inequalities $a_{\alpha_j} \geq 0$ for $j = 1, 2, \dots, n$ (see the paragraph before Lemma 4.2 in [13]). The number of inequalities in (*) is $k - n$, thus we have k inequalities altogether defining $C_{\mathbf{i}}$. There is therefore a matrix $P_{\mathbf{i}} \in M_k(\mathbb{Z})$ such that

$$C_{\mathbf{i}} = \{\mathbf{a} \in \mathbb{Z}^k \,:\, P_{\mathbf{i}}\mathbf{a} \geq 0\},$$

where, for $\mathbf{z} \in \mathbb{Z}^k$, $\mathbf{z} \geq 0$ means that each entry in $\mathbf{z}$ is nonnegative.

There is a description of $C_{\mathbf{i}}$ in [13] which will be useful to us. We shall need the chamber diagram (chamber ansatz) for $\mathbf{i}$ defined in [2, §§1.4, 2.3]. We take $n+1$ strings, numbered from top to bottom, and write $\mathbf{i}$ from left to right along the bottom of the diagram. Above a letter i_j in $\mathbf{i}$, the i_jth and (i_j+1)st strings from the top above i_j cross. Thus, for example, in the case $n = 3$ with $\mathbf{i} = (1, 3, 2, 1, 3, 2)$, the chamber diagram is shown in Figure 3.

We denote the chamber diagram of $\mathbf{i}$ by $\mathrm{CD}(\mathbf{i})$. We shall be concerned with the bounded chambers of the diagram. A *chamber* will be defined as a pair $(c, \mathbf{i})$, where c is a bounded component of the complement of $\mathrm{CD}(\mathbf{i})$. Each chamber $(c, \mathbf{i})$ can be labelled with the numbers of the strings passing

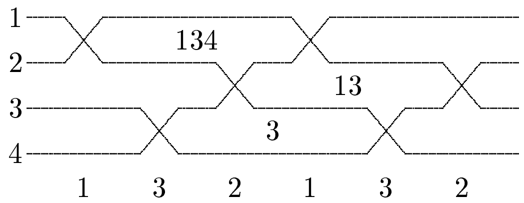


Figure 3: Chamber Diagram

below it, denoted $l(c, \mathbf{i})$. Following [2], we call such a label a chamber set. For example, the chamber sets corresponding to the 3 bounded chambers in Figure 3 are $134, 3, 13$.

There is another way to think of the set of bounded chambers in a chamber diagram. We recall that a quiver of type A_n is a directed graph such that the underlying undirected graph is the Dynkin diagram of type A_n. Following [13, §5.1], we define a *partial quiver* P of type A_n to be a quiver of type A_n which has some (or none) of its arrows replaced by undirected edges in such a way that the subgraph obtained by deleting undirected edges and vertices incident only with undirected edges is non-empty and connected. We shall now number the edges of a partial quiver of type A_n from 2 to n, starting from the right, as in Figure 4.

Figure 4: Edge Numbering of the Dynkin Diagram.

If P is a partial quiver of type A_n, we denote by $l(P)$ the subset of $[1, n+1]$ defined as follows: Let $l_1(P) = \{j \in [2, n] \, : \, \text{edge } j \text{ of } P \text{ is an } L\}$ (this means that edge j has an arrow pointing to the left). If the rightmost directed edge of P is an R, and this is in position i, then let $l_2(P) = [1, i-1]$. Otherwise $l_2(P)$ is the empty set. If the leftmost directed edge of P is an R, and this is edge j, let $l_3(P) = [j+1, n+1]$. Otherwise $l_3(P)$ is the empty set. We then define $l(P) = l_1(P) \cup l_2(P) \cup l_3(P)$.

It is shown in [13, §5.4] that the map l is a bijection from the set of partial quivers of type A_n to the set of all chamber sets. For example, the chamber sets associated to the 8 partial quivers of type A_3 are shown in Figure 5.

We now consider spanning vectors for the Lusztig cone $C_{\mathbf{i}}$. The $k \times k$ matrix $P_{\mathbf{i}}$ has an inverse $Q_{\mathbf{i}}$ with entries in $\mathbb{N}$ (see [13, §4.2]). We have $k = \frac{1}{2}n(n+1)$. Note that $k - n = \frac{1}{2}n(n-1)$ of the rows of $P_{\mathbf{i}}$ correspond to inequalities arising from consecutive occurrences of a letter in $\mathbf{i}$. Each such consecutive pair corresponds naturally to a bounded chamber $(c, \mathbf{i})$. Thus for each chamber $(c, \mathbf{i})$, there is a corresponding row of $P_{\mathbf{i}}$ and therefore a corresponding column of $Q_{\mathbf{i}}$. Now the columns of $Q_{\mathbf{i}}$ give spanning vectors of $C_{\mathbf{i}}$ in the sense that the elements of $C_{\mathbf{i}}$ are nonnegative linear combinations of such spanning vectors. We denote this spanning vector by $\mathbf{a}(c, \mathbf{i})$. The remaining n rows of $P_{\mathbf{i}}$ correspond to inequalities $a_{\alpha_j} \geq 0$ for α_j a simple root. We denote the corresponding spanning vectors by $\mathbf{a}(j, \mathbf{i})$. Thus the k spanning vectors of $C_{\mathbf{i}}$ are:

$$\begin{array}{ll} \mathbf{a}(c, \mathbf{i}) & (k - n \text{ vectors}), \\ \mathbf{a}(j, \mathbf{i}) & (n \text{ vectors}). \end{array}$$

Partial Quiver	Chamber Set
$\circ \xleftarrow{L} \circ \xleftarrow{L} \circ$	$\{2,3\}$
$\circ \xleftarrow{L} \circ \xrightarrow{R} \circ$	$\{1,3\}$
$\circ \xrightarrow{R} \circ \xleftarrow{L} \circ$	$\{2,4\}$
$\circ \xrightarrow{R} \circ \xrightarrow{R} \circ$	$\{1,4\}$
$\circ \xleftarrow{L} \circ \text{——} \circ$	$\{3\}$
$\circ \text{——} \circ \xleftarrow{L} \circ$	$\{2\}$
$\circ \xrightarrow{R} \circ \text{——} \circ$	$\{1,2,4\}$
$\circ \text{——} \circ \xrightarrow{R} \circ$	$\{1,3,4\}$

Figure 5: Partial Quivers of Type A_3

It is shown in [13] how to obtain the spanning vectors $\mathbf{a}(c,\mathbf{i})$, $\mathbf{a}(j,\mathbf{i})$. We recall that the coordinates of $\mathbf{a} = (a_1, a_2, \dots, a_k)$ correspond naturally to the positive roots $\alpha^1, \alpha^2, \dots, \alpha^k$, given the reduced expression $\mathbf{i}$ for w_0. It is possible to find the spanning vectors by attaching to each positive root α^m a multiplicity a_m, which will be the appropriate coordinate of the spanning vector $\mathbf{a}$. It is shown in [13] that the multiplicity function $\alpha^m \mapsto a_m$ for $\mathbf{a}(c,\mathbf{i})$ depends only upon the partial quiver P with $l(P) = l(c,\mathbf{i})$, and that the multiplicity function for $\mathbf{a}(j,\mathbf{i})$ depends only upon j. These multiplicity functions are constructed by the following algorithm.

For $j \in [1,n]$, the multiplicity of a positive root $\alpha_{pq} = \alpha_p + \alpha_{p+1} + \cdots + \alpha_{q-1}$ is 1 if $1 \leq p \leq j < j+1 \leq q \leq n+1$ and is 0 otherwise. This determines the spanning vector $\mathbf{a}(j,\mathbf{i})$, whose components are all either 0 or 1.

For a partial quiver P the multiplicity of a positive root α_{pq} is given as follows. We consider the components of P, i.e. the maximal connected subquivers all of whose arrows point in the same direction. For each component Y of P let $a(Y)$ be the number of the leftmost edge to the right of Y and let $b(Y)$ be the number of the rightmost edge to the left of Y. We have $a(Y) < b(Y)$. The component Y determines a set $\Phi^+(Y)$ of positive roots α_{pq} such that $1 \leq p \leq a(Y) < b(Y) \leq q \leq n+1$. Let m_{pq} be the number of components Y of P such that $\alpha_{pq} \in \Phi^+(Y)$. Then the multiplicity a_{pq} of α_{pq} is given by $a_{pq} = \lceil \frac{1}{2} m_{pq} \rceil$, which is the smallest integer m with $\frac{1}{2} m_{pq} \leq m$. The multiplicity a_{pq} is the coordinate of $\mathbf{a}(c,\mathbf{i})$ corresponding to the positive root α_{pq}.

4 Transforms of the Lusztig cones and their spanning vectors

We now bring into play the parametrization of the canonical basis $\mathbf{B}$ arising from Kashiwara's approach. We recall there is a bijection

$$\psi_{\mathbf{i}} : \mathbf{B} \to Y_{\mathbf{i}}$$

between the canonical basis and the string cone $Y_{\mathbf{i}}$. Now it has been shown by the second author [15] and independently by Premat [17] that $C_{\mathbf{i}} \subseteq Y_{\mathbf{i}}$, i.e. that the Lusztig cone lies in the string cone. Thus there is a corresponding subset $\psi_{\mathbf{i}}^{-1}(C_{\mathbf{i}})$ of the canonical basis $\mathbf{B}$.

We also have a bijection

$$\phi_{\mathbf{i}} : \mathbf{B} \to \mathbb{N}^k$$

and a transition function

$$S_{\mathbf{i}}^{\mathbf{j}} : Y_{\mathbf{i}} \to \mathbb{N}^k$$

given by $S_{\mathbf{i}}^{\mathbf{j}} = \phi_{\mathbf{j}}\psi_{\mathbf{i}}^{-1}$. We consider the question: what is the subset $S_{\mathbf{i}}^{\mathbf{j}}(C_{\mathbf{i}})$ of $\mathbb{N}^k$ corresponding to the Lusztig cone $C_{\mathbf{i}}$? We state the following conjecture:

Conjecture 4.1 *For any reduced expression* $\mathbf{i}$ *of* w_0*,* $S_{\mathbf{i}}^{\mathbf{j}}(C_{\mathbf{i}})$ *is a region of linearity of Lusztig's piecewise-linear function* $R = R_{\mathbf{j}}^{\mathbf{j}'}$*.*

We now describe an algorithm proved by the authors, as yet unpublished, which gives the transforms $S_{\mathbf{i}}^{\mathbf{j}}(\mathbf{a}(c,\mathbf{i}))$, $S_{\mathbf{i}}^{\mathbf{j}}(\mathbf{a}(j,\mathbf{i}))$ of the spanning vectors of $C_{\mathbf{i}}$.

We first define a (p,q,r,s)-rectangle. Suppose we begin with the array of numbers as in Figure 6.

$$\begin{array}{ccccccccccccc}
\cdot & \cdot & \cdot & 0 & 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & 1 & 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 2 & 2 & 2 & 2 & 2 & 2 & \cdot & \cdot & \cdot \\
 & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\
 & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & &
\end{array}$$

Figure 6: Array of numbers for defining a (p,q,r,s) rectangle.

Let $(p,q,r,s) \in \mathbb{N}^4$. A (p,q,r,s)-rectangle is a rectangle in this array, with top vertex on line p, middle vertices on lines q,r, and bottom vertex on line s and with all vertices lying half-way between two numbers. All lines of the rectangle are at an angle of $\pm\pi/4$ to the horizontal. This forces $p < q < s$, $p < r < s$ and $p+s = q+r$. We take only alternate columns of numbers in the rectangle, starting with the first column if q is odd, and with the second column if q is even. A $(0,3,5,8)$-rectangle is shown in Figure 7. Let P be a partial quiver. Let Y be a component of P and let $a = a(Y)$ and let $b = b(Y)$ be the integers defined above. Let $\rho(Y)$ be a $(0, a, n+2-b, n+a-b+2)$-rectangle if Y has type L, and a $(b-a-1, b-1, n+1-a, n+1)$-rectangle if Y is of type R.

Now let $j \in [1,n]$. Let $a = j$ and $b = j+1$. Let $\rho(j)$ be a $(0,a,n+2-b,n+a-b+2) = (0,j,n+1-j,n+1)$-rectangle (note that this is also a $(b-a-1, b-1, n+1-a, n+1)$-rectangle).

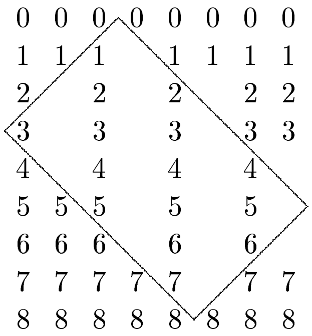


Figure 7: A $(0,3,5,8)$ rectangle.

We define the diagram $E(P)$ of P in the following way. We go through the components Y of P one by one, from left to right. It is possible to fit the rectangles $\rho(Y)$ together as follows. If a component of type L is followed by a component of type R, the corresponding rectangles share leftmost corners, and if a component of type R is followed by a component of type L, they share rightmost corners. In each case, it is easy to see that when the rectangles are superimposed, sharing a common leftmost or rightmost corner, the overlapping numbers agree. The resulting diagram is defined to be $E(P)$. It is convenient to label each rectangle by the corresponding component of P; on the left hand corner if Y is of type R, and on the right hand corner if Y is of type L.

We also define the diagram $E(j)$ for each $j \in [1,n]$. $E(j)$ is defined to be the rectangle $\rho(j)$.

Example
We consider the case when $n=6$ and $P = LRLL-$. Then P has three components, $L_1 = L----$, $R_1 = -R---$ and $L_2 = --LL-$. The rectangles $\rho(Y)$ are as in Figure 8. These rectangles fit

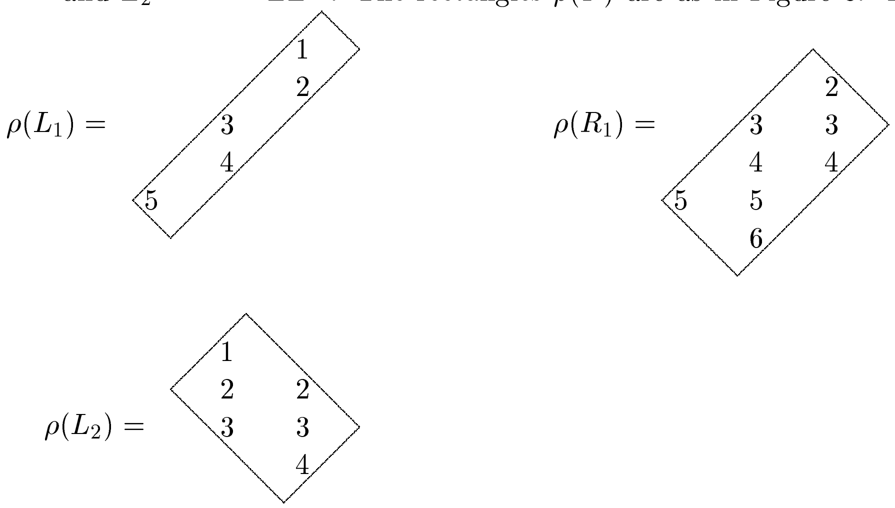


Figure 8: The rectangles $\rho(Y)$.

together to give the diagram $E(P)$. The left hand corners of $\rho(L_1), \rho(R_1)$ match, and so do the right hand corners of $\rho(R_1), \rho(L_2)$. The diagram $E(P)$ is shown in Figure 9. If P is a partial quiver, the sides

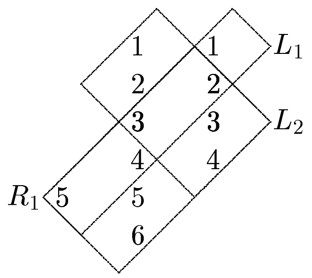


Figure 9: The diagram $E(P)$.

of the rectangles $\rho(Y)$ for Y a component of P divide $E(P)$ into diagonal rows of smaller rectangles called boxes. Consider the boxes in a diagonal running from top left to bottom right. The number of boxes in such a diagonal will be odd for the first p diagonals, starting from the top, for some p, and even for the remaining diagonals (or vice versa). Let G be the line in the diagram dividing the two adjacent diagonals containing an even and an odd number of boxes. Similarly, we can consider the boxes in a diagonal running from top right to bottom left. Again, it is possible to draw a line H separating the diagonals with an odd number of boxes from those with an even number. Let C be the point of intersection of the lines G, H. We call C the centre of the diagram $E(P)$. Let V be the vertical line through C. In our example, the picture is shown in Figure 10. Let X be an extremal left or right

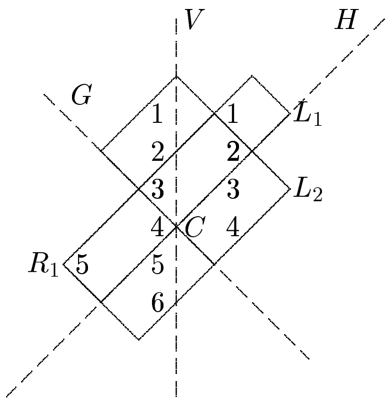


Figure 10: The complete diagram.

corner of $E(P)$. Let $R(X)$ be the rectangle which has the given corner as a vertex and whose edges through this point extend as far as possible in the figure (the vertex of the rectangle opposite to the given corner point may not be explicitly shown in the figure). The vertical line V divides $R(X)$ into two parts; let $\Phi^+(X)$ be the set of positive roots obtained from the part of $R(X)$ on the same side of V as X, by reading the numbers downwards in vertical lines. Thus $i, i+1, \ldots, j$ gives the positive root $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$. Let $\Phi^+(P)$ be the union of $\Phi^+(X)$ for all left and right corners X. Thus in the example in Figure 9 we have:

$$
\begin{aligned}
\Phi^+(L_1) &= \{\alpha_1 + \alpha_2\} \\
\Phi^+(L_2) &= \{\alpha_2 + \alpha_3 + \alpha_4\} \\
\Phi^+(R_1) &= \{\alpha_5, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6\} \\
\Phi^+(X_0) &= \{\alpha_1 + \alpha_2 + \alpha_3\}
\end{aligned}
$$

where X_0 is the only unlabelled left or right corner. Finally,

$$\Phi^+(P) = \{\alpha_1 + \alpha_2, \alpha_2 + \alpha_3 + \alpha_4, \alpha_5, \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2 + \alpha_3\}.$$

We also define a set $\Phi^+(j)$ for $j \in [1, n]$. $\Phi^+(j)$ is the set of positive roots obtained from the rectangle $E(j) = \rho(j)$ by reading the numbers downwards in vertical lines.

Theorem 4.2 *(Carter and Marsh). Suppose the coordinates of the vector $S_{\mathbf{i}}^{\mathbf{j}}(\mathbf{a}(c, \mathbf{i}))$ are labelled by the positive roots Φ^+ by means of the reduced expression $\mathbf{j}$ of w_0. Let P be the partial quiver $P(c, \mathbf{i})$. Then the coordinate of $S_{\mathbf{i}}^{\mathbf{j}}(\mathbf{a}(c, \mathbf{i}))$ labelled by $\alpha \in \Phi^+$ is 1 if $\alpha \in \Phi^+(P)$ and is 0 if $\alpha \notin \Phi^+(P)$. Also, the coordinate of $S_{\mathbf{i}}^{\mathbf{j}}(\mathbf{a}(j, \mathbf{i}))$ labelled by $\alpha \in \Phi^+$ is 1 if $\alpha \in \Phi^+(j)$ and is 0 if $\alpha \notin \Phi^+(j)$.*

The proof of this result depends on Theorem 5.17 in [15], together with the description of the reparametrization function associated with Lusztig's parametrization provided in [2]. It is hoped that this proof will appear in due course. In the present paper we shall illustrate it in type A_4.

5 Regions of linearity of the function R in type A_4

We shall now consider in detail the case when $\mathbf{g}$ has type A_4. The Dynkin diagram will be labelled as in Figure 1. In this case, $\mathbf{j}, \mathbf{j}'$ are the reduced words for w_0 given by

$$\begin{aligned} \mathbf{j} &= (1, 3, 2, 4, 1, 3, 2, 4, 1, 3) \\ \mathbf{j}' &= (2, 4, 1, 3, 2, 4, 1, 3, 2, 4) \end{aligned}$$

The piecewise-linear function $R = R_{\mathbf{j}}^{\mathbf{j}'}$ can be written as a composition of functions corresponding to a sequence of reduced words beginning with $\mathbf{j}$ and ending with $\mathbf{j}'$ such that consecutive words differ by a braid relation. When $s_i s_j$ is replaced by $s_j s_i$ the corresponding pair of components a, b is replaced by b, a. When $s_i s_j s_i$ is replaced by $s_j s_i s_j$ the corresponding triple of components a, b, c is replaced by a', b', c' where:

$$(a', b', c') = \begin{cases} (b + c - a, a, b) & \text{if } a \leq c \\ (b, c, b + a - c) & \text{if } a \geq c. \end{cases}$$

See [9, §2]. We list in Table I a sequence of reduced words of this kind from $\mathbf{j}$ to $\mathbf{j}'$, underlying the long braid relations which are used. These long braid relations are denoted by the letters $A, B, C, D, E, F, G, H, I, J$.

Table I

Sequence of reduced words for w_0.

1	3	2	4	1	3	2	4	1	3	
3	1	2	4	1	3	2	4	1	3	
3	$\underline{1}$	$\underline{2}$	$\underline{1}$	4	3	2	4	1	3	A
3	2	1	2	4	3	2	4	1	3	
3	2	1	4	$\underline{2}$	$\underline{3}$	$\underline{2}$	4	1	3	B
3	2	1	4	3	2	3	4	1	3	
3	2	1	4	3	2	$\underline{3}$	$\underline{4}$	$\underline{3}$	1	C
3	2	1	4	3	2	4	3	4	1	
3	2	1	$\underline{4}$	$\underline{3}$	$\underline{4}$	2	3	4	1	D
3	2	1	3	4	3	2	3	4	1	
$\underline{3}$	$\underline{2}$	$\underline{3}$	1	4	3	2	3	4	1	E
2	3	2	1	4	$\underline{3}$	$\underline{2}$	$\underline{3}$	4	1	F
2	3	2	1	4	2	3	2	4	1	
2	3	$\underline{2}$	$\underline{1}$	$\underline{2}$	4	3	2	4	1	G
2	3	1	2	1	4	3	2	4	1	
2	3	1	2	4	1	3	2	4	1	
2	3	1	2	4	3	1	2	4	1	
2	3	1	2	4	3	$\underline{1}$	$\underline{2}$	$\underline{1}$	4	H
2	3	1	2	4	3	2	1	2	4	
2	3	1	4	$\underline{2}$	$\underline{3}$	$\underline{2}$	1	2	4	I
2	3	1	4	3	2	3	1	2	4	
2	1	$\underline{3}$	$\underline{4}$	$\underline{3}$	2	3	1	2	4	J
2	1	4	3	4	2	3	1	2	4	
2	4	1	3	4	2	3	1	2	4	
2	4	1	3	2	4	3	1	2	4	
2	4	1	3	2	4	1	3	2	4	

Let $v \in \mathbb{R}^{10}$ and let $R(v)$ be the image of v under $R = R_{\mathsf{j}}^{\mathsf{j}'}$. Since

$$(a', b', c') = \begin{cases} (b, c, b) + (c - a, a - c, 0) & \text{if } a \leq c \\ (b, c, b) + (0, 0, a - c) & \text{if } a \geq c, \end{cases}$$

we consider the function $g(v)$ obtained from v by applying the map $(a, b, c) \mapsto (b, c, b)$ at each length 3 braid relation and $(a, b) \mapsto (b, a)$ at each length 2 braid relation. We write

$$R(v) = g(v) + \varepsilon(v).$$

In our case we write $v = (a, b, c, d, e, f, g, h, i, j)$. By following the sequence of steps in Table I we see that $g(v) = (c, h, e, j, g, h, i, j, g, h)$.

We also define vectors α_X for $X \in \{A, B, C, D, E, F, G, H, I, J\}$, by $\alpha_X(v) = \xi - \eta$ where ξ, η are the first and third components of the triple in the vector to which the braid relation at X is applied. These vectors α_X can be read off from Table II and are as follows:

$$\begin{aligned}
\alpha_A(v) &= a - e \\
\alpha_B(v) &= c - g \\
\alpha_C(v) &= f - j \\
\alpha_D(v) &= d - h \\
\alpha_E(v) &= b - f \\
\alpha_F(v) &= f - j \\
\alpha_G(v) &= c - g \\
\alpha_H(v) &= e - i \\
\alpha_I(v) &= 0 \\
\alpha_J(v) &= f - j
\end{aligned}$$

We note that the vectors $\alpha_A, \alpha_B, \alpha_C, \alpha_D, \alpha_E, \alpha_H$ are linearly independent and that the remaining vectors are expressible in terms of these by

$$\alpha_J = \alpha_F = \alpha_C, \quad \alpha_G = \alpha_B, \quad \alpha_I = 0.$$

Now for each of the 10 values of X we have a choice between 2 linear functions. It may appear from this that the total number of regions of Lusztig's function R is 2^{10}. This is not so, however, for some of the systems of inequalities determining the sequence of choices may be inconsistent. Calculation along the lines outlined in [5] shows that there are 204 consistent choices. Again it is possible for two different consistent sequences to give the same linear function R. Thus we obtain an equivalence relation on the set of consistent sequences. Calculation shows that there are 144 equivalence classes. Thus the function R in type A_4 has 144 regions of linearity.

Each region of linearity may be defined by a system of independent inequalities of the form

$$\begin{array}{ll} & \sum_X n_X \alpha_X \geq 0 \\ \text{or} & \sum_X n_X \alpha_X \leq 0 \end{array}$$

where $n_X \in \mathbb{N}$.

Table II

Determination of vector $g(v)$.

a	b	c	d	e	f	g	h	i	j	
b	a	c	d	e	f	g	h	i	j	
b	$\underline{a}$	$\underline{c}$	$\underline{e}$	d	f	g	h	i	j	A
b	c	e	c	d	f	g	h	i	j	
b	c	e	d	$\underline{c}$	$\underline{f}$	$\underline{g}$	h	i	j	B
b	c	e	d	f	g	f	h	i	j	
b	c	e	d	f	g	$\underline{f}$	$\underline{h}$	$\underline{j}$	i	C
b	c	e	d	f	g	h	j	h	i	
b	c	e	$\underline{d}$	$\underline{f}$	$\underline{h}$	g	j	h	i	D
b	c	e	f	h	f	g	j	h	i	
$\underline{b}$	$\underline{c}$	$\underline{f}$	e	h	f	g	j	h	i	E
c	f	c	e	h	$\underline{f}$	$\underline{g}$	$\underline{j}$	h	i	F
c	f	c	e	h	g	j	g	h	i	
c	f	$\underline{c}$	$\underline{e}$	$\underline{g}$	h	j	g	h	i	G
c	f	e	g	e	h	j	g	h	i	
c	f	e	g	h	e	j	g	h	i	
c	f	e	g	h	j	e	g	h	i	
c	f	e	g	h	j	$\underline{e}$	$\underline{g}$	$\underline{i}$	h	H
c	f	e	g	h	j	g	i	g	h	
c	f	e	h	$\underline{g}$	$\underline{j}$	$\underline{g}$	i	g	h	I
c	f	e	h	j	g	j	i	g	h	
c	e	$\underline{f}$	$\underline{h}$	$\underline{j}$	g	j	i	g	h	J
c	e	h	j	h	g	j	i	g	h	
c	h	e	j	h	g	j	i	g	h	
c	h	e	j	g	h	j	i	g	h	
c	h	e	j	g	h	i	j	g	h	

We list the defining inequalities for the 144 regions in Table III. We use the following notation. The line

$$ABC, BCDEH || A, E, BCDE, BEH, AB^2CDEH$$

will denote the region:

$$\alpha_A + \alpha_B + \alpha_C \leq 0 \qquad \alpha_A \geq 0$$

$$\alpha_B + \alpha_C + \alpha_D + \alpha_E + \alpha_H \leq 0 \qquad \alpha_E \geq 0$$

$$\alpha_B + \alpha_C + \alpha_D + \alpha_E \geq 0$$

$$\alpha_B + \alpha_E + \alpha_H \geq 0$$

$$\alpha_A + 2\alpha_B + \alpha_C + \alpha_D + \alpha_E + \alpha_H \geq 0$$

The values of $R(v)$ for v in each of these 144 regions of linearity can be obtained from Table IV. We have $R(v) = g(v) + \varepsilon(v)$ where $g(v) = (c, h, e, j, g, h, i, j, g, h)$ and $\varepsilon(v)$ is given in Table IV.

Table III
Regions of linearity of R.

Region of linearity	Defining inequalities
1	A, BC, CD, E, BH ‖ BCD
2	A, BC, E, BH ‖ B, CD
3	ABC, CD, E, BH ‖ A, BCD
4	A, BC, E, BCDH ‖ BCD, BH
5	A, CD, E, BH ‖ BC, D
6	A, BC, CD, BEH ‖ BCD, E
7	A, BC, BEH ‖ B, CD, E
8	ABC, E, BH ‖ A, B, CD
9	ABC, E, BCDH ‖ A, BCD, BH
10	A, E, BCDH ‖ BC, D, BH
11	A, CD, BEH ‖ BC, D, E
12	A, B, C, BEH ‖ CD, BE
13	B, ABC, E, H ‖ AB, CD

Region of linearity	Defining inequalities
14	ABC, BCD, E, H ‖ ABCD, BH
15	A, D, E, BCH ‖ BC, BH
16	A, C, D, BEH ‖ BC, DE
17	A, B, BEH ‖ C, D, BE
18	AB, C, BEH ‖ A, CD, BE
19	ABC, BE, H ‖ AB, CD, E
20	B, ABC, E ‖ AB, CD, H
21	ABC, BCD, E ‖ B, ABCD, H
22	BCD, E, H ‖ ABC, D, BH
23	D, E, BCH ‖ A, BC, BH
24	A, DE, BCH ‖ BC, E, BH
25	A, C, BDEH ‖ BC, DE, BEH
26	A, D, BEH ‖ B, C, DE
27	A, B, D, BEH ‖ C, BDE
28	AB, C, BE, H ‖ CD, ABE
29	B, ABC, CD, E ‖ ABCD, H
30	BC, D, E, H ‖ ABC, BH
31	A, C, DE, BH ‖ BC, BEH
32	A, BDEH ‖ B, C, DE, BEH
33	AB, BEH ‖ A, C, D, BE
34	ABC, BE ‖ AB, CD, E, H
35	BCD, E ‖ B, ABC, D, H
36	DE, BCH ‖ A, BC, E, BH

Region of linearity	Defining inequalities
37	C, DE, BH ‖ A, BC, BEH
38	A, DE, BH ‖ B, C, BEH
39	A, B, BDEH ‖ C, BDE, BEH
40	AB, D, BEH ‖ A, C, BDE
41	AB, BE, H ‖ C, D, ABE
42	AB, C, BE ‖ CD, ABE, H
43	ABC, CD, BE ‖ ABCD, E, H
44	B, CD, E ‖ ABC, D, H
45	BC, D, E ‖ B, ABC, H
46	BC, DE, H ‖ ABC, E, BH
47	DE, BH ‖ A, B, C, BEH
48	AB, BDEH ‖ A, C, BDE, BEH
49	AB, BE ‖ C, D, ABE, H
50	CD, BE ‖ ABC, D, E, H
51	BC, DE ‖ B, ABC, E, H
52	B, DE, H ‖ AB, C, BEH
53	AB, BDE, H ‖ C, ABDE, BEH
54	AB, D, BE ‖ C, ABDE, H
55	C, D, BE ‖ ABC, DE, H
56	B, C, DE ‖ ABC, BE, H
57	B, DE ‖ AB, C, BE, H
58	BDE, H ‖ AB, C, DE, BEH
59	AB, BDE ‖ C, BE, ABDE, H

Region of linearity	Defining inequalities
60	D, BE $\parallel$ AB, C, DE, H
61	C, BDE $\parallel$ ABC, BE, DE, H
62	BDE $\parallel$ AB, C, BE, DE, H
63	B, AB, C, BE, ABE, H $\parallel$ CD
64	ABC, CD, BCD, E, H, BH $\parallel$ ABCD
65	BC, ABC, BCD, ABCD, E, H $\parallel$ BH
66	A, C, D, DE, BH, BEH $\parallel$ BC
67	A, C, BC, CD, BCD, BEH $\parallel$ BCDE
68	BC, ABC, BCDE, ABCDE, H $\parallel$ E, BH
69	B, C, ABC, BCDE, ABCDE $\parallel$ BE, H
70	BC, ABC, BCD, ABCD, E $\parallel$ B, H
71	B, AB, C, BE, ABE $\parallel$ CD, H
72	B, AB, BDE, ABDE, H $\parallel$ C, BEH
73	B, AB, BE, ABE, H $\parallel$ C, D
74	A, D, DE, BH, BEH $\parallel$ B, C
75	C, D, DE, BH, BEH $\parallel$ A, BC
76	B, D, DE, H, BEH $\parallel$ AB, C
77	C, BC, DE, H, BH $\parallel$ ABC, BEH
78	CD, BCD, E, H, BH $\parallel$ ABC, D
79	B, AB, D, BE, ABDE $\parallel$ C, H
80	B, C, D, BE, DE $\parallel$ ABC, H
81	C, ABC, CD, ABCD, BE $\parallel$ ABCDE, H
82	C, D, BCDE, H, BEH $\parallel$ ABC, DE

Region of linearity	Defining inequalities
83	AB, D, BDE, H, BEH ‖ C, ABDE
84	A, C, BC, BCDEH, B^2CDEH ‖ BCDE, BEH
85	C, ABC, CD, ABCD, BEH ‖ A, BCDE
86	C, ABC, BCDE, H, AB^2CDEH ‖ ABCDE, BEH
87	ABC, CD, BCDE, H, BEH ‖ ABCD, E
88	BC, ABC, BCDE, ABCDE ‖ B, E, H
89	D, DE, BH, BEH ‖ A, B, C
90	B, AB, BDE, ABDE ‖ C, BE, H
91	B, AB, BE, ABE ‖ C, D, H
92	B, D, DE, BE ‖ AB, C, H
93	C, ABC, BCDE, AB^2CDE ‖ BE, ABCDE, H
94	C, BCDE, H, BDEH ‖ ABC, DE, BEH
95	D, BDE, H, BEH ‖ AB, C, DE
96	CD, BCDE, H, BEH ‖ ABC, D, E
97	C, ABC, BCDEH, AB^2CDEH ‖ A, BCDE, BEH
98	A, D, E ‖ B, BC, BH, BCH
99	B, E, H ‖ AB, ABC, D, CD
100	CD, E, BH ‖ A, ABC, D, BCD
101	A, E, BH ‖ B, BC, D, CD
102	A, BC, E ‖ B, BCD, BH, BCDH
103	ABC, CD, BEH ‖ A, ABCD, E, BCDE
104	ABC, BCDE, H ‖ E, ABCDE, BEH, AB^2CDEH
105	A, BC, BCDEH ‖ E, BCDE, BEH, B^2CDEH

Region of linearity	Defining inequalities
106	C, D, BEH ‖ A, ABC, DE, BCDE
107	A, B, C ‖ BE, BCDE, BEH, BCDEH
108	B, E ‖ AB, ABC, D, CD, H
109	A, E ‖ B, BC, D, BH, BCDH
110	D, E ‖ A, B, BC, BH, BCH
111	A, DE ‖ B, BC, E, BH, BCH
112	E, BCDH ‖ A, ABC, D, BCD, BH
113	E, BH ‖ A, B, ABC, D, CD
114	ABC, E ‖ A, B, BCD, BH, BCDH
115	ABC, BEH ‖ A, AB, CD, E, BE
116	ABC, BCDEH ‖ A, E, BCDE, BEH, AB^2CDEH
117	BE, H ‖ AB, ABC, D, CD, E
118	A, BEH ‖ B, BC, D, CD, E
119	CD, BEH ‖ A, ABC, D, E, BCDE
120	D, BEH ‖ A, AB, C, DE, BDE
121	C, BDEH ‖ A, ABC, DE, BCDE, BEH
122	BCDE, H ‖ ABC, E, DE, BEH, BDEH
123	A, BCDEH ‖ BC, E, DE, BEH, BDEH
124	ABC, BCDE ‖ ABCDE, AB^2CDE, E, BE, H
125	AB, C ‖ A, BE, BCDE, BEH, BCDEH
126	A, BC ‖ B, E, BCDE, BEH, BCDEH
127	A, B ‖ C, BE, BDE, BEH, BDEH
128	DE ‖ A, B, BC, E, BH, BCH

Region of linearity	Defining inequalities
129	BDEH ‖ A, AB, C, DE, BDE, BEH
130	AB ‖ A, C, BE, BDE, BEH, BDEH
131	BE ‖ AB, ABC, D, CD, E, H
132	BCDE ‖ ABC, E, BE, DE, BDE, H
133	C, BC, ABC, BCDE, ABCDE, H, BH ‖ BEH
134	B, C, ABC, CD, ABCD, BE, ABCDE ‖ H
135	B, AB, D, BDE, ABDE, H, BEH ‖ C
136	C, BC, D, DE, H, BH, BEH ‖ ABC
137	C, ABC, CD, ABCD, BCDE, H, BEH ‖ ABCDE
138	E ‖ A, B, ABC, D, BCD, BH, BCDH
139	BEH ‖ A, AB, ABC, D, CD, E, BE
140	BCDEH ‖ A, ABC, E, DE, BCDE, BEH, BDEH
141	ABC ‖ A, AB, E, BE, BCDE, BEH, BCDEH
142	A ‖ B, BC, E, DE, BEH, BDEH, BCDEH
143	C, BC, ABC, CD, BCD, ABCD, BCDE, ABCDE, H, BH, BEH ‖ -
144	- ‖ A, AB, ABC, E, BE, DE, BDE, BCDE, BEH, BDEH, BCDEH.

Table IV

The linear function $\varepsilon(v)$ on each region.

Region of linearity	$\varepsilon(v)$
1	$(-\alpha_A-\alpha_E,-\alpha_B+\alpha_D-\alpha_E-\alpha_H,\alpha_A,\alpha_C+\alpha_E,-\alpha_C-\alpha_D,-\alpha_B-\alpha_C,$ $\alpha_B+\alpha_C+\alpha_D+\alpha_H,\alpha_B+2\alpha_C+\alpha_D,0,0),$
2	$(-\alpha_A-\alpha_E,-\alpha_B+\alpha_D-\alpha_E-\alpha_H,\alpha_A,\alpha_C+\alpha_E,0,-\alpha_B-\alpha_C,\alpha_B+\alpha_H,$ $\alpha_B+\alpha_C,\alpha_C+\alpha_D,0),$

Region of linearity	$\varepsilon(v)$
3	$(-\alpha_E, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, 0, \alpha_C + \alpha_E, -\alpha_C - \alpha_D, -\alpha_A - \alpha_B - \alpha_C,$ $\alpha_B + \alpha_C + \alpha_D + \alpha_H, \alpha_A + \alpha_B + 2\alpha_C + \alpha_D, 0, 0),$
4	$(-\alpha_A - \alpha_E, \alpha_D - \alpha_E, \alpha_A, \alpha_C + \alpha_E, -\alpha_B - \alpha_C - \alpha_D - \alpha_H, -\alpha_B - \alpha_C,$ $\alpha_B + \alpha_C + \alpha_D + \alpha_H, 2\alpha_B + 2\alpha_C + \alpha_D + \alpha_H, 0, 0),$
5	$(-\alpha_A - \alpha_E, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, \alpha_A, \alpha_C + \alpha_E, -\alpha_C - \alpha_D, 0,$ $\alpha_B + \alpha_C + \alpha_D + \alpha_H, \alpha_C + \alpha_D, 0, \alpha_B + \alpha_C),$
6	$(-\alpha_A, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, \alpha_A, \alpha_C, -\alpha_C - \alpha_D, -\alpha_B - \alpha_C,$ $\alpha_B + \alpha_C + \alpha_D + \alpha_E + \alpha_H, \alpha_B + 2\alpha_C + \alpha_D, 0, 0),$
7	$(-\alpha_A, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, \alpha_A, \alpha_C, 0, -\alpha_B - \alpha_C, \alpha_B + \alpha_E + \alpha_H, \alpha_B + \alpha_C,$ $\alpha_C + \alpha_D, 0),$
8	$(-\alpha_E, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, 0, \alpha_C + \alpha_E, 0, -\alpha_A - \alpha_B - \alpha_C, \alpha_B + \alpha_H,$ $\alpha_A + \alpha_B + \alpha_C, \alpha_C + \alpha_D, 0),$
9	$(-\alpha_E, \alpha_D - \alpha_E, 0, \alpha_C + \alpha_E, -\alpha_B - \alpha_C - \alpha_D - \alpha_H, -\alpha_A - \alpha_B - \alpha_C,$ $\alpha_B + \alpha_C + \alpha_D + \alpha_H, \alpha_A + 2\alpha_B + 2\alpha_C + \alpha_D + \alpha_H, 0, 0),$
10	$(-\alpha_A - \alpha_E, \alpha_D - \alpha_E, \alpha_A, \alpha_C + \alpha_E, -\alpha_B - \alpha_C - \alpha_D - \alpha_H, 0,$ $\alpha_B + \alpha_C + \alpha_D + \alpha_H, \alpha_B + \alpha_C + \alpha_D + \alpha_H, 0, \alpha_B + \alpha_C),$
11	$(-\alpha_A, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, \alpha_A, \alpha_C, -\alpha_C - \alpha_D, 0,$ $\alpha_B + \alpha_C + \alpha_D + \alpha_E + \alpha_H, \alpha_C + \alpha_D, 0, \alpha_B + \alpha_C),$
12	$(-\alpha_A, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, \alpha_A, -\alpha_B + \alpha_C, \alpha_B, -\alpha_C, \alpha_B + \alpha_E + \alpha_H, \alpha_C,$ $\alpha_C + \alpha_D, 0),$
13	$(-\alpha_E, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, -\alpha_B, \alpha_C + \alpha_E, \alpha_B, -\alpha_A - \alpha_B - \alpha_C, \alpha_H,$ $\alpha_A + \alpha_B + \alpha_C, \alpha_C + \alpha_D, 0),$
14	$(-\alpha_E, \alpha_D - \alpha_E, -\alpha_B - \alpha_C - \alpha_D, \alpha_C + \alpha_E, -\alpha_H, -\alpha_A - \alpha_B - \alpha_C, \alpha_H,$

Region of linearity	$\varepsilon(v)$
	$\alpha_A + 2\alpha_B + 2\alpha_C + \alpha_D + \alpha_H, 0, 0)$,
15	$(-\alpha_A - \alpha_D - \alpha_E, \alpha_D - \alpha_E, \alpha_A, \alpha_C + \alpha_E, -\alpha_B - \alpha_C - \alpha_H, \alpha_D, \alpha_B + \alpha_C + \alpha_H,$
	$\alpha_B + \alpha_C + \alpha_H, 0, \alpha_B + \alpha_C)$,
16	$(-\alpha_A, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, \alpha_A, \alpha_C - \alpha_D, -\alpha_C, \alpha_D,$
	$\alpha_B + \alpha_C + \alpha_D + \alpha_E + \alpha_H, \alpha_C, 0, \alpha_B + \alpha_C)$,
17	$(-\alpha_A, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, \alpha_A, -\alpha_B + \alpha_C, \alpha_B, 0, \alpha_B + \alpha_E + \alpha_H, 0,$
	$\alpha_C + \alpha_D, \alpha_C)$,
18	$(0, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, 0, -\alpha_A - \alpha_B + \alpha_C, \alpha_A + \alpha_B, -\alpha_C, \alpha_B + \alpha_E + \alpha_H,$
	$\alpha_C, \alpha_C + \alpha_D, 0)$,
19	$(0, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, -\alpha_B - \alpha_E, \alpha_C, \alpha_B + \alpha_E, -\alpha_A - \alpha_B - \alpha_C, \alpha_H,$
	$\alpha_A + \alpha_B + \alpha_C, \alpha_C + \alpha_D, 0)$,
20	$(-\alpha_E, -\alpha_B + \alpha_D - \alpha_E, -\alpha_B, \alpha_C + \alpha_E, \alpha_B, -\alpha_A - \alpha_B - \alpha_C, 0, \alpha_A + \alpha_B + \alpha_C,$
	$\alpha_C + \alpha_D + \alpha_H, 0)$,
21	$(-\alpha_E, \alpha_D - \alpha_E, -\alpha_B - \alpha_C - \alpha_D, \alpha_C + \alpha_E, 0, -\alpha_A - \alpha_B - \alpha_C, 0,$
	$\alpha_A + 2\alpha_B + 2\alpha_C + \alpha_D, \alpha_H, 0)$,
22	$(-\alpha_E, \alpha_D - \alpha_E, -\alpha_B - \alpha_C - \alpha_D, \alpha_C + \alpha_E, -\alpha_H, 0, \alpha_H, \alpha_B + \alpha_C + \alpha_D + \alpha_H,$
	$0, \alpha_A + \alpha_B + \alpha_C)$,
23	$(-\alpha_D - \alpha_E, \alpha_D - \alpha_E, 0, \alpha_C + \alpha_E, -\alpha_B - \alpha_C - \alpha_H, \alpha_D, \alpha_B + \alpha_C + \alpha_H,$
	$\alpha_B + \alpha_C + \alpha_H, 0, \alpha_A + \alpha_B + \alpha_C)$,
24	$(-\alpha_A - \alpha_D - \alpha_E, \alpha_D, \alpha_A, \alpha_C, -\alpha_B - \alpha_C - \alpha_H, \alpha_D + \alpha_E, \alpha_B + \alpha_C + \alpha_H,$
	$\alpha_B + \alpha_C + \alpha_H, 0, \alpha_B + \alpha_C)$,
25	$(-\alpha_A, \alpha_D, \alpha_A, -\alpha_B + \alpha_C - \alpha_D - \alpha_E - \alpha_H, -\alpha_C, \alpha_B + \alpha_D + \alpha_E + \alpha_H,$
	$\alpha_B + \alpha_C + \alpha_D + \alpha_E + \alpha_H, \alpha_C, 0, \alpha_B + \alpha_C)$,

Region of linearity	$\varepsilon(v)$
26	$(-\alpha_A, -\alpha_B+\alpha_D-\alpha_E-\alpha_H, \alpha_A, \alpha_C-\alpha_D, 0, \alpha_D, \alpha_B+\alpha_D+\alpha_E+\alpha_H, 0, \alpha_C, \alpha_B+\alpha_C)$,
27	$(-\alpha_A, -\alpha_B+\alpha_D-\alpha_E-\alpha_H, \alpha_A, -\alpha_B+\alpha_C-\alpha_D, \alpha_B, \alpha_D, \alpha_B+\alpha_D+\alpha_E+\alpha_H, 0, \alpha_C, \alpha_C)$,
28	$(0, -\alpha_B+\alpha_D-\alpha_E-\alpha_H, -\alpha_B-\alpha_E, -\alpha_A-\alpha_B+\alpha_C, \alpha_A+2\alpha_B+\alpha_E, -\alpha_C, \alpha_H, \alpha_C, \alpha_C+\alpha_D, 0)$,
29	$(-\alpha_E, -\alpha_B+\alpha_D-\alpha_E, -\alpha_B-\alpha_C-\alpha_D, \alpha_C+\alpha_E, \alpha_B, -\alpha_A-\alpha_B-\alpha_C, 0, \alpha_A+\alpha_B+2\alpha_C+\alpha_D, \alpha_H, 0)$,
30	$(-\alpha_D-\alpha_E, \alpha_D-\alpha_E, -\alpha_B-\alpha_C, \alpha_C+\alpha_E, -\alpha_H, \alpha_D, \alpha_H, \alpha_B+\alpha_C+\alpha_H, 0, \alpha_A+\alpha_B+\alpha_C)$,
31	$(-\alpha_A-\alpha_D-\alpha_E, \alpha_D, \alpha_A, -\alpha_B+\alpha_C-\alpha_H, -\alpha_C, \alpha_B+\alpha_D+\alpha_E+\alpha_H, \alpha_B+\alpha_C+\alpha_H, \alpha_C, 0, \alpha_B+\alpha_C)$,
32	$(-\alpha_A, \alpha_D, \alpha_A, -\alpha_B+\alpha_C-\alpha_D-\alpha_E-\alpha_H, 0, \alpha_B+\alpha_D+\alpha_E+\alpha_H, \alpha_B+\alpha_D+\alpha_E+\alpha_H, 0, \alpha_C, \alpha_B+\alpha_C)$,
33	$(0, -\alpha_B+\alpha_D-\alpha_E-\alpha_H, 0, -\alpha_A-\alpha_B+\alpha_C, \alpha_A+\alpha_B, 0, \alpha_B+\alpha_E+\alpha_H, 0, \alpha_C+\alpha_D, \alpha_C)$,
34	$(0, -\alpha_B+\alpha_D-\alpha_E, -\alpha_B-\alpha_E, \alpha_C, \alpha_B+\alpha_E, -\alpha_A-\alpha_B-\alpha_C, 0, \alpha_A+\alpha_B+\alpha_C, \alpha_C+\alpha_D+\alpha_H, 0)$,
35	$(-\alpha_E, \alpha_D-\alpha_E, -\alpha_B-\alpha_C-\alpha_D, \alpha_C+\alpha_E, 0, 0, 0, \alpha_B+\alpha_C+\alpha_D, \alpha_H, \alpha_A+\alpha_B+\alpha_C)$,
36	$(-\alpha_D-\alpha_E, \alpha_D, 0, \alpha_C, -\alpha_B-\alpha_C-\alpha_H, \alpha_D+\alpha_E, \alpha_B+\alpha_C+\alpha_H, \alpha_B+\alpha_C+\alpha_H, 0, \alpha_A+\alpha_B+\alpha_C)$,
37	$(-\alpha_D-\alpha_E, \alpha_D, 0, -\alpha_B+\alpha_C-\alpha_H, -\alpha_C, \alpha_B+\alpha_D+\alpha_E+\alpha_H,$

Region of linearity	$\varepsilon(v)$
	$\alpha_B+\alpha_C+\alpha_H,\alpha_C,0,\alpha_A+\alpha_B+\alpha_C),$
38	$(-\alpha_A-\alpha_D-\alpha_E,\alpha_D,\alpha_A,-\alpha_B+\alpha_C-\alpha_H,0,\alpha_B+\alpha_D+\alpha_E+\alpha_H,\alpha_B+\alpha_H,$
	$0,\alpha_C,\alpha_B+\alpha_C),$
39	$(-\alpha_A,\alpha_D,\alpha_A,-2\alpha_B+\alpha_C-\alpha_D-\alpha_E-\alpha_H,\alpha_B,\alpha_B+\alpha_D+\alpha_E+\alpha_H,$
	$\alpha_B+\alpha_D+\alpha_E+\alpha_H,0,\alpha_C,\alpha_C),$
40	$(0,-\alpha_B+\alpha_D-\alpha_E-\alpha_H,0,-\alpha_A-\alpha_B+\alpha_C-\alpha_D,\alpha_A+\alpha_B,\alpha_D,$
	$\alpha_B+\alpha_D+\alpha_E+\alpha_H,0,\alpha_C,\alpha_C),$
41	$(0,-\alpha_B+\alpha_D-\alpha_E-\alpha_H,-\alpha_B-\alpha_E,-\alpha_A-\alpha_B+\alpha_C,\alpha_A+2\alpha_B+\alpha_E,0,$
	$\alpha_H,0,\alpha_C+\alpha_D,\alpha_C),$
42	$(0,-\alpha_B+\alpha_D-\alpha_E,-\alpha_B-\alpha_E,-\alpha_A-\alpha_B+\alpha_C,\alpha_A+2\alpha_B+\alpha_E,-\alpha_C,0,$
	$\alpha_C,\alpha_C+\alpha_D+\alpha_H,0),$
43	$(0,-\alpha_B+\alpha_D-\alpha_E,-\alpha_B-\alpha_C-\alpha_D-\alpha_E,\alpha_C,\alpha_B+\alpha_E,-\alpha_A-\alpha_B-\alpha_C,0,$
	$\alpha_A+\alpha_B+2\alpha_C+\alpha_D,\alpha_H,0),$
44	$(-\alpha_E,-\alpha_B+\alpha_D-\alpha_E,-\alpha_B-\alpha_C-\alpha_D,\alpha_C+\alpha_E,\alpha_B,0,0,\alpha_C+\alpha_D,\alpha_H,$
	$\alpha_A+\alpha_B+\alpha_C),$
45	$(-\alpha_D-\alpha_E,\alpha_D-\alpha_E,-\alpha_B-\alpha_C,\alpha_C+\alpha_E,0,\alpha_D,0,\alpha_B+\alpha_C,\alpha_H,$
	$\alpha_A+\alpha_B+\alpha_C),$
46	$(-\alpha_D-\alpha_E,\alpha_D,-\alpha_B-\alpha_C,\alpha_C,-\alpha_H,\alpha_D+\alpha_E,\alpha_H,\alpha_B+\alpha_C+\alpha_H,0,$
	$\alpha_A+\alpha_B+\alpha_C),$
47	$(-\alpha_D-\alpha_E,\alpha_D,0,-\alpha_B+\alpha_C-\alpha_H,0,\alpha_B+\alpha_D+\alpha_E+\alpha_H,\alpha_B+\alpha_H,0,$
	$\alpha_C,\alpha_A+\alpha_B+\alpha_C),$
48	$(0,\alpha_D,0,-\alpha_A-2\alpha_B+\alpha_C-\alpha_D-\alpha_E-\alpha_H,\alpha_A+\alpha_B,$
	$\alpha_B+\alpha_D+\alpha_E+\alpha_H,\alpha_B+\alpha_D+\alpha_E+\alpha_H,0,\alpha_C,\alpha_C),$

Region of linearity	$\varepsilon(v)$
49	$(0, -\alpha_B + \alpha_D - \alpha_E, -\alpha_B - \alpha_E, -\alpha_A - \alpha_B + \alpha_C, \alpha_A + 2\alpha_B + \alpha_E, 0, 0, 0,$ $\alpha_C + \alpha_D + \alpha_H, \alpha_C),$
50	$(0, -\alpha_B + \alpha_D - \alpha_E, -\alpha_B - \alpha_C - \alpha_D - \alpha_E, \alpha_C, \alpha_B + \alpha_E, 0, 0, \alpha_C + \alpha_D,$ $\alpha_H, \alpha_A + \alpha_B + \alpha_C),$
51	$(-\alpha_D - \alpha_E, \alpha_D, -\alpha_B - \alpha_C, \alpha_C, 0, \alpha_D + \alpha_E, 0, \alpha_B + \alpha_C, \alpha_H,$ $\alpha_A + \alpha_B + \alpha_C),$
52	$(-\alpha_D - \alpha_E, \alpha_D, -\alpha_B, -\alpha_B + \alpha_C - \alpha_H, \alpha_B, \alpha_B + \alpha_D + \alpha_E + \alpha_H, \alpha_H, 0, \alpha_C,$ $\alpha_A + \alpha_B + \alpha_C),$
53	$(0, \alpha_D, -\alpha_B - \alpha_D - \alpha_E, -\alpha_A - 2\alpha_B + \alpha_C - \alpha_D - \alpha_E - \alpha_H,$ $\alpha_A + 2\alpha_B + \alpha_D + \alpha_E, \alpha_B + \alpha_D + \alpha_E + \alpha_H, \alpha_H, 0, \alpha_C, \alpha_C),$
54	$(0, -\alpha_B + \alpha_D - \alpha_E, -\alpha_B - \alpha_D - \alpha_E, -\alpha_A - \alpha_B + \alpha_C - \alpha_D,$ $\alpha_A + 2\alpha_B + \alpha_D + \alpha_E, \alpha_D, 0, 0, \alpha_C + \alpha_H, \alpha_C),$
55	$(0, -\alpha_B + \alpha_D - \alpha_E, -\alpha_B - \alpha_C - \alpha_D - \alpha_E, \alpha_C - \alpha_D, \alpha_B + \alpha_D + \alpha_E, \alpha_D,$ $0, \alpha_C, \alpha_H, \alpha_A + \alpha_B + \alpha_C),$
56	$(-\alpha_D - \alpha_E, \alpha_D, -\alpha_B - \alpha_C, -\alpha_B + \alpha_C, \alpha_B, \alpha_B + \alpha_D + \alpha_E, 0, \alpha_C, \alpha_H,$ $\alpha_A + \alpha_B + \alpha_C),$
57	$(-\alpha_D - \alpha_E, \alpha_D, -\alpha_B, -\alpha_B + \alpha_C, \alpha_B, \alpha_B + \alpha_D + \alpha_E, 0, 0, \alpha_C + \alpha_H,$ $\alpha_A + \alpha_B + \alpha_C),$
58	$(0, \alpha_D, -\alpha_B - \alpha_D - \alpha_E, -\alpha_B + \alpha_C - \alpha_D - \alpha_E - \alpha_H, \alpha_B + \alpha_D + \alpha_E,$ $\alpha_B + \alpha_D + \alpha_E + \alpha_H, \alpha_H, 0, \alpha_C, \alpha_A + \alpha_B + \alpha_C),$
59	$(0, \alpha_D, -\alpha_B - \alpha_D - \alpha_E, -\alpha_A - 2\alpha_B + \alpha_C - \alpha_D - \alpha_E,$ $\alpha_A + 2\alpha_B + \alpha_D + \alpha_E, \alpha_B + \alpha_D + \alpha_E, 0, 0, \alpha_C + \alpha_H, \alpha_C),$
60	$(0, -\alpha_B + \alpha_D - \alpha_E, -\alpha_B - \alpha_D - \alpha_E, \alpha_C - \alpha_D, \alpha_B + \alpha_D + \alpha_E, \alpha_D, 0,$

Region of linearity	$\varepsilon(v)$
	$0, \alpha_C + \alpha_H, \alpha_A + \alpha_B + \alpha_C)$,
61	$(0, \alpha_D, -\alpha_B - \alpha_C - \alpha_D - \alpha_E, -\alpha_B + \alpha_C - \alpha_D - \alpha_E, \alpha_B + \alpha_D + \alpha_E,$ $\alpha_B + \alpha_D + \alpha_E, 0, \alpha_C, \alpha_H, \alpha_A + \alpha_B + \alpha_C)$,
62	$(0, \alpha_D, -\alpha_B - \alpha_D - \alpha_E, -\alpha_B + \alpha_C - \alpha_D - \alpha_E, \alpha_B + \alpha_D + \alpha_E,$ $\alpha_B + \alpha_D + \alpha_E, 0, 0, \alpha_C + \alpha_H, \alpha_A + \alpha_B + \alpha_C)$,
63	$(-\alpha_A - \alpha_B - \alpha_E, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, \alpha_A, \alpha_C + \alpha_E, \alpha_B, -\alpha_C, \alpha_H, \alpha_C,$ $\alpha_C + \alpha_D, 0)$,
64	$(-\alpha_E, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, -\alpha_B - \alpha_C - \alpha_D, \alpha_C + \alpha_E, \alpha_B, -\alpha_A - \alpha_B - \alpha_C,$ $\alpha_H, \alpha_A + \alpha_B + 2\alpha_C + \alpha_D, 0, 0)$,
65	$(-\alpha_A - \alpha_B - \alpha_C - \alpha_D - \alpha_E, \alpha_D - \alpha_E, \alpha_A, \alpha_C + \alpha_E, -\alpha_H, \alpha_D, \alpha_H,$ $\alpha_B + \alpha_C + \alpha_H, 0, 0)$,
66	$(-\alpha_A - \alpha_D - \alpha_E, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, \alpha_A, \alpha_C + \alpha_E, -\alpha_C, \alpha_D,$ $\alpha_B + \alpha_C + \alpha_H, \alpha_C, 0, \alpha_B + \alpha_C)$,
67	$(-\alpha_A, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, \alpha_A, -\alpha_B - \alpha_D, \alpha_B, \alpha_D,$ $\alpha_B + \alpha_C + \alpha_D + \alpha_E + \alpha_H, \alpha_C, 0, 0)$,
68	$(-\alpha_A - \alpha_B - \alpha_C - \alpha_D - \alpha_E, \alpha_D, \alpha_A, \alpha_C, -\alpha_H, \alpha_D + \alpha_E, \alpha_H, \alpha_B + \alpha_C + \alpha_H,$ $0, 0)$,
69	$(-\alpha_A - \alpha_B - \alpha_C - \alpha_D - \alpha_E, \alpha_D, \alpha_A, -\alpha_B + \alpha_C, \alpha_B, \alpha_B + \alpha_D + \alpha_E, 0, \alpha_C,$ $\alpha_H, 0)$,
70	$(-\alpha_A - \alpha_B - \alpha_C - \alpha_D - \alpha_E, \alpha_D - \alpha_E, \alpha_A, \alpha_C + \alpha_E, 0, \alpha_D, 0, \alpha_B + \alpha_C,$ $\alpha_H, 0)$,
71	$(-\alpha_A - \alpha_B - \alpha_E, -\alpha_B + \alpha_D - \alpha_E, \alpha_A, \alpha_C + \alpha_E, \alpha_B, -\alpha_C, 0, \alpha_C,$ $\alpha_C + \alpha_D + \alpha_H, 0)$,

Region of linearity	$\varepsilon(v)$
72	$(-\alpha_A - \alpha_B - \alpha_D - \alpha_E, \alpha_D, \alpha_A, -\alpha_B + \alpha_C - \alpha_H, \alpha_B, \alpha_B + \alpha_D + \alpha_E + \alpha_H, \alpha_H, 0, \alpha_C, \alpha_C)$,
73	$(-\alpha_A - \alpha_B - \alpha_E, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, \alpha_A, \alpha_C + \alpha_E, \alpha_B, 0, \alpha_H, 0, \alpha_C + \alpha_D, \alpha_C)$,
74	$(-\alpha_A - \alpha_D - \alpha_E, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, \alpha_A, \alpha_C + \alpha_E, 0, \alpha_D, \alpha_B + \alpha_H, 0, \alpha_C, \alpha_B + \alpha_C)$,
75	$(-\alpha_D - \alpha_E, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, 0, \alpha_C + \alpha_E, -\alpha_C, \alpha_D, \alpha_B + \alpha_C + \alpha_H, \alpha_C, 0, \alpha_A + \alpha_B + \alpha_C)$,
76	$(-\alpha_D - \alpha_E, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, -\alpha_B, \alpha_C + \alpha_E, \alpha_B, \alpha_D, \alpha_H, 0, \alpha_C, \alpha_A + \alpha_B + \alpha_C)$,
77	$(-\alpha_D - \alpha_E, \alpha_D, -\alpha_B - \alpha_C, -\alpha_B + \alpha_C - \alpha_H, \alpha_B, \alpha_B + \alpha_D + \alpha_E + \alpha_H, \alpha_H, \alpha_C, 0, \alpha_A + \alpha_B + \alpha_C)$,
78	$(-\alpha_E, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, -\alpha_B - \alpha_C - \alpha_D, \alpha_C + \alpha_E, \alpha_B, 0, \alpha_H, \alpha_C + \alpha_D, 0, \alpha_A + \alpha_B + \alpha_C)$,
79	$(-\alpha_A - \alpha_B - \alpha_D - \alpha_E, -\alpha_B + \alpha_D - \alpha_E, \alpha_A, \alpha_C + \alpha_E, \alpha_B, \alpha_D, 0, 0, \alpha_C + \alpha_H, \alpha_C)$,
80	$(-\alpha_D - \alpha_E, -\alpha_B + \alpha_D - \alpha_E, -\alpha_B - \alpha_C, \alpha_C + \alpha_E, \alpha_B, \alpha_D, 0, \alpha_C, \alpha_H, \alpha_A + \alpha_B + \alpha_C)$,
81	$(0, -\alpha_B + \alpha_D - \alpha_E, -\alpha_B - \alpha_C - \alpha_D - \alpha_E, -\alpha_A - \alpha_B - \alpha_D, \alpha_A + 2\alpha_B + \alpha_C + \alpha_D + \alpha_E, \alpha_D, 0, \alpha_C, \alpha_H, 0)$,
82	$(0, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, -\alpha_B - \alpha_C - \alpha_D - \alpha_E, \alpha_C - \alpha_D, \alpha_B + \alpha_D + \alpha_E, \alpha_D, \alpha_H, \alpha_C, 0, \alpha_A + \alpha_B + \alpha_C)$,
83	$(0, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, -\alpha_B - \alpha_D - \alpha_E, -\alpha_A - \alpha_B + \alpha_C - \alpha_D,$

Region of linearity	$\varepsilon(v)$
	$\alpha_A + 2\alpha_B + \alpha_D + \alpha_E, \alpha_D, \alpha_H, 0, \alpha_C, \alpha_C),$
84	$(-\alpha_A, \alpha_D, \alpha_A, -2\alpha_B - \alpha_D - \alpha_E - \alpha_H, \alpha_B, \alpha_B + \alpha_D + \alpha_E + \alpha_H,$
	$\alpha_B + \alpha_C + \alpha_D + \alpha_E + \alpha_H, \alpha_C, 0, 0),$
85	$(0, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, 0, -\alpha_A - \alpha_B - \alpha_D, \alpha_A + \alpha_B, \alpha_D,$
	$\alpha_B + \alpha_C + \alpha_D + \alpha_E + \alpha_H, \alpha_C, 0, 0),$
86	$(0, \alpha_D, -\alpha_B - \alpha_C - \alpha_D - \alpha_E, -\alpha_A - 2\alpha_B - \alpha_D - \alpha_E - \alpha_H,$
	$\alpha_A + 2\alpha_B + \alpha_C + \alpha_D + \alpha_E, \alpha_B + \alpha_D + \alpha_E + \alpha_H, \alpha_H, \alpha_C, 0, 0),$
87	$(0, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, -\alpha_B - \alpha_C - \alpha_D - \alpha_E, \alpha_C, \alpha_B + \alpha_E,$
	$-\alpha_A - \alpha_B - \alpha_C, \alpha_H, \alpha_A + \alpha_B + 2\alpha_C + \alpha_D, 0, 0),$
88	$(-\alpha_A - \alpha_B - \alpha_C - \alpha_D - \alpha_E, \alpha_D, \alpha_A, \alpha_C, 0, \alpha_D + \alpha_E, 0, \alpha_B + \alpha_C, \alpha_H, 0),$
89	$(-\alpha_D - \alpha_E, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, 0, \alpha_C + \alpha_E, 0, \alpha_D, \alpha_B + \alpha_H, 0, \alpha_C,$
	$\alpha_A + \alpha_B + \alpha_C),$
90	$(-\alpha_A - \alpha_B - \alpha_D - \alpha_E, \alpha_D, \alpha_A, -\alpha_B + \alpha_C, \alpha_B, \alpha_B + \alpha_D + \alpha_E, 0, 0,$
	$\alpha_C + \alpha_H, \alpha_C),$
91	$(-\alpha_A - \alpha_B - \alpha_E, -\alpha_B + \alpha_D - \alpha_E, \alpha_A, \alpha_C + \alpha_E, \alpha_B, 0, 0, 0,$
	$\alpha_C + \alpha_D + \alpha_H, \alpha_C),$
92	$(-\alpha_D - \alpha_E, -\alpha_B + \alpha_D - \alpha_E, -\alpha_B, \alpha_C + \alpha_E, \alpha_B, \alpha_D, 0, 0, \alpha_C + \alpha_H,$
	$\alpha_A + \alpha_B + \alpha_C),$
93	$(0, \alpha_D, -\alpha_B - \alpha_C - \alpha_D - \alpha_E, -\alpha_A - 2\alpha_B - \alpha_D - \alpha_E,$
	$\alpha_A + 2\alpha_B + \alpha_C + \alpha_D + \alpha_E, \alpha_B + \alpha_D + \alpha_E, 0, \alpha_C, \alpha_H, 0),$
94	$(0, \alpha_D, -\alpha_B - \alpha_C - \alpha_D - \alpha_E, -\alpha_B + \alpha_C - \alpha_D - \alpha_E - \alpha_H, \alpha_B + \alpha_D + \alpha_E,$
	$\alpha_B + \alpha_D + \alpha_E + \alpha_H, \alpha_H, \alpha_C, 0, \alpha_A + \alpha_B + \alpha_C),$
95	$(0, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, -\alpha_B - \alpha_D - \alpha_E, \alpha_C - \alpha_D, \alpha_B + \alpha_D + \alpha_E, \alpha_D,$

Region of linearity	$\varepsilon(v)$
	$\alpha_H, 0, \alpha_C, \alpha_A+\alpha_B+\alpha_C)$,
96	$(0, -\alpha_B+\alpha_D-\alpha_E-\alpha_H, -\alpha_B-\alpha_C-\alpha_D-\alpha_E, \alpha_C, \alpha_B+\alpha_E, 0, \alpha_H,$
	$\alpha_C+\alpha_D, 0, \alpha_A+\alpha_B+\alpha_C)$,
97	$(0, \alpha_D, 0, -\alpha_A-2\alpha_B-\alpha_D-\alpha_E-\alpha_H, \alpha_A+\alpha_B, \alpha_B+\alpha_D+\alpha_E+\alpha_H,$
	$\alpha_B+\alpha_C+\alpha_D+\alpha_E+\alpha_H, \alpha_C, 0, 0)$,
98	$(-\alpha_A-\alpha_D-\alpha_E, \alpha_D-\alpha_E, \alpha_A, \alpha_C+\alpha_E, 0, \alpha_D, 0, 0, \alpha_B+\alpha_C+\alpha_H,$
	$\alpha_B+\alpha_C)$,
99	$(-\alpha_E, -\alpha_B+\alpha_D-\alpha_E-\alpha_H, -\alpha_B, \alpha_C+\alpha_E, \alpha_B, 0, \alpha_H, 0, \alpha_C+\alpha_D,$
	$\alpha_A+\alpha_B+\alpha_C)$,
100	$(-\alpha_E, -\alpha_B+\alpha_D-\alpha_E-\alpha_H, 0, \alpha_C+\alpha_E, -\alpha_C-\alpha_D, 0,$
	$\alpha_B+\alpha_C+\alpha_D+\alpha_H, \alpha_C+\alpha_D, 0, \alpha_A+\alpha_B+\alpha_C)$,
101	$(-\alpha_A-\alpha_E, -\alpha_B+\alpha_D-\alpha_E-\alpha_H, \alpha_A, \alpha_C+\alpha_E, 0, 0, \alpha_B+\alpha_H, 0,$
	$\alpha_C+\alpha_D, \alpha_B+\alpha_C)$,
102	$(-\alpha_A-\alpha_E, \alpha_D-\alpha_E, \alpha_A, \alpha_C+\alpha_E, 0, -\alpha_B-\alpha_C, 0, \alpha_B+\alpha_C,$
	$\alpha_B+\alpha_C+\alpha_D+\alpha_H, 0)$,
103	$(0, -\alpha_B+\alpha_D-\alpha_E-\alpha_H, 0, \alpha_C, -\alpha_C-\alpha_D, -\alpha_A-\alpha_B-\alpha_C,$
	$\alpha_B+\alpha_C+\alpha_D+\alpha_E+\alpha_H, \alpha_A+\alpha_B+2\alpha_C+\alpha_D, 0, 0)$,
104	$(0, \alpha_D, -\alpha_B-\alpha_C-\alpha_D-\alpha_E, \alpha_C, -\alpha_H, -\alpha_A-\alpha_B-\alpha_C, \alpha_H,$
	$\alpha_A+2\alpha_B+2\alpha_C+\alpha_D+\alpha_E+\alpha_H, 0, 0)$,
105	$(-\alpha_A, \alpha_D, \alpha_A, \alpha_C, -\alpha_B-\alpha_C-\alpha_D-\alpha_E-\alpha_H, -\alpha_B-\alpha_C,$
	$\alpha_B+\alpha_C+\alpha_D+\alpha_E+\alpha_H, 2\alpha_B+2\alpha_C+\alpha_D+\alpha_E+\alpha_H, 0, 0)$,
106	$(0, -\alpha_B+\alpha_D-\alpha_E-\alpha_H, 0, \alpha_C-\alpha_D, -\alpha_C, \alpha_D,$
	$\alpha_B+\alpha_C+\alpha_D+\alpha_E+\alpha_H, \alpha_C, 0, \alpha_A+\alpha_B+\alpha_C)$,

Region of linearity	$\varepsilon(v)$
107	$(-\alpha_A, \alpha_D, \alpha_A, -\alpha_B + \alpha_C, \alpha_B, -\alpha_C, 0, \alpha_C, \alpha_B + \alpha_C + \alpha_D + \alpha_E + \alpha_H, 0),$
108	$(-\alpha_E, -\alpha_B + \alpha_D - \alpha_E, -\alpha_B, \alpha_C + \alpha_E, \alpha_B, 0, 0, 0, \alpha_C + \alpha_D + \alpha_H,$ $\alpha_A + \alpha_B + \alpha_C),$
109	$(-\alpha_A - \alpha_E, \alpha_D - \alpha_E, \alpha_A, \alpha_C + \alpha_E, 0, 0, 0, 0, \alpha_B + \alpha_C + \alpha_D + \alpha_H,$ $\alpha_B + \alpha_C),$
110	$(-\alpha_D - \alpha_E, \alpha_D - \alpha_E, 0, \alpha_C + \alpha_E, 0, \alpha_D, 0, 0, \alpha_B + \alpha_C + \alpha_H,$ $\alpha_A + \alpha_B + \alpha_C),$
111	$(-\alpha_A - \alpha_D - \alpha_E, \alpha_D, \alpha_A, \alpha_C, 0, \alpha_D + \alpha_E, 0, 0, \alpha_B + \alpha_C + \alpha_H, \alpha_B + \alpha_C),$
112	$(-\alpha_E, \alpha_D - \alpha_E, 0, \alpha_C + \alpha_E, -\alpha_B - \alpha_C - \alpha_D - \alpha_H, 0, \alpha_B + \alpha_C + \alpha_D + \alpha_H,$ $\alpha_B + \alpha_C + \alpha_D + \alpha_H, 0, \alpha_A + \alpha_B + \alpha_C),$
113	$(-\alpha_E, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, 0, \alpha_C + \alpha_E, 0, 0, \alpha_B + \alpha_H, 0, \alpha_C + \alpha_D,$ $\alpha_A + \alpha_B + \alpha_C),$
114	$(-\alpha_E, \alpha_D - \alpha_E, 0, \alpha_C + \alpha_E, 0, -\alpha_A - \alpha_B - \alpha_C, 0, \alpha_A + \alpha_B + \alpha_C,$ $\alpha_B + \alpha_C + \alpha_D + \alpha_H, 0),$
115	$(0, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, 0, \alpha_C, 0, -\alpha_A - \alpha_B - \alpha_C, \alpha_B + \alpha_E + \alpha_H,$ $\alpha_A + \alpha_B + \alpha_C, \alpha_C + \alpha_D, 0),$
116	$(0, \alpha_D, 0, \alpha_C, -\alpha_B - \alpha_C - \alpha_D - \alpha_E - \alpha_H, -\alpha_A - \alpha_B - \alpha_C,$ $\alpha_B + \alpha_C + \alpha_D + \alpha_E + \alpha_H, \alpha_A + 2\alpha_B + 2\alpha_C + \alpha_D + \alpha_E + \alpha_H, 0, 0),$
117	$(0, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, -\alpha_B - \alpha_E, \alpha_C, \alpha_B + \alpha_E, 0, \alpha_H, 0, \alpha_C + \alpha_D,$ $\alpha_A + \alpha_B + \alpha_C),$
118	$(-\alpha_A, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, \alpha_A, \alpha_C, 0, 0, \alpha_B + \alpha_E + \alpha_H, 0, \alpha_C + \alpha_D,$ $\alpha_B + \alpha_C),$
119	$(0, -\alpha_B + \alpha_D - \alpha_E - \alpha_H, 0, \alpha_C, -\alpha_C - \alpha_D, 0, 0, \alpha_B + \alpha_C + \alpha_D + \alpha_E + \alpha_H,$

Region of linearity	$\varepsilon(v)$
	$\alpha_C+\alpha_D,0,\alpha_A+\alpha_B+\alpha_C),$
120	$(0,-\alpha_B+\alpha_D-\alpha_E-\alpha_H,0,\alpha_C-\alpha_D,0,\alpha_D,\alpha_B+\alpha_D+\alpha_E+\alpha_H,0,$
	$\alpha_C,\alpha_A+\alpha_B+\alpha_C),$
121	$(0,\alpha_D,0,-\alpha_B+\alpha_C-\alpha_D-\alpha_E-\alpha_H,-\alpha_C,\alpha_B+\alpha_D+\alpha_E+\alpha_H,$
	$\alpha_B+\alpha_C+\alpha_D+\alpha_E+\alpha_H,\alpha_C,0,\alpha_A+\alpha_B+\alpha_C),$
122	$(0,\alpha_D,-\alpha_B-\alpha_C-\alpha_D-\alpha_E,\alpha_C,-\alpha_H,0,\alpha_H,\alpha_B+\alpha_C+\alpha_D+\alpha_E+\alpha_H,$
	$0,\alpha_A+\alpha_B+\alpha_C),$
123	$(-\alpha_A,\alpha_D,\alpha_A,\alpha_C,-\alpha_B-\alpha_C-\alpha_D-\alpha_E-\alpha_H,0,\alpha_B+\alpha_C+\alpha_D+\alpha_E+\alpha_H,$
	$\alpha_B+\alpha_C+\alpha_D+\alpha_E+\alpha_H,0,\alpha_B+\alpha_C),$
124	$(0,\alpha_D,-\alpha_B-\alpha_C-\alpha_D-\alpha_E,\alpha_C,0,-\alpha_A-\alpha_B-\alpha_C,0,$
	$\alpha_A+2\alpha_B+2\alpha_C+\alpha_D+\alpha_E,\alpha_H,0),$
125	$(0,\alpha_D,0,-\alpha_A-\alpha_B+\alpha_C,\alpha_A+\alpha_B,-\alpha_C,0,\alpha_C,$
	$\alpha_B+\alpha_C+\alpha_D+\alpha_E+\alpha_H,0),$
126	$(-\alpha_A,\alpha_D,\alpha_A,\alpha_C,0,-\alpha_B-\alpha_C,0,\alpha_B+\alpha_C,\alpha_B+\alpha_C+\alpha_D+\alpha_E+\alpha_H,0),$
127	$(-\alpha_A,\alpha_D,\alpha_A,-\alpha_B+\alpha_C,\alpha_B,0,0,0,\alpha_B+\alpha_C+\alpha_D+\alpha_E+\alpha_H,\alpha_C),$
128	$(-\alpha_D-\alpha_E,\alpha_D,0,\alpha_C,0,\alpha_D+\alpha_E,0,0,\alpha_B+\alpha_C+\alpha_H,\alpha_A+\alpha_B+\alpha_C),$
129	$(0,\alpha_D,0,-\alpha_B+\alpha_C-\alpha_D-\alpha_E-\alpha_H,0,\alpha_B+\alpha_D+\alpha_E+\alpha_H,$
	$\alpha_B+\alpha_D+\alpha_E+\alpha_H,0,\alpha_C,\alpha_A+\alpha_B+\alpha_C),$
130	$(0,\alpha_D,0,-\alpha_A-\alpha_B+\alpha_C,\alpha_A+\alpha_B,0,0,0,\alpha_B+\alpha_C+\alpha_D+\alpha_E+\alpha_H,$
	$\alpha_C),$
131	$(0,-\alpha_B+\alpha_D-\alpha_E,-\alpha_B-\alpha_E,\alpha_C,\alpha_B+\alpha_E,0,0,0,\alpha_C+\alpha_D+\alpha_H,$
	$\alpha_A+\alpha_B+\alpha_C),$
132	$(0,\alpha_D,-\alpha_B-\alpha_C-\alpha_D-\alpha_E,\alpha_C,0,0,0,\alpha_B+\alpha_C+\alpha_D+\alpha_E,\alpha_H,$

Region of linearity	$\varepsilon(v)$
	$\alpha_A+\alpha_B+\alpha_C)$,
133	$(-\alpha_A-\alpha_B-\alpha_C-\alpha_D-\alpha_E,\alpha_D,\alpha_A,-\alpha_B+\alpha_C-\alpha_H,\alpha_B,$ $\alpha_B+\alpha_D+\alpha_E+\alpha_H,\alpha_H,\alpha_C,0,0)$,
134	$(-\alpha_A-\alpha_B-\alpha_C-\alpha_D-\alpha_E,-\alpha_B+\alpha_D-\alpha_E,\alpha_A,\alpha_C+\alpha_E,\alpha_B,\alpha_D,0,\alpha_C,$ $\alpha_H,0)$,
135	$(-\alpha_A-\alpha_B-\alpha_D-\alpha_E,-\alpha_B+\alpha_D-\alpha_E-\alpha_H,\alpha_A,\alpha_C+\alpha_E,\alpha_B,\alpha_D,\alpha_H,0,$ $\alpha_C,\alpha_C)$,
136	$(-\alpha_D-\alpha_E,-\alpha_B+\alpha_D-\alpha_E-\alpha_H,-\alpha_B-\alpha_C,\alpha_C+\alpha_E,\alpha_B,\alpha_D,\alpha_H,\alpha_C,0,$ $\alpha_A+\alpha_B+\alpha_C)$,
137	$(0,-\alpha_B+\alpha_D-\alpha_E-\alpha_H,-\alpha_B-\alpha_C-\alpha_D-\alpha_E,-\alpha_A-\alpha_B-\alpha_D,$ $\alpha_A+2\alpha_B+\alpha_C+\alpha_D+\alpha_E,\alpha_D,\alpha_H,\alpha_C,0,0)$,
138	$(-\alpha_E,\alpha_D-\alpha_E,0,\alpha_C+\alpha_E,0,0,0,0,\alpha_B+\alpha_C+\alpha_D+\alpha_H,$ $\alpha_A+\alpha_B+\alpha_C)$,
139	$(0,-\alpha_B+\alpha_D-\alpha_E-\alpha_H,0,\alpha_C,0,0,\alpha_B+\alpha_E+\alpha_H,0,\alpha_C+\alpha_D,$ $\alpha_A+\alpha_B+\alpha_C)$,
140	$(0,\alpha_D,0,\alpha_C,-\alpha_B-\alpha_C-\alpha_D-\alpha_E-\alpha_H,0,\alpha_B+\alpha_C+\alpha_D+\alpha_E+\alpha_H,$ $\alpha_B+\alpha_C+\alpha_D+\alpha_E+\alpha_H,0,\alpha_A+\alpha_B+\alpha_C)$,
141	$(0,\alpha_D,0,\alpha_C,0,-\alpha_A-\alpha_B-\alpha_C,0,\alpha_A+\alpha_B+\alpha_C,$ $\alpha_B+\alpha_C+\alpha_D+\alpha_E+\alpha_H,0)$,
142	$(-\alpha_A,\alpha_D,\alpha_A,\alpha_C,0,0,0,0,\alpha_B+\alpha_C+\alpha_D+\alpha_E+\alpha_H,\alpha_B+\alpha_C)$,
143	$(-\alpha_A-\alpha_B-\alpha_C-\alpha_D-\alpha_E,-\alpha_B+\alpha_D-\alpha_E-\alpha_H,\alpha_A,\alpha_C+\alpha_E,\alpha_B,\alpha_D,\alpha_H,$ $\alpha_C,0,0)$,
144	$(0,\alpha_D,0,\alpha_C,0,0,0,0,\alpha_B+\alpha_C+\alpha_D+\alpha_E+\alpha_H,\alpha_A+\alpha_B+\alpha_C)$.

We note that not all the regions of linearity are defined by the same number of inequalities. There are 62 regions (1 to 62) defined by 6 inequalities, 70 regions (63 to 132) defined by 7 inequalities, 10 regions defined by 8 inequalities (133 to 142) and 2 regions (143 to 144) defined by 11 inequalities.

We note in particular that the number (62) of regions defined by the minimum number of inequalities is equal to the number of equivalence classes of reduced expressions for w_0. We shall show that there is in fact a natural bijection between these two sets.

Now the vectors which parametrize the PBW-basis elements of U^- have all coordinates in $\mathbb{N}$. Thus we shall add the inequalities

$$a \geq 0,\ b \geq 0,\ c \geq 0,\ d \geq 0,\ e \geq 0,\ f \geq 0,\ g \geq 0,\ h \geq 0,\ i \geq 0,\ j \geq 0$$

to those of Table III.

We define a *simplicial region* to be the subset of $\mathbb{R}^{10}$ defined by one of the sets of 6 inequalities in Table III together with the above inequalities asserting that all coordinates are nonnegative. Of these 10 latter inequalities there will be 4 from which the remaining 6 follow using the 6 inequalities from Table III. Thus each simplicial region will have 10 walls, 6 of which are given by Table III and the remaining 4 will have form that some coordinate is greater than or equal to zero. The 62 simplicial regions together with their walls are shown in Table V (where, for example, $abgh$ indicates the four walls $a \geq 0$, $b \geq 0$, $g \geq 0$ and $h \geq 0$).

Table V

The simplicial regions and their walls.

Region	Main walls	Coordinate walls
1	[A, BC, CD, E, BH] ‖ [BCD]	abgh
2	[A, BC, E, BH] ‖ [B, CD]	abgh
3	[ABC, CD, E, BH] ‖ [A, BCD]	begh
4	[A, BC, E, BCDH] ‖ [BCD, BH]	abgh
5	[A, CD, E, BH] ‖ [BC, D]	abgh
6	[A, BC, CD, BEH] ‖ [BCD, E]	afgh
7	[A, BC, BEH] ‖ [B, CD, E]	afgh
8	[ABC, E, BH] ‖ [A, B, CD]	begh
9	[ABC, E, BCDH] ‖ [A, BCD, BH]	begh
10	[A, E, BCDH] ‖ [BC, D, BH]	abgh

Region	Main walls	Coordinate walls
11	[A, CD, BEH] ∥ [BC, D, E]	afgh
12	[A, B, C, BEH] ∥ [CD, BE]	acfh
13	[B, ABC, E, H] ∥ [AB, CD]	bceh
14	[ABC, BCD, E, H] ∥ [ABCD, BH]	begh
15	[A, D, E, BCH] ∥ [BC, BH]	abdg
16	[A, C, D, BEH] ∥ [BC, DE]	adfg
17	[A, B, BEH] ∥ [C, D, BE]	achj
18	[AB, C, BEH] ∥ [A, CD, BE]	cefh
19	[ABC, BE, H] ∥ [AB, CD, E]	cefh
20	[B, ABC, E] ∥ [AB, CD, H]	bchi
21	[ABC, BCD, E] ∥ [B, ABCD, H]	bghi
22	[BCD, E, H] ∥ [ABC, D, BH]	begh
23	[D, E, BCH] ∥ [A, BC, BH]	bdeg
24	[A, DE, BCH] ∥ [BC, E, BH]	adfg
25	[A, C, BDEH] ∥ [BC, DE, BEH]	adfg
26	[A, D, BEH] ∥ [B, C, DE]	adgj
27	[A, B, D, BEH] ∥ [C, BDE]	acdj
28	[AB, C, BE, H] ∥ [CD, ABE]	cefh
29	[B, ABC, CD, E] ∥ [ABCD, H]	bchi
30	[BC, D, E, H] ∥ [ABC, BH]	bdeg
31	[A, C, DE, BH] ∥ [BC, BEH]	adfg
32	[A, BDEH] ∥ [B, C, DE, BEH]	adgj
33	[AB, BEH] ∥ [A, C, D, BE]	cehj

Region	Main walls	Coordinate walls
34	[ABC, BE] ‖ [AB, CD, E, H]	cfhi
35	[BCD, E] ‖ [B, ABC, D, H]	bghi
36	[DE, BCH] ‖ [A, BC, E, BH]	defg
37	[C, DE, BH] ‖ [A, BC, BEH]	defg
38	[A, DE, BH] ‖ [B, C, BEH]	adgj
39	[A, B, BDEH] ‖ [C, BDE, BEH]	acdj
40	[AB, D, BEH] ‖ [A, C, BDE]	cdej
41	[AB, BE, H] ‖ [C, D, ABE]	cehj
42	[AB, C, BE] ‖ [CD, ABE, H]	cfhi
43	[ABC, CD, BE] ‖ [ABCD, E, H]	cfhi
44	[B, CD, E] ‖ [ABC, D, H]	bchi
45	[BC, D, E] ‖ [B, ABC, H]	bdgi
46	[BC, DE, H] ‖ [ABC, E, BH]	defg
47	[DE, BH] ‖ [A, B, C, BEH]	degj
48	[AB, BDEH] ‖ [A, C, BDE, BEH]	cdej
49	[AB, BE] ‖ [C, D, ABE, H]	chij
50	[CD, BE] ‖ [ABC, D, E, H]	cfhi
51	[BC, DE] ‖ [B, ABC, E, H]	dfgi
52	[B, DE, H] ‖ [AB, C, BEH]	cdej
53	[AB, BDE, H] ‖ [C, ABDE, BEH]	cdej
54	[AB, D, BE] ‖ [C, ABDE, H]	cdij
55	[C, D, BE] ‖ [ABC, DE, H]	cdfi
56	[B, C, DE] ‖ [ABC, BE, H]	cdfi

Region	Main walls	Coordinate walls
57	[B, DE] ‖ [AB, C, BE, H]	cdij
58	[BDE, H] ‖ [AB, C, DE, BEH]	cdej
59	[AB, BDE] ‖ [C, BE, ABDE, H]	cdij
60	[D, BE] ‖ [AB, C, DE, H]	cdij
61	[C, BDE] ‖ [ABC, BE, DE, H]	cdfi
62	[BDE] ‖ [AB, C, BE, DE, H]	cdij

In order to define a natural bijection between the set of equivalence relations of reduced words for w_0 and the set of simplicial regions we consider the transforms under $S_{\mathbf{i}}^{\mathbf{j}}$ of the spanning vectors of $C_{\mathbf{i}}$ considered in §4. We have one such vector v_P for each partial quiver P and one vector v_j for each $j = 1, 2, 3, 4$. These vectors v_P, v_j can, for example, be calculated by the rectangle algorithm described in §4. These vectors are listed in Table VI.

We consider the incidence properties relating these vectors v_P and v_j to the walls of the simplicial regions. There are altogether 32 hyperplanes in $\mathbb{R}^{10}$ which arise as walls of simplicial regions. 22 of these are hyperplanes appearing in Table III and 10 are the hyperplanes obtained by putting one coordinate equal to 0. There are 22 vectors of form v_P and 4 vectors of form v_j. These are obtained from the rectangle algorithm which associates with each partial quiver P or each $j \in \{1, 2, 3, 4\}$ a set of positive roots. Such a set of positive roots is translated into a vector with components 0 or 1 by means of the ordering

$$(\alpha_1, \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_3 + \alpha_4, \alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \alpha_2, \alpha_4, \alpha_2)$$

determined by the reduced word $\mathbf{j} = (1, 3, 2, 4, 1, 3, 2, 4, 1, 3)$. In Table VII we indicate which vectors v_P and v_j lie on which walls of simplicial regions. A cross in a particular position indicates that the given vector does not lie on the given wall.

Table VI

Vectors obtained by the rectangle algorithm.

Partial quiver P	Vector v_P
LLL	(1,0,0,0,0,0,0,0,0,0)
LLR	(0,0,1,0,0,0,0,0,1,0)
LRL	(1,1,0,0,0,0,1,1,0,0)
RLL	(1,0,0,1,0,0,0,0,0,1)

Partial quiver P	Vector v_P
LRR	(0,1,0,0,0,0,0,1,1,0)
RLR	(0,0,1,1,0,0,0,0,1,1)
RRL	(1,0,0,0,0,0,1,0,0,0)
RRR	(0,0,0,0,0,0,0,0,1,0)
LL-	(0,0,0,0,0,0,0,1,0,0)
LR-	(0,1,0,0,0,1,0,0,0,0)
RL-	(0,0,0,0,0,1,0,0,0,1)
RR-	(0,0,0,1,0,0,0,0,0,0)
- LL	(1,0,0,0,0,0,0,0,0,1)
- LR	(0,0,1,0,0,0,0,0,1,1)
- RL	(1,1,0,0,0,0,1,0,0,0)
- RR	(0,1,0,0,0,0,0,0,1,0)
L- -	(0,1,0,0,0,0,0,1,0,0)
R- -	(0,0,0,1,0,0,0,0,0,1)
- L-	(0,0,1,0,0,0,0,0,0,1)
- R-	(0,1,0,0,0,0,1,0,0,0)
- - L	(1,0,0,0,1,0,0,0,0,0)
- - R	(0,0,0,0,1,0,0,0,1,0)

Integer j	Vector v_j
1	(1,0,0,0,1,0,0,0,1,0)
2	(0,0,1,0,0,0,1,0,0,0)
3	(0,1,0,0,0,1,0,0,0,1)
4	(0,0,0,1,0,0,0,1,0,0)

Table VII

Incidence table: spanning vectors and walls of simplicial regions.

x indicates that the vector lies on the given wall.

	L	L	L	R	L	R	R	R	L	L	R	R	-	-	-	-	L	R	-	-	-	-				
	L	L	R	L	R	L	R	R	L	R	L	R	L	L	R	R	-	-	L	R	-	-	1	2	3	4
	L	R	L	L	R	R	L	R	-	-	-	-	L	R	L	R	-	-	-	-	L	R				
A	.	x	.	.	x	x	.	x	x	x	x	x	.	x	.	x	x	x	x	x	x	.	x	x	x	x
B	x	.	.	x	x	.	.	x	x	x	x	x	x	.	.	x	x	x	.	.	x	x	x	x	x	x
C	x	x	x	.	x	.	x	x	x	.	x	x	.	.	x	x	x	.	.	x	x	x	x	x	x	x
D	x	x	.	.	.	.	x	x	.	x	x	.	x	x	x	x	.	.	x	x	x	x	x	x	x	x
E	x	x	.	x	.	x	x	x	x	x	.	x	x	x	.	.	.	x	x	.	x	x	x	x	x	x
H	x	.	x	x	.	.	x	.	x	x	x	x	x	.	x	.	x	x	x	x	.	x	x	x	x	x
AB	.	.	x	.	x	.	x	x	x	x	x	x	.	.	x	x	x	x	.	.	x	.	x	x	x	x
BC	x	.	.	.	x	x	.	x	x	.	x	x	.	x	.	x	x	.	x	.	x	x	x	x	x	x
CD	x	x	.	x	.	x	x	x	.	.	x	.	.	.	x	x	.	x	.	x	x	x	x	x	x	x
DE	x	x	x	.	x	.	x	x	.	x	.	.	x	x	.	.	x	.	x	.	x	x	x	x	x	x
BE	x	.	x	x	.	.	.	x	x	x	.	x	x	.	x	.	.	x	.	x	x	x	x	x	x	x
BH	x	x	.	x	.	x	.	.	x	x	x	x	x	x	.	.	x	x	.	.	.	x	x	x	x	x
ABC	.	.	x	x	x	x	x	x	x	.	x	x	x	x	x	x	x	.	x	.	x	.	x	x	x	x
BCD	x	.	.	x	.	.	.	x	.	.	x	.	.	x	.	x	.	x	x	.	x	x	x	x	x	x
ABE	.	.	.	.	.	.	x	x	x	x	.	x	.	.	.	.	.	x	.	x	x	.	x	x	x	x
BDE	x	.	.	.	x	.	.	x	.	x	.	.	x	.	x	.	x	.	.	x	x	x	x	x	x	x
BCH	x	x	.	.	.	.	.	.	x	.	x	x	.	.	.	.	x	.	x	.	.	x	x	x	x	x
BEH	x	x	x	x	x	x	.	.	x	x	.	x	x	x	x	x	.	x	.	x	.	x	x	x	x	x
ABCD	.	.	.	.	.	.	x	x	.	.	x	.	x	x	x	x	.	x	x	.	x	.	x	x	x	x
ABDE	.	.	x	.	x	.	x	x	.	x	.	.	.	.	.	.	x	.	.	x	x	.	x	x	x	x
BCDH	x	x	.	x	.	x	.	.	.	.	x	.	.	.	.	.	.	x	x	.	.	x	x	x	x	x
BDEH	x	x	.	.	.	.	.	.	.	x	.	.	x	x	x	x	x	.	.	x	.	x	x	x	x	x
a	.	x	.	.	x	x	.	x	x	x	x	x	.	x	.	x	x	x	x	x	.	x	.	x	x	x
b	x	x	.	x	.	x	x	x	x	.	x	x	x	x	.	.	.	x	x	.	x	x	x	x	.	x
c	x	.	x	x	x	.	x	x	x	x	x	x	x	.	x	x	x	x	.	x	x	x	x	.	x	x
d	x	x	x	.	x	.	x	x	x	x	x	.	x	x	x	x	x	.	x	x	x	x	x	x	x	.
e	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x	.	.	.	x	x	x
f	x	x	x	x	x	x	x	x	x	.	.	x	x	x	x	x	x	x	x	x	x	x	x	x	.	x
g	x	x	.	x	x	x	.	x	x	x	x	x	x	x	.	x	x	x	x	.	x	x	x	.	x	x
h	x	x	.	x	.	x	x	x	.	x	x	x	x	x	x	x	.	x	x	x	x	x	x	x	x	.
i	x	.	x	x	.	.	x	.	x	x	x	x	x	.	x	.	x	x	x	x	x	.	.	x	x	x
j	x	x	x	.	x	.	x	x	x	x	.	x	.	.	x	x	x	.	.	x	x	x	x	x	.	x

The incidence table enables us to describe a bijection between equivalence classes of reduced words and simplicial regions. Let $\mathcal{W}$ be the set of hyperplanes in $\mathbb{R}^{10}$ which arise as walls of simplicial regions. We have seen that $|\mathcal{W}| = 22 + 10 = 32$.

Proposition 5.1 *Let* $\mathbf{i}$ *be a reduced expression for* w_0. *Let* $\mathcal{P}(\mathbf{i})$ *be the set of* 6 *partial quivers obtained from* $\mathbf{i}$ *using the chamber sets as in* §3. *Consider the* 10 *vectors* $\{v_P,\ P \in \mathcal{P}(\mathbf{i});\ v_j,\ j = 1,2,3,4\}$. *For each vector in the set consider the set of* 9 *remaining vectors. Then there is a unique hyperplane* $W \in \mathcal{W}$ *such that these* 9 *vectors lie on* W. *Moreover, the set of* 10 *elements of* $\mathcal{W}$ *obtained in this way are the* 10 *boundary walls of a simplicial region* $X_{\mathbf{i}}$. *The original* 10 *vectors* $v_P,\ P \in \mathcal{P}(\mathbf{i});\ v_j,\ j = 1,2,3,4$ *all lie in* $X_{\mathbf{i}}$. *Also, the map* $\mathbf{i} \to X_{\mathbf{i}}$ *gives a bijection between equivalence classes of reduced words and simplicial regions.*

The bijection $\mathbf{i} \to X_{\mathbf{i}}$ is described in Table VIII.

Corollary 5.2 *The vectors* $v_P,\ P \in \mathcal{P}(\mathbf{i});\ v_j,\ j = 1,2,3,4$ *are spanning vectors of the region* $X_{\mathbf{i}}$.

Proof: This follows from the fact that each vector lies in $X_{\mathbf{i}}$ and that each wall of $X_{\mathbf{i}}$ contains 9 of the 10 vectors. □

Table VIII

The correspondence between reduced words and simplicial regions.

Simplicial region	Reduced word
1	[1, 3, 2, 1, 3, 4, 3, 2, 1, 3]
2	[1, 3, 2, 1, 3, 4, 2, 3, 2, 1]
3	[3, 2, 1, 2, 3, 4, 3, 2, 1, 3]
4	[1, 3, 2, 4, 1, 3, 2, 4, 1, 3]
5	[1, 3, 2, 3, 4, 3, 2, 1, 2, 3]
6	[1, 2, 3, 2, 1, 4, 3, 2, 1, 3]
7	[1, 2, 3, 2, 1, 4, 2, 3, 2, 1]
8	[3, 2, 1, 2, 3, 4, 2, 3, 2, 1]
9	[3, 2, 1, 2, 4, 3, 2, 4, 1, 3]

Simplicial region	Reduced word
10	[1, 3, 2, 4, 3, 2, 1, 2, 4, 3]
11	[1, 2, 3, 2, 4, 3, 2, 1, 2, 3]
12	[1, 2, 3, 1, 2, 1, 4, 3, 2, 1]
13	[3, 2, 1, 3, 2, 3, 4, 3, 2, 1]
14	[3, 2, 1, 4, 3, 2, 3, 4, 1, 3]
15	[1, 3, 4, 3, 2, 3, 1, 2, 4, 3]
16	[1, 2, 3, 4, 3, 2, 3, 1, 2, 3]
17	[1, 2, 3, 1, 2, 4, 3, 2, 1, 2]
18	[2, 1, 2, 3, 2, 1, 4, 3, 2, 1]
19	[2, 3, 2, 1, 2, 3, 4, 3, 2, 1]
20	[3, 2, 1, 3, 4, 2, 3, 2, 4, 1]
21	[3, 2, 1, 4, 3, 2, 4, 3, 4, 1]
22	[3, 2, 4, 3, 2, 1, 2, 3, 4, 3]
23	[3, 4, 3, 2, 1, 2, 3, 2, 4, 3]
24	[1, 4, 3, 4, 2, 3, 1, 2, 4, 3]
25	[1, 2, 4, 3, 4, 2, 3, 1, 2, 3]
26	[1, 2, 3, 4, 3, 2, 1, 2, 3, 2]
27	[1, 2, 3, 4, 3, 1, 2, 1, 3, 2]
28	[2, 3, 1, 2, 1, 3, 4, 3, 2, 1]
29	[3, 2, 1, 3, 4, 3, 2, 3, 4, 1]
30	[3, 4, 3, 2, 3, 1, 2, 3, 4, 3]
31	[1, 4, 3, 2, 3, 4, 3, 1, 2, 3]

Simplicial region	Reduced word
32	[1, 2, 4, 3, 4, 2, 1, 2, 3, 2]
33	[2, 1, 2, 3, 2, 4, 3, 2, 1, 2]
34	[2, 3, 2, 1, 4, 2, 3, 2, 4, 1]
35	[3, 2, 4, 3, 2, 1, 2, 4, 3, 4]
36	[4, 3, 4, 2, 1, 2, 3, 2, 4, 3]
37	[4, 3, 2, 1, 2, 3, 4, 3, 2, 3]
38	[1, 4, 3, 2, 3, 4, 1, 2, 3, 2]
39	[1, 2, 4, 3, 4, 1, 2, 1, 3, 2]
40	[2, 1, 2, 3, 4, 3, 2, 1, 3, 2]
41	[2, 3, 1, 2, 3, 4, 3, 2, 1, 2]
42	[2, 3, 1, 2, 1, 4, 3, 2, 4, 1]
43	[2, 3, 2, 1, 4, 3, 2, 3, 4, 1]
44	[3, 2, 3, 4, 3, 2, 1, 2, 3, 4]
45	[3, 4, 3, 2, 3, 1, 2, 4, 3, 4]
46	[4, 3, 4, 2, 3, 1, 2, 3, 4, 3]
47	[4, 3, 2, 1, 2, 3, 4, 2, 3, 2]
48	[2, 1, 2, 4, 3, 4, 2, 1, 3, 2]
49	[2, 3, 1, 2, 4, 3, 2, 1, 2, 4]
50	[2, 3, 2, 4, 3, 2, 1, 2, 3, 4]
51	[4, 3, 4, 2, 3, 1, 2, 4, 3, 4]
52	[4, 3, 2, 1, 3, 2, 3, 4, 3, 2]
53	[2, 1, 4, 3, 2, 3, 4, 1, 3, 2]

Simplicial region	Reduced word
54	[2, 3, 4, 3, 1, 2, 1, 3, 2, 4]
55	[2, 3, 4, 3, 2, 3, 1, 2, 3, 4]
56	[4, 3, 2, 3, 4, 3, 1, 2, 3, 4]
57	[4, 3, 2, 3, 4, 1, 2, 3, 2, 4]
58	[4, 2, 3, 2, 1, 2, 3, 4, 3, 2]
59	[4, 2, 3, 4, 1, 2, 1, 3, 2, 4]
60	[2, 3, 4, 3, 2, 1, 2, 3, 2, 4]
61	[4, 2, 3, 2, 4, 3, 1, 2, 3, 4]
62	[4, 2, 3, 2, 4, 1, 2, 3, 2, 4]

6 The transition function $S_{\mathbf{i}}^{\mathbf{j}}$ on the Lusztig cone $C_{\mathbf{i}}$

We shall now describe an algorithm for calculating $S_{\mathbf{i}}^{\mathbf{j}}(\mathbf{a})$ for $\mathbf{a} \in C_{\mathbf{i}}$. As before $\mathbf{j}$ is the reduced word $(1,3,2,4,1,3,2,4,1,3)$. The algorithm is best indicated by an example. We illustrate in the case when $\mathbf{i} = (3,2,1,4,3,2,3,4,1,3)$. The Lusztig cone for this reduced word $\mathbf{i}$ is given by

$$C_{\mathbf{i}} = \{\mathbf{a} \in \mathbb{Z}^{10} \,:\, P_{\mathbf{i}}\mathbf{a} \geq 0\},$$

where

$$P_{\mathbf{i}} = \begin{pmatrix} -1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Thus $\mathbf{a} \in C_{\mathbf{i}}$ if and only if $\mathbf{a} = a\mathbf{c}_1 + b\mathbf{c}_2 + c\mathbf{c}_3 + d\mathbf{c}_4 + e\mathbf{c}_5 + f\mathbf{c}_6 + g\mathbf{c}_7 + h\mathbf{c}_8 + i\mathbf{c}_9 + j\mathbf{c}_{10}$, where $\mathbf{c}_1, \mathbf{c}_2, \dots \mathbf{c}_{10}$ are the columns of $Q_{\mathbf{i}} = P_{\mathbf{i}}^{-1}$ and $a \geq 0,\ b \geq 0, \dots, j \geq 0$. We have

$$Q_{\mathbf{i}} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}.$$

Thus a typical point $\mathbf{a} \in C_{\mathbf{i}}$ is given by

$$\begin{aligned} \mathbf{a} \;=\;& (i, a+d+f+h+i, a+b+d+e+f+g+h+i, a+b+c+i+j, \\ & a+b+c+d+f+h+i+j, a+b+c+d+e+f+g+h+i+j, g, f+g+h, j, h), \end{aligned}$$

where $a \geq 0,\ b \geq 0, \dots, j \geq 0$.

We now consider $S_{\mathbf{i}}^{\mathbf{j}}(\mathbf{a})$. We recall that $S_{\mathbf{i}}^{\mathbf{j}} = \phi_{\mathbf{j}} \psi_{\mathbf{i}}^{-1}$ where $\psi_{\mathbf{i}} : \mathbf{B} \to Y_{\mathbf{i}}$ and $\phi_{\mathbf{j}} : B \to \mathbb{N}^k$ are bijections. Let $\psi_{\mathbf{i}}^{-1}(\mathbf{a}) = b$ and $\phi_{\mathbf{i}}^{-1}(b) = \mathbf{c}$. Then $S_{\mathbf{i}}^{\mathbf{j}}(\mathbf{a}) = \mathbf{c}$. We also recall that

$$b \equiv \widetilde{F}_{i_1}^{a_1} \widetilde{F}_{i_2}^{a_2} \cdots \widetilde{F}_{i_k}^{a_k} \cdot 1 \mod v\mathcal{L}', \quad \text{and}$$
$$b \equiv F_{\mathbf{j}}^{\mathbf{c}} \mod v\mathcal{L},$$

where $\mathcal{L}$ is the $\mathbb{Z}[v]$-lattice spanned by $\mathbf{B}$ and $\mathcal{L}'$ is the $\mathcal{A}$-lattice spanned by $\mathbf{B}$. Since $\mathbb{Z}[v] \subseteq \mathcal{A}$ we have $\mathcal{L} \subseteq \mathcal{L}'$. Thus

$$\widetilde{F}_{i_1}^{a_1} \widetilde{F}_{i_2}^{a_2} \cdots \widetilde{F}_{i_k}^{a_k} \cdot 1 \equiv F_{\mathbf{j}}^{\mathbf{c}} \mod v\mathcal{L}'.$$

We shall describe an algorithm for obtaining the vector $\mathbf{c}$ determined by this condition. This is based on the fact that, since the reduced expression $\mathbf{j}$ is adapted to the quiver Q_1 in Figure 11, (in the sense of [8]), the actions of $\widetilde{F}_1$ and $\widetilde{F}_3$ on elements $F_{\mathbf{j}}^{\mathbf{c}}$, taken $\mod v\mathcal{L}'$, are given by a simple formula. By [9, Cor. 2.5] we have

$$\begin{aligned} \widetilde{F}_1^x F_{\mathbf{j}}^{\mathbf{c}} &\equiv F_{\mathbf{j}}^{\mathbf{c}'} \mod v\mathcal{L}', \\ \widetilde{F}_3^x F_{\mathbf{j}}^{\mathbf{c}} &\equiv F_{\mathbf{j}}^{\mathbf{c}''} \mod v\mathcal{L}', \end{aligned}$$

where

$$\begin{aligned} \mathbf{c}' &= \mathbf{c} + (x, 0, 0, 0, 0, 0, 0, 0, 0, 0), \\ \mathbf{c}'' &= \mathbf{c} + (0, x, 0, 0, 0, 0, 0, 0, 0, 0). \end{aligned}$$

Similarly, the reduced expression $\mathbf{j}'$ is adapted to the quiver Q_2 in Figure 11, so the actions of $\widetilde{F}_2$ and $\widetilde{F}_4$ on elements $F_{\mathbf{j}'}^{\mathbf{c}}$, taken mod $v\mathcal{L}'$, are given by

$$\begin{aligned} \widetilde{F}_2^x F_{\mathbf{j}'}^{\mathbf{c}} &\equiv F_{\mathbf{j}'}^{\mathbf{c}'} \mod v\mathcal{L}', \\ \widetilde{F}_4^x F_{\mathbf{j}'}^{\mathbf{c}} &\equiv F_{\mathbf{j}'}^{\mathbf{c}''} \mod v\mathcal{L}', \end{aligned}$$

where

$$\begin{aligned}
\mathbf{c}' &= \mathbf{c} + (x,0,0,0,0,0,0,0,0,0), \\
\mathbf{c}'' &= \mathbf{c} + (0,x,0,0,0,0,0,0,0,0).
\end{aligned}$$

Q_1 Q_2

Figure 11: Two quivers of type A_4.

In order to calculate $\widetilde{F}_3^{a_1}\widetilde{F}_2^{a_2}\widetilde{F}_1^{a_3}\widetilde{F}_4^{a_4}\widetilde{F}_3^{a_5}\widetilde{F}_2^{a_6}\widetilde{F}_3^{a_7}\widetilde{F}_4^{a_8}\widetilde{F}_1^{a_9}\widetilde{F}_3^{a_{10}}\cdot 1 \mod v\mathcal{L}'$ we shall write the element being acted on at each stage in the form $F_{\mathbf{j}}^{\mathbf{c}}$ or $F_{\mathbf{j}'}^{\mathbf{c}'}$ as appropriate, in order to be able to use the above formulae. The elements can be transformed from form $F_{\mathbf{j}}^{\mathbf{c}}$ to $F_{\mathbf{j}'}^{\mathbf{c}'}$ by using the transition function $R = R_{\mathbf{j}}^{\mathbf{j}'}$ and from form $F_{\mathbf{j}'}^{\mathbf{c}'}$ to $F_{\mathbf{j}}^{\mathbf{c}}$ by $R^{-1} = R_{\mathbf{j}'}^{\mathbf{j}}$. It follows from the description of the transition function in [8] and the form of $\mathbf{j}$ and $\mathbf{j}'$ that $R_{\mathbf{j}'}^{\mathbf{j}} = \tau R_{\mathbf{j}}^{\mathbf{j}'}\tau$ where $\tau = (1\,2)(3\,4)(5\,6)(7\,8)(9\,10)$. Thus $R^{-1} = \tau R \tau$ where $\tau(a,b,c,d,e,f,g,h,i,j) = (b,a,d,c,f,e,h,g,j,i)$.

We now apply this algorithm to our given example where

$$(a_1,a_2,a_3,a_4,a_5,a_6,a_7,a_8,a_9,a_{10}) = (i, a+d+f+h+i, a+b+d+e+f+g+h+i, a+b+c+i+j,$$

$$a+b+c+d+f+h+i+j, a+b+c+d+e+f+g+h+i+j, g, f+g+h, j, h).$$

We have

$$\widetilde{F}_3^{a_{10}} \cdot 1 = \widetilde{F}_3^{a_{10}} \cdot F_{\mathbf{j}}^{\mathbf{0}} \equiv F_{\mathbf{j}}^{\mathbf{c_1}} \mod v\mathcal{L}',$$

where $\mathbf{c_1} = (0,h,0,0,0,0,0,0,0,0)$. Also,

$$\widetilde{F}_1^{a_9} F_{\mathbf{j}}^{\mathbf{c_1}} \equiv F_{\mathbf{j}}^{\mathbf{c_2}} \mod v\mathcal{L}',$$

where $\mathbf{c_2} = (j,h,0,0,0,0,0,0,0,0)$. One can then check that $\mathbf{c_2}$ lies in region 144 of §5 and it follows from Table IV that $F_{\mathbf{j}}^{\mathbf{c_2}} \equiv F_{\mathbf{j}'}^{\mathbf{c_3}} \mod v\mathcal{L}'$, where $\mathbf{c_3} = (0,0,0,0,0,0,0,0,h,j)$. Hence $\widetilde{F}_4^{a_8} F_{\mathbf{j}'}^{\mathbf{c_3}} \equiv F_{\mathbf{j}'}^{\mathbf{c_4}} \mod v\mathcal{L}'$, where $\mathbf{c_4} = (0,f+g+h,0,0,0,0,0,0,h,j)$.

We now apply $R_{\mathbf{j}'}^{\mathbf{j}}$ to $\mathbf{c_4}$. Applying τ to this vector we obtain $\mathbf{c_5} = (f+g+h,0,0,0,0,0,0,0,j,h)$. We check that this vector lies in region 136 of §5. Hence $R_{\mathbf{j}}^{\mathbf{j}'}(\mathbf{c_5}) = \mathbf{c_6}$, where $\mathbf{c_6} = (0,j,h,0,0,0,0,0,0,f+g)$. Applying τ again we obtain $\mathbf{c_7} = (j,0,0,h,0,0,0,0,f+g,0)$. Hence

$$F_{\mathbf{j}'}^{\mathbf{c_4}} \equiv F_{\mathbf{j}}^{\mathbf{c_7}} \mod v\mathcal{L}'.$$

Then

$$\widetilde{F}_3^{a_7} F_{\mathbf{j}}^{\mathbf{c_7}} \equiv F_{\mathbf{j}}^{\mathbf{c_8}} \mod v\mathcal{L}',$$

where $\mathbf{c_8} = (j,g,0,h,0,0,0,0,f+g,0)$. This vector lies in region 139 of §5. Hence

$$F_{\mathbf{j}}^{\mathbf{c_8}} \equiv F_{\mathbf{j}'}^{\mathbf{c_9}} \mod v\mathcal{L}',$$

where $\mathbf{c_9} = (0, f+h, 0,0,0,0,g,0,h,j)$. Then

$$\widetilde{F}_2^{a_6} F_{\mathbf{j}'}^{\mathbf{c_9}} \equiv F_{\mathbf{j}'}^{\mathbf{c_{10}}} \mod v\mathcal{L}',$$

where $\mathbf{c_{10}} = (a+b+c+d+e+f+g+h+i+j, f+h, 0,0,0,0,g,0,h,j)$.

We now apply $R_{\mathbf{j}'}^{\mathbf{j}}$ to $\mathbf{c_{10}}$. Applying τ we obtain $\mathbf{c_{11}} = (f+h, a+b+c+d+e+f+g+h+i+j, 0,0,0,0,0,g,j,h)$. This lies in region 144 of §5. Hence $R_{\mathbf{j}}^{\mathbf{j}'}(\mathbf{c_{11}}) = \mathbf{c_{12}}$, where $\mathbf{c_{12}} = (0,0,0,0,0,g,j,h,a+b+c+d+e+f+i, f+g)$. Applying τ again we obtain $\mathbf{c_{13}} = (0,0,0,0,g,0,h,j,f+g,a+b+c+d+e+f+i)$. Hence

$$F_{\mathbf{j}'}^{\mathbf{c_{10}}} \equiv F_{\mathbf{j}}^{\mathbf{c_{13}}} \mod v\mathcal{L}'.$$

Then $\widetilde{F}_3^{a_5} F_{\mathbf{j}}^{\mathbf{c_{13}}} \equiv F_{\mathbf{j}}^{\mathbf{c_{14}}} \mod v\mathcal{L}'$, where $\mathbf{c_{14}} = (0, a+b+c+d+f+h+i+j, 0,0,g,0,h,j,f+g,a+b+c+d+e+f+i)$. This lies in region 133 of §5. Hence

$$F_{\mathbf{j}}^{\mathbf{c_{14}}} \equiv F_{\mathbf{j}'}^{\mathbf{c_{15}}} \mod v\mathcal{L}',$$

where $\mathbf{c_{15}} = (e+g, 0,0,f+h, 0, a+b+c+d+i+j, g, 0, h, j)$. Thus

$$\widetilde{F}_4^{a_4} F_{\mathbf{j}'}^{\mathbf{c_{15}}} \equiv F_{\mathbf{j}'}^{\mathbf{c_{16}}} \mod v\mathcal{L}',$$

where $\mathbf{c_{16}} = (e+g, a+b+c+i+j, 0, f+h, 0, a+b+c+d+i+j, g, 0, h, j)$.

We now apply $R_{\mathbf{j}'}^{\mathbf{j}}$ to $\mathbf{c_{16}}$. Applying τ gives $\mathbf{c_{17}} = (a+b+c+i+j, e+g, f+h, 0, a+b+c+d+i+j, 0,0,g,j,h)$. This lies in region 142 of §5. Hence $R_{\mathbf{j}}^{\mathbf{j}'}(\mathbf{c_{17}}) = \mathbf{c_{18}}$, where $\mathbf{c_{18}} = (d+f+h, 0, a+b+c+i+j, 0,0,g,j,h,a+b+c+d+e+f+i, f+g)$. Applying τ again gives $\mathbf{c_{19}} = (0, d+f+h, 0, a+b+c+i+j, g, 0, h, j, f+g, a+b+c+d+e+f+i)$. Hence

$$F_{\mathbf{j}'}^{\mathbf{c_{16}}} \equiv F_{\mathbf{j}}^{\mathbf{c_{19}}} \mod v\mathcal{L}'.$$

We have $\widetilde{F}_1^{a_3} F_{\mathbf{j}}^{\mathbf{c_{19}}} \equiv F_{\mathbf{j}}^{\mathbf{c_{20}}} \mod v\mathcal{L}'$, where $\mathbf{c_{20}} = (a+b+d+e+f+g+h+i, d+f+h, 0, a+b+c+i+j, g, 0, h, j, f+g, a+b+c+d+e+f+i)$. This lies in region 104. Hence

$$F_{\mathbf{j}}^{\mathbf{c_{20}}} \equiv F_{\mathbf{j}'}^{\mathbf{c_{21}}} \mod v\mathcal{L}',$$

where $\mathbf{c_{21}} = (0, a+b+c+i+j, e+g, 0, f+h, c+j, g, a+b+d+i, h, j)$. Thus

$$\widetilde{F}_2^{a_2} F_{\mathbf{j}'}^{\mathbf{c_{21}}} \equiv F_{\mathbf{j}'}^{\mathbf{c_{22}}} \mod v\mathcal{L}',$$

where $\mathbf{c_{22}} = (a+d+f+h+i, a+b+c+i+j, e+g, 0, f+h, c+j, g, a+b+d+i, h, j)$.

We now apply $R_{\mathbf{j}'}^{\mathbf{j}}$ to $\mathbf{c_{22}}$. Applying τ gives $\mathbf{c_{23}} = (a+b+c+i+j, a+d+f+h+i, 0, e+g, c+j, f+h, a+b+d+i, g, j, h)$. This lies in region 49. Hence $R_{\mathbf{j}}^{\mathbf{j}'}(\mathbf{c_{23}}) = \mathbf{c_{24}}$, where $\mathbf{c_{24}} = (0, b+e+g, b+c+j, d+f+h, a+i, g, j, h, a+b+c+d+e+f+i, f+g)$. Applying τ again gives $\mathbf{c_{25}} = (b+e+g, 0, d+f+h, b+c+j, g, a+i, h, j, f+g, a+b+c+d+e+f+i)$. Thus

$$F_{\mathbf{j}'}^{\mathbf{c_{22}}} \equiv F_{\mathbf{j}}^{\mathbf{c_{25}}} \mod v\mathcal{L}'.$$

Thus $\widetilde{F}_3^{a_1} F_{\mathbf{j}}^{\mathbf{c_{25}}} \equiv F_{\mathbf{j}}^{\mathbf{c_{26}}} \mod v\mathcal{L}'$, where $\mathbf{c_{26}} = (b+e+g, i, d+f+h, b+c+j, g, a+i, h, j, f+g, a+b+c+d+e+f+i)$.

Thus we finally arrive at the conclusion that

$$\widetilde{F}_3^{a_1}\widetilde{F}_2^{a_2}\widetilde{F}_1^{a_3}\widetilde{F}_4^{a_4}\widetilde{F}_3^{a_5}\widetilde{F}_2^{a_6}\widetilde{F}_3^{a_7}\widetilde{F}_4^{a_8}\widetilde{F}_1^{a_9}\widetilde{F}_3^{a_{10}}\cdot 1 \equiv F_{\mathbf{j}}^{\mathbf{c}} \mod v\mathcal{L}',$$

where $\mathbf{c} = (b+e+g, i, d+f+h, b+c+j, g, a+i, h, j, f+g, a+b+c+d+e+f+i)$.

This procedure can be carried out for each reduced expression $\mathbf{i}$ for w_0, i.e. for representatives of the 62 commutation classes of such reduced expressions. This was carried out by computer and in each case the coordinates of $\mathbf{c}$ are seen to be positive linear combinations of $a, b, c, d, e, f, g, h, i, j$. Thus we have obtained:

Proposition 6.1 *Let $\mathbf{i}$ be a reduced expression for w_0 in type A_4. Then $S_{\mathbf{i}}^{\mathbf{j}}(C_{\mathbf{i}}) = X_{\mathbf{i}}$. Moreover the map $S_{\mathbf{i}}^{\mathbf{j}} : C_{\mathbf{i}} \to X_{\mathbf{i}}$ is linear.*

Proof: We know from Corollary 5.2 that $S_{\mathbf{i}}^{\mathbf{j}}$ maps the spanning vectors of $C_{\mathbf{i}}$ to spanning vectors of $X_{\mathbf{i}}$. Since $S_{\mathbf{i}}^{\mathbf{j}}$ is linear on $C_{\mathbf{i}}$ it follows that $S_{\mathbf{i}}^{\mathbf{j}}(C_{\mathbf{i}}) = X_{\mathbf{i}}$. □

7 An Isomorphism of Graphs

We shall now show that the bijection given by $\mathbf{i} \mapsto X_{\mathbf{i}}$ in Proposition 5.1 between equivalence classes of reduced words for w_0 and simplicial regions is an isomorphism of graphs.

We first introduce a graph structure on the set of equivalence classes of reduced words for w_0. We say that two such classes are adjacent if there exist reduced words $\mathbf{i}, \mathbf{j}$ in these classes such that $\mathbf{j}$ can be obtained from $\mathbf{i}$ by applying a long braid relation.

For example, the classes containing the reduced words $(2,3,1,2,1,3,4,3,2,1)$, $(2,3,1,2,3,4,3,2,1,2)$ are adjacent. For the former is equivalent to $(2,3,1,2,3,4,3,1,2,1)$, which can be obtained from the latter by applying the long braid relation $212 \to 121$.

Secondly, we introduce a graph structure on the set of simplicial regions. Two simplicial regions are said to be adjacent if they have a common wall with equation $Q = 0$ and there is a bijection between their remaining walls such that corresponding walls have equations $Q_i = 0$, $Q_i + \lambda_i Q = 0$ for some $\lambda_i \in \mathbb{R}$.

For example, the simplicial regions 28 and 41 are adjacent. We see from Table V that their walls are as follows:

$$\begin{array}{lll} 28 & [AB, C, BE, H]|||[CD, ABE] & cefh \\ 41 & [AB, BE, H]|||[C, D, ABE] & cehj. \end{array}$$

The common wall is taken as C, i.e. $\alpha_C = 0$. The bijection between the remaining walls satisfies $CD \leftrightarrow D$, $f \leftrightarrow j$ and $X \leftrightarrow X$ for all other walls X. This bijection is of the required form since $\alpha_C(v) = f - j$.

We shall obtain an isomorphism of graphs by showing that $\mathbf{i}$, $\mathbf{j}$ are adjacent if and only if $X_{\mathbf{i}}, X_{\mathbf{j}}$ are adjacent. We first need the following property of a pair of reduced words:

Proposition 7.1 *Two classes of reduced words* $[\mathbf{i}], [\mathbf{i}']$ *for* w_0 *differ by a simple long braid relation if and only if their two families of chamber sets differ by just one chamber set.*

We illustrate this proposition with the reduced words $(2,3,1,2,3,4,3,1,2,1)$ and $(2,3,1,2,3,4,3,2,1,2)$, which differ by a long braid relation $121 \to 212$. Their respective chamber diagrams are shown in Figure 12, and their respective chamber sets are:

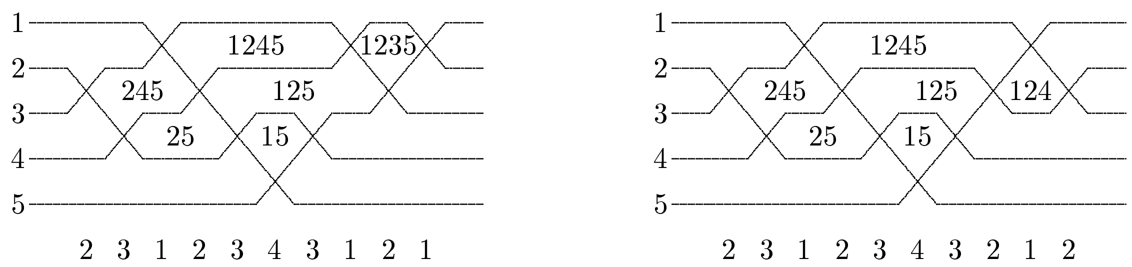


Figure 12: Two words differing by a long braid relation.

$$25,\ 15,\ 245,\ 125,\ 1245,\ 1235,$$

and

$$25,\ 15,\ 245,\ 125,\ 1245,\ 124.$$

Proof: First suppose that the reduced words $\mathbf{i}, \mathbf{i}'$ differ by a single long braid relation. Consider the crossings in their respective chamber diagrams. The only difference in the order of the crossings is in a set of three consecutive crossings. We may assume

$$\begin{aligned} \mathbf{i} &= \dots k, k+1, k, \dots \\ \mathbf{i}' &= \dots k+1, k, k+1, \dots \end{aligned}$$

where the dotted letters are the same in $\mathbf{i}, \mathbf{i}'$. The corresponding crossings have form:

$$\begin{aligned} \mathbf{i}: &\quad \dots (pq), (pr), (qr) \dots \\ \mathbf{i}': &\quad \dots (qr), (pr), (pq) \dots \end{aligned}$$

where the dotted crossings are the same for $\mathbf{i}, \mathbf{i}'$, and $p < q < r$. Now the three crossings $(pq), (pr), (qr)$ bound a chamber in each chamber diagram. This chamber has form as shown in Figure 13.

The chamber sets corresponding to this chamber in $\mathrm{CD}(\mathbf{i})$ and $\mathrm{CD}(\mathbf{i}')$ are different, but all other chamber sets in $\mathrm{CD}(\mathbf{i})$, $\mathrm{CD}(\mathbf{i}')$ are the same. Thus the families of chamber sets in $\mathrm{CD}(\mathbf{i})$, $\mathrm{CD}(\mathbf{i}')$ differ by just one chamber set.

Now suppose conversely that the chamber sets in $\mathrm{CD}(\mathbf{i})$, $\mathrm{CD}(\mathbf{i}')$ differ by just one chamber set. Let C be the chamber in $\mathrm{CD}(\mathbf{i})$ whose chamber set does not appear in $\mathrm{CD}(\mathbf{i})$. We note that each crossing (ij) appears just once in each chamber diagram and, if $i < j$, the chamber set for the chamber on the left of (ij) gives the chamber set on the right of (ij) replacing j by i.

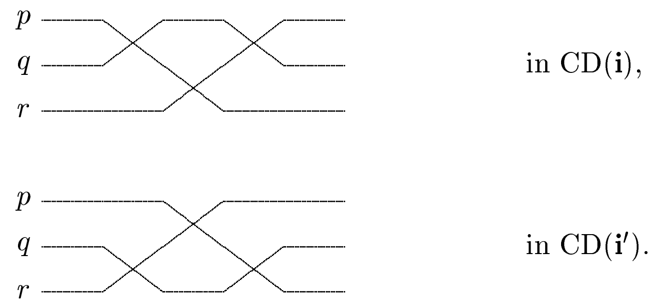


Figure 13: Chamber bounded by $(pq), (pr)$ and (qr).

(a) Suppose first that chambers C, C' occupy the same positions in their respective chamber diagrams. Let the left and right hand crossings of C be $(ij), (kl)$, respectively, with $i < j$, $k < l$. Since all other crossings are the same in the two chamber diagrams, the left and right hand end crossings of C' must be $(kl), (ij)$, respectively. In $\mathrm{CD}(\mathbf{i})$ string k is above string j at crossing (ij), whereas string j is above string k at crossing (kl). Thus the crossing (jk) must be above chamber C in $\mathrm{CD}(\mathbf{i})$. A similar argument show that crossing (jk) is below chamber C' in $\mathrm{CD}(\mathbf{i}')$. Since crossing (jk) appears in the same row in both chamber diagrams we obtain a contradiction. Thus C, C' cannot occupy the same positions in $\mathrm{CD}(\mathbf{i})$, $\mathrm{CD}(\mathbf{i}')$ respectively.

(b) Consider the sequence of chambers in the row of $\mathrm{CD}(\mathbf{i})$ containing C. This sequence has form

$$\dots C_1, C, C_2, \dots$$

and their chamber sets have form

$$\dots S(C_1), S(C), S(C_2), \dots .$$

When C is removed, we know that $S(C_1), S(C_2)$ will be consecutive chamber sets in the same row of $\mathrm{CD}(\mathbf{i}')$. Thus $S(C_1), S(C_2)$ differ by a single number, i.e.

$$|S(C_1) \cap S(C_2)| = |S(C_1)| - 1 = |S(C_2)| - 1.$$

This implies that the left and right hand end crossings of C must have form

$$\begin{array}{rll} & (ij),\ (jk) & \text{for } i < j < k, \\ \text{or} & (jk),\ (ij) & \text{for } i < j < k. \end{array}$$

So, in passing from $\dots, C_1, C, C_2, \dots$ to $\dots, C_1, C_2, \dots$, crossings (ij) and (jk) are removed and crossing (ik) is added. Since the set of crossings in $\mathrm{CD}(\mathbf{i})$, $\mathrm{CD}(\mathbf{i}')$ is the same, the crossing (ik) must be removed from $\mathrm{CD}(\mathbf{i})$ in adjoining C' and crossings (ij) and (jk) added. Thus the row of $\mathrm{CD}(\mathbf{i}')$ containing C' is the row of $\mathrm{CD}(\mathbf{i})$ containing the crossing (ik).

(c) Suppose the left and right end crossings of C are $(ij), (jk)$ respectively with $i < j < k$. Then the crossing (ik) must be below the level of C in $\mathrm{CD}(\mathbf{i})$. Alternatively, if the left and right end crossings of C are $(jk), (ij)$, respectively with $i < j < k$, the crossing (ik) will be above the level of C in $\mathrm{CD}(\mathbf{i})$. These two possibilities are illustrated in Figure 14.

Case 1

Case 2

Figure 14: The possibilities for crossing (ik).

(d) We next show that crossing (ik) in $\mathrm{CD}(\mathbf{i})$ lies either in the row immediately below that of $(ij),(jk)$, or the row immediately above.

It will be sufficient to consider Case 1 and show that (ik) is in the row below $(ij),(jk)$. A similar argument will work in Case 2. So suppose if possible that (ik) is not in the line immediately below (ij) and (jk). Then there will be a crossing (im) with $i<m$ between (ij) and (ik). Crossing (jm) cannot occur between (ij) and (jk) since then (ij) and (jk) could not bound the same chamber. So there must be a crossing (mk) between (ik) and (jk). Also all crossings involving m between (im) and (mk) are on a level below the level of C. See Figure 15.

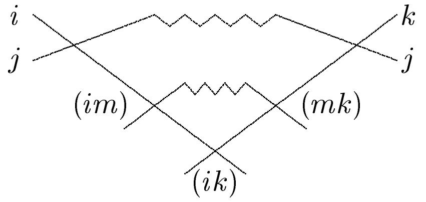


Figure 15: All crossings involving m between (im) and (mk) are on a level below the level of C.

Now consider the chamber diagram $\mathrm{CD}(\mathbf{i}')$. The crossings involving m in $\mathrm{CD}(\mathbf{i}')$ are in the same positions as in $\mathrm{CD}(\mathbf{i})$ since $m \notin \{i,j,k\}$. Crossing (ik) lies to the right of (im) and to the left of (mk) in $\mathrm{CD}(\mathbf{i}')$.

Now string i remains below string m to the right of (im) and string k remains below string m to the left of (mk) in $\mathrm{CD}(\mathbf{i}')$. Thus crossing (ik) lies below string m in $\mathrm{CD}(\mathbf{i}')$. Since all crossings involving m between (im) and (mk) are on a level below that of C, the level of crossing (ik) in $\mathrm{CD}(\mathbf{i}')$ is below the level of C in $\mathrm{CD}(\mathbf{i})$. But the level of (ik) in $\mathrm{CD}(\mathbf{i}')$ is the same as the level of C in $\mathrm{CD}(\mathbf{i})$, so we have a contradiction. Thus crossing (ik) in $\mathrm{CD}(\mathbf{i})$ lies on the line below (ij) and (jk). In Case 2 (ik) lies on the line above (ij) and (jk).

(e) We show next there is no crossing involving j between (ij) and (jk) in $\mathrm{CD}(\mathbf{i})$. Suppose we are in Case 1. If there is a crossing (mj) with $m < j$ between (ij) and (jk) then crossing (mk) would be to the right of (mj) and the left of (jk). But then (ij) and (jk) could not bound the same chamber. Similarly if there is a crossing (jm) with $j < m$ between (ij) and (jk) then crossing (im) would be to the right of (ij) and to the left of (jm). Then (ij) and (jk) could not bound the same chamber. A similar argument applies in Case 2.

(f) Now we show there is no crossing involving i between (ij) and (ik) and no crossing involving k between (ik) and (jk). Suppose we are in Case 1. Suppose there were a crossing (im), $i < m$, between (ij) and (ik). Then string m remains below string j and above string i between (im) and (jk). Since string i is in the row below string j at the crossing (ik), by (e), we have a contradiction. Similarly there is no crossing (mk), $m < k$, between (ik) and (jk). A similar argument holds in Case 2.

(g) We now know that the chambers C and C' are as in Figure 16.

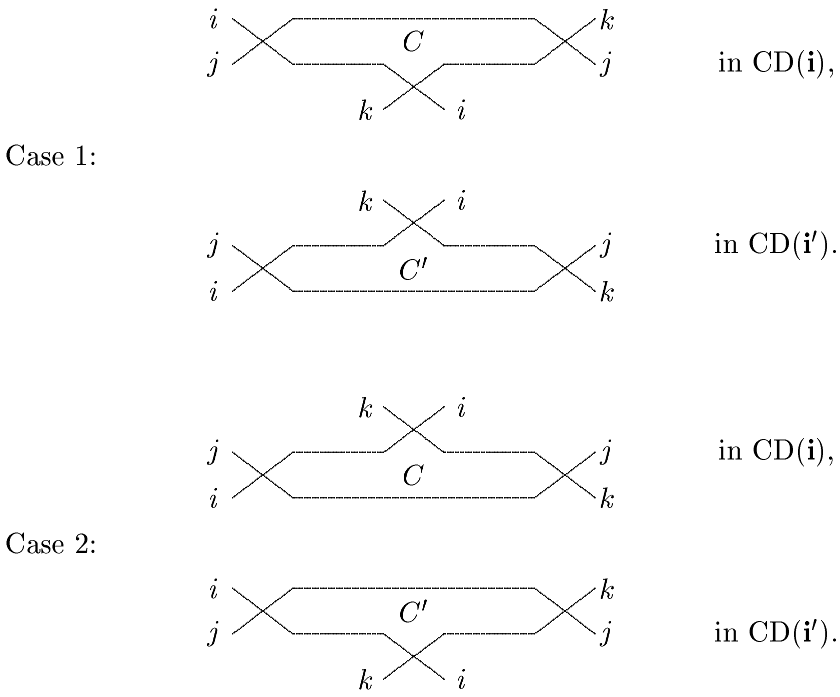


Figure 16: The chambers C and C'.

In Case 1 we may choose a total order on the crossings in $\mathrm{CD}(\mathbf{i})$ and $\mathrm{CD}(\mathbf{i}')$ compatible with the diagrams of the form

$$\begin{array}{ll} \dots (ij),(ik),(jk),\dots & \text{in } \mathrm{CD}(\mathbf{i}) \\ \dots (jk),(ik),(ij),\dots & \text{in } \mathrm{CD}(\mathbf{i}'), \end{array}$$

where the crossings agree apart from the given triples. The crossings in this order give rise to reduced words

$$\begin{array}{ccc} \cdots & s_p s_{p+1} s_p & \cdots \\ \cdots & s_{p+1} s_p s_{p+1} & \cdots \end{array}$$

commutation equivalent to $\mathbf{i}, \mathbf{i}'$ respectively and differing by a long braid relation. A similar agreement holds in Case 2, so the proposition is proved.

Proposition 7.2 *Let $\mathbf{i}, \mathbf{i}'$ be reduced words for w_0. Their commutation classes $[\mathbf{i}], [\mathbf{i}']$ are adjacent if and only if the simplicial regions $X_{\mathbf{i}}, X_{\mathbf{i}'}$ are adjacent.*

Proof: Suppose $[\mathbf{i}], [\mathbf{i}']$ are adjacent. Then $\mathbf{i}, \mathbf{i}'$ both determine 6 partial quivers and these two sets of 6 have 5 in common, by Proposition 7.1. Let these partial quivers be

$$\begin{aligned} \mathcal{P}(\mathbf{i}) &= \{P_1, P_2, P_3, P_4, P_5, P_6\}, \quad \text{and} \\ \mathcal{P}(\mathbf{i}') &= \{P_1, P_2, P_3, P_4, P_5, P_6'\}. \end{aligned}$$

Then, by Proposition 5.1, $X_{\mathbf{i}}$ has spanning vectors $\{v_P,\ P \in \mathcal{P}(\mathbf{i}),\ v_1, v_2, v_3, v_4\}$ and $X_{\mathbf{i}'}$ has spanning vectors $\{v_P,\ P \in \mathcal{P}(\mathbf{i}'),\ v_1, v_2, v_3, v_4\}$. Let W be the common wall of $X_{\mathbf{i}}, X_{\mathbf{i}'}$, spanned by vectors $v_{P_1}, v_{P_2}, v_{P_3}, v_{P_4}, v_{P_5}$ and v_1, v_2, v_3, v_4. Let W_i, $i = 1, 2, \dots, 9$, be the walls of $X_{\mathbf{i}}$ containing all spanning vectors of $X_{\mathbf{i}}$ except one, where the omitted spanning vector is not v_{P_6}. Let W_i', $i = 1, 2, \dots, 9$, be the corresponding walls of $X_{\mathbf{i}'}$. Let W be given by equation $Q = 0$ and W_i by equation $Q_i = 0$. Then we have

$$W_i \cap W_{i'} = W_i \cap W \qquad \text{and}$$

$$W_i \cap W = \{\mathbf{x} : Q(\mathbf{x}) = 0,\ Q_i(\mathbf{x}) = 0\}.$$

Since $W_i \cap W \subseteq W_i'$, the corresponding ideals $I(W_i \cap W)$, $I(W_i')$ satisfy $I(W_i') \subseteq I(W_i \cap W)$. Let W_i' have equation $Q_i' = 0$. Then Q_i' lies in the ideal generated by Q and Q_i. The only linear polynomials in this ideal are those of the form $\lambda Q + \lambda_i Q_i$ with $\lambda, \lambda_i \in \mathbb{R}$. Thus Q_i' has form $\lambda Q + \lambda_i Q_i$. Now W_i' is not equal to W so $\lambda_i \neq 0$. Without loss of generality we may choose $\lambda_i = 1$, so $Q_i' = Q_i + \lambda Q$ for some $\lambda \in \mathbb{R}$. This shows that the simplicial regions $X_{\mathbf{i}}, X_{\mathbf{i}'}$ are adjacent.

Conversely, suppose $\mathbf{i}, \mathbf{i}'$ are reduced words such that the regions $X_{\mathbf{i}}, X_{\mathbf{i}'}$ are adjacent. Thus they have a common wall W given by equation $Q = 0$ and a $1-1$ correspondence between their remaining walls

$$W_i \leftrightarrow W_i' \qquad i = 1, 2, \dots, 9,$$

where W_i has equation $Q_i = 0$ and W_i' has equation $Q_i + \lambda_i Q = 0$ for $\lambda_i \in \mathbb{R}$. Let L_i, L_i' be the 1-dimensional subspaces of $\mathbb{R}^{10}$ given by

$$L_i = \cap_{j, j \neq i} W_j \cap W, \qquad L_i' = \cap_{j, j \neq i} W_j' \cap W.$$

Both L_i and L'_i are given by the equations $Q = 0$, $Q_j = 0$, $j \in \{1, 2, \dots, 9\}$, $j \neq i$. Thus $L_i = L'_i$. Hence the regions $X_{\mathbf{i}}, X_{\mathbf{i}'}$ have 9 common spanning vectors, since a spanning vector has all coordinates 0 or 1 so is determined by the 1-dimensional subspace containing it. Four of these common spanning vectors are the vectors v_1, v_2, v_3, v_4 of Proposition 5.1. The remaining spanning vectors of $X_{\mathbf{i}}, X_{\mathbf{i}'}$ have form v_P, for $P \in \mathcal{P}(\mathbf{i})$, $P \in \mathcal{P}(\mathbf{i}')$ respectively. Since distinct partial quivers P give distinct vectors v_P we see that $\mathcal{P}(\mathbf{i})$ and $\mathcal{P}(\mathbf{i}')$ must have 5 partial quivers in common. Thus by Proposition 7.1 the commutation classes $[\mathbf{i}]$ and $[\mathbf{i}']$ of reduced words must be adjacent. □

This proposition shows that the correspondence $[\mathbf{i}] \to X_{\mathbf{i}}$ is an isomorphism of graphs. This graph, with 62 vertices, is shown in Figure 17. There is an involution ι on the set of reduced expressions for w_0, taking a reduced expression $\mathbf{i} = (i_1, i_2, \dots, i_{10})$ to $(5 - i_1, 5 - i_2, \dots, 5 - i_{10})$ (thus applying the graph automorphism of the Dynkin diagram of type A_4). This induces an involution on the set of commutation classes of reduced expressions, and thus on the set of simplicial regions. We have numbered the simplicial regions in such a way that ι takes region m to region $63 - m$, for $m = 1, 2, \dots 62$. In fact, ι induces an automorphism of the graph of simplicial regions.

8 A Correspondence of Monomials

We shall now show how monomials in the Kashiwara operators given by vectors in a Lusztig cone are related to the corresponding monomials in divided powers in the generators F_i of U^-. To do this we shall relate both to elements of a basis in U^- of PBW-type. For this purpose we shall need to consider root vectors of U^-. We follow ideas of Xi in [19].

Let $\mathbf{l} = (l_1, l_2, \dots, l_k)$ be the reduced expression for w_0 given by

$$(l_1, l_2, \dots, l_k) = (n, n-1, n, n-2, n-1, n, \dots, 1, 2, \dots, n).$$

The elements of the corresponding basis $B_{\mathbf{l}}$ of PBW-type are:

$$F_{\mathbf{l}}^{\mathbf{c}} = F_{l_1}^{(c_1)} T''_{l_1,-1}(F_{l_2}^{(c_2)}) \cdots T''_{l_1,-1} T''_{l_2,-1} \cdots T''_{l_{k-1},-1}(F_{l_k}^{(c_k)}).$$

The corresponding sequence of positive roots is:

$$\alpha_{l_1}, s_{l_1}(\alpha_{l_2}), s_{l_1} s_{l_2}(\alpha_{l_3}), \dots, s_{l_1} s_{l_2} \cdots s_{l_{k-1}}(\alpha_{l_k}).$$

We write $\alpha_{ij} = \alpha_i + \cdots + \alpha_{j-1}$ for $i < j$. For each p with $1 \leq p \leq k$ we have $s_{l_1} s_{l_2} \cdots s_{l_{p-1}}(\alpha_{l_p}) = \alpha_{ij}$ for some $i < j$, and each α_{ij} appears in this way for precisely one p. We then define

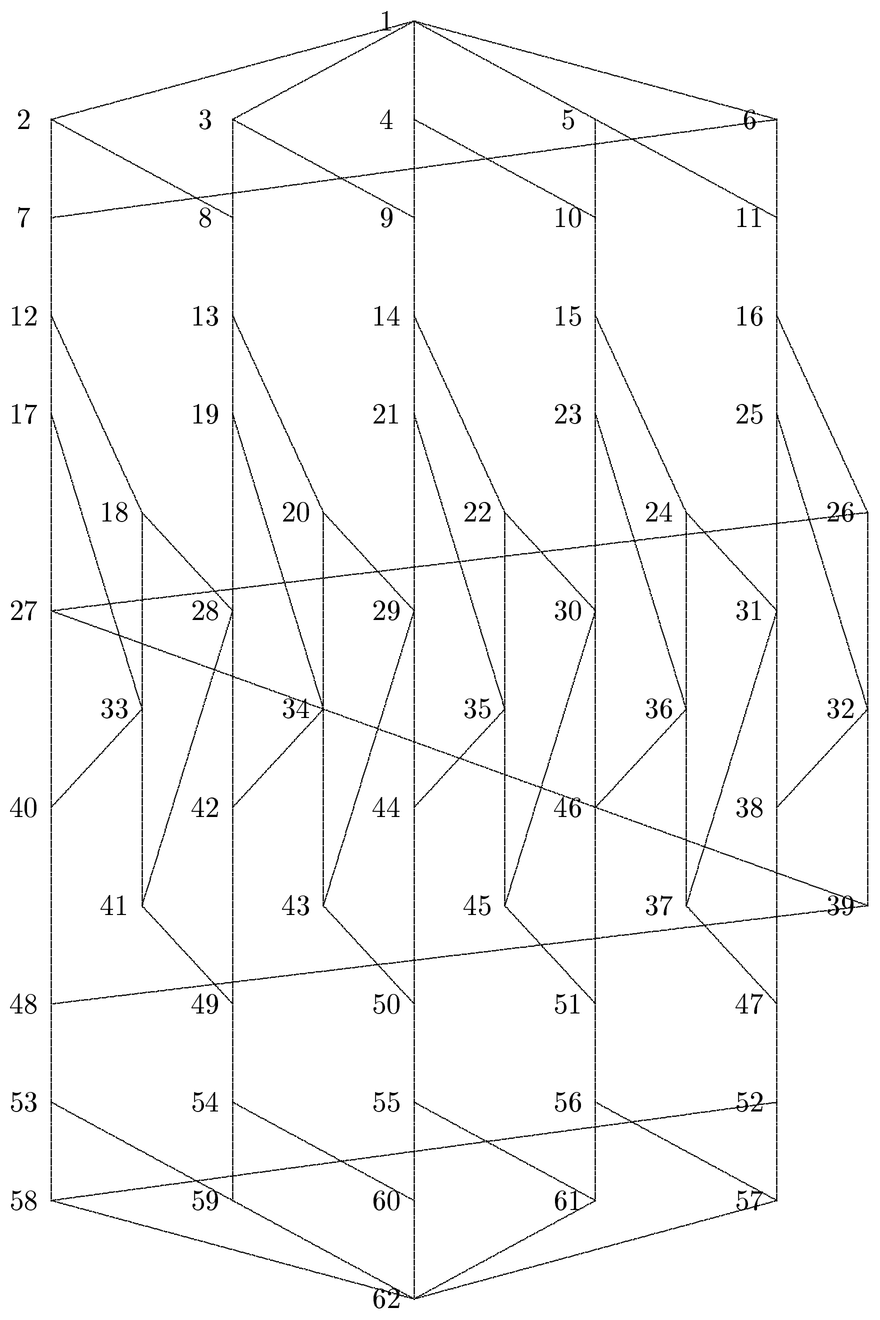
$$F_{ij} = T''_{l_1,-1} T''_{l_2,-1} \cdots T''_{l_{p-1},-1}(F_{l_p}).$$

Each F_{ij} is an element of U^- of weight $-\alpha_{ij}$. The elements F_{ij} will be called the root vectors of U^-.

Lemma 8.1 *We have:*

$$F_{ij} = (-1)^{j-i-1} T_i^{-1} T_{i+1}^{-1} \cdots T_{j-2}^{-1} F_{j-1}$$

for $i < j$.



The graph of simplicial regions in Type A_4

Figure 17

Proof: Let p be the positive integer such that $s_{l_1}s_{l_2}\cdots s_{l_{p-1}}(\alpha_{l_p}) = \alpha_{ij}$. Then we have:

$$s_{l_1}s_{l_2}\cdots s_{l_p} = s_n s_{n-1} s_n \cdots s_{i+1}s_{i+2}\cdots s_n s_i s_{i+1}\cdots s_{n+i+1-j}.$$

It is readily checked that the word

$$s_{j-2}s_{j-3}\cdots s_i s_n s_{n-1}s_n\cdots s_{i+1}s_{i+2}\cdots s_n s_i s_{i+1}\cdots s_{n+i+1-j}$$

is reduced and that this element of W transforms $\alpha_{n+i+1-j}$ into α_{j-1}. It follows from Lusztig [8, §1.3(c)], that

$$T''_{j-2,-1}T''_{j-3,-1}\cdots T''_{i,-1}T''_{n,-1}T''_{n-1,-1}T''_{n,-1}\cdots T''_{n+i+1-j}(F_{n+i+1-j}) = F_{j-1}.$$

This asserts that:

$$T''_{j-2,-1}T''_{j-3,-1}\cdots T''_{i,-1}(F_{ij}) = F_{j-1},$$

and so

$$\begin{aligned} F_{ij} &= (T''_{i,-1})^{-1}(T''_{i+1,-1})^{-1}\cdots(T''_{j-2,-1})^{-1}(F_{j-1}) \\ &= r_i^{-1}T_i^{-1}r_{i+1}^{-1}T_{i+1}^{-1}\cdots r_{j-2}^{-1}T_{j-2}^{-1}(F_{j-1}) \end{aligned}$$

since $T''_{i,-1} = T_i r_i$. It follows, using the definition of r_i, that

$$F_{ij} = (-1)^{j-i-1}T_i^{-1}T_{i+1}^{-1}\cdots T_{j-2}^{-1}(F_{j-1}),$$

as required. □

Lemma 8.2 *We have*

$$F_{ij} = F_i F_{i+1,j} - vF_{i+1,j}F_i,$$

for $i+1<j$.

Proof: First note that

$$F_{i,i+2} = -T_i^{-1}(F_{i+1}) = F_iF_{i+1} - vF_{i+1}F_i.$$

Assuming the required result inductively for $F_{i',j'}$ with $j'-i' < j-i$ we have:

$$\begin{aligned} F_{ij} &= -T_i^{-1}F_{i+1,j} \\ &= -T_i^{-1}(F_{i+1}F_{i+2,j} - vF_{i+2,j}F_{i+1}) \\ &= (F_iF_{i+1} - vF_{i+1}F_i)F_{i+2,j} - vF_{i+2,j}(F_iF_{i+1} - vF_{i+1}F_i) \\ &= F_i(F_{i+1}F_{i+2,j} - vF_{i+2,j}F_{i+1}) - v(F_{i+1}F_{i+2,j} - vF_{i+2,j}F_{i+1})F_i \\ &= F_iF_{i+1,j} - vF_{i+1,j}F_i. \square \end{aligned}$$

We now obtain commutation relations between root vectors and their divided powers analagous to those obtained by Xi in [19, §5.6]:

Proposition 8.3

$$
\begin{array}{lll}
(a)\ F_{pq}^{(M)}F_{rs}^{(N)} & = & F_{rs}^{(N)}F_{pq}^{(M)},\ \text{if } q<r \text{ or } r<p<q<s,\\
(b)\ F_{pq}^{(M)}F_{rs}^{(N)} & = & v^{MN}F_{rs}^{(N)}F_{pq}^{(M)},\ \text{if } r<p<q=s,\\
(c)\ F_{pq}^{(M)}F_{rs}^{(N)} & = & v^{-MN}F_{rs}^{(N)}F_{pq}^{(M)},\ \text{if } p=r<q<s \text{ or } p<r<q=s,\\
(d)\ F_{pq}^{(M)}F_{rs}^{(N)} & = & \displaystyle\sum_{0\le t\le \min(M,N)} v^{(M-t)(N-t)}F_{rs}^{(N-t)}F_{ps}^{(t)}F_{pq}^{(M-t)},\ \text{if } q=r, \quad \textit{and}\\
(e)\ F_{pq}^{(M)}F_{rs}^{(N)} & = & F_{rs}^{(N)}F_{pq}^{(M)},\ \text{if } s<p \text{ or } p<r<s<q.\\
(f)\ F_{pq}^{(M)}F_{rs}^{(N)} & = & \displaystyle\sum_{0\le t\le \min(M,N)} v^{-\frac{1}{2}t(t-1)}(v^{-1}-v)^t[t]!F_{rq}^{(t)}F_{rs}^{(N-t)}F_{pq}^{(M-t)}F_{ps}^{(t)},\ \text{if } p<r<q<s.
\end{array}
$$

Note that (e) is a restatement of (a). Using the relations (a)–(e) we may now obtain our main result of this section:

Theorem 8.4 *Suppose* $\mathbf{i}$ *is a reduced expression for* w_0*, and suppose* $\mathbf{a}\in C_{\mathbf{i}}$*. Then we have*

$$\widetilde{F}_{i_1}^{a_1}\widetilde{F}_{i_2}^{a_2}\cdots\widetilde{F}_{i_k}^{a_k}\cdot 1 \equiv F_{i_1}^{(a_1)}F_{i_2}^{(a_2)}\cdots F_{i_k}^{(a_k)} \mod v\mathcal{L}'.$$

Proof: Let $b=F_{i_1}^{(a_1)}F_{i_2}^{(a_2)}\cdots F_{i_k}^{(a_k)}\in U^-$. Then it is shown by Marsh in [14, §4.1] that in type A_4 for $\mathbf{a}\in C_{\mathbf{i}}$ we have $b\in\mathbf{B}$. So there is a unique $\mathbf{c}\in\mathbb{N}^k$ such that $b\equiv F_{\mathbf{l}}^{\mathbf{c}} \mod v\mathcal{L}$ (with $\mathbf{l}$ as above).

Since $b\in\mathcal{L}$ and $B_{\mathbf{l}}$ is a $\mathbb{Z}[v]$-basis of $\mathcal{L}$, b is a linear combination of elements $F_{\mathbf{l}}^{\mathbf{d}}$ for $\mathbf{d}\in\mathbb{N}^k$, with coefficients in $\mathbb{Z}[v]$. We shall obtain information about this expression for b by means of an algorithm.

This algorithm is as follows. It deals with expressions of the form

$$\sum_{\mathbf{p}=(p_1,p_2,\ldots,p_t)\in P} \lambda_{\mathbf{p},\mathbf{a}}F_{\gamma_1}^{(f_1)}F_{\gamma_2}^{(f_2)}\cdots F_{\gamma_u}^{(f_u)}$$

where $\gamma_1,\gamma_2,\ldots,\gamma_u$ are positive roots, P is a subset of $\mathbb{N}^t$ defined by certain linear inequalities, $\mathbf{a}=(a_1,a_2,\ldots,a_k)$ is as above, $f_1,f_2,\ldots,f_u$ are certain linear functions in $a_1,a_2,\ldots,a_k$ and $p_1,p_2,\ldots,p_t$, and $\lambda_{\mathbf{p},\mathbf{a}}\in\mathbb{Z}[v]$.

The element $F_{i_1}^{(a_1)}F_{i_2}^{(a_2)}\cdots F_{i_k}^{(a_k)}$ is an expression of this form with just one summand, and the algorithm starts with this expression.

Let $\alpha^p=s_{l_1}s_{l_2}\cdots s_{l_{p-1}}(\alpha_{l_p})$. Then $\alpha^1,\alpha^2,\ldots,\alpha^k$ is the total ordering on the positive roots determined by $\mathbf{l}$. We write $\alpha^p<\alpha^q$ if $p<q$. The aim of the algorithm is to transform $F_{i_1}^{(a_1)}F_{i_2}^{(a_2)}\cdots F_{i_k}^{(a_k)}$ into an expression of the above general form in which the positive roots appearing satisfy $\gamma_1<\gamma_2<\cdots<\gamma_u$. Consider adjacent pairs $F_{\gamma_i}^{(f_i)}F_{\gamma_{i+1}}^{(f_{i+1})}$ in each monomial in the sum, where $\gamma_i\neq\gamma_{i+1}$. Find the first adjacent pair for which $\gamma_i>\gamma_{i+1}$. Then, by using one of the relations (a)–(e) of Proposition 8.3 it is possible to rewrite $F_{\gamma_i}^{(f_i)}F_{\gamma_{i+1}}^{(f_{i+1})}$. In fact just one of these relations can be used. Applying this procedure to the first adjacent pair in each monomial gives us another expression of the same general form, so that the algorithm can be repeated.

Whenever an adjacent pair $F_{\gamma_i}^{(f_i)}F_{\gamma_{i+1}}^{(f_{i+1})}$ is obtained with $\gamma_i = \gamma_{i+1}$ it is replaced by the equivalent expression $\begin{bmatrix} f_i + f_{i+1} \\ f_i \end{bmatrix} F_{\gamma_i}^{(f_i+f_{i+1})}$.

If the algorithm terminates we shall have an expression of b as a $\mathbb{Z}[v]$-combination of elements in $B_{\mathbf{l}}$. It is not a priori clear that the algorithm will terminate. However, the algorithm was implemented in Maple for each reduced expression $\mathbf{i}$ for w_0 in type A_4, and did in fact terminate in each case. The implementation did not calculate the coefficients $\lambda_{\mathbf{p},\mathbf{a}}$ explicitly, but it did calculate the smallest power of v appearing in each $\lambda_{\mathbf{p},\mathbf{a}}$ (which by construction must appear with coefficient 1). This number is a quadratic expression in $p_1, p_2, \ldots, p_t,\ a_1, a_2, \ldots, a_k$.

Next, consider the monomial

$$\widetilde{F}_{i_1}^{a_1}\widetilde{F}_{i_2}^{a_2}\cdots\widetilde{F}_{i_k}^{a_k}\cdot 1$$

in the Kashiwara root operators $\widetilde{F}_i$. It was shown in §6 that the map $S_{\mathbf{i}}^{\mathbf{j}}$ is linear on $C_{\mathbf{i}}$, where $\mathbf{j} = (1,3,2,4,1,3,2,4,1,3)$ and $\mathbf{i}$ is any reduced expression for w_0. In fact, a similar calculation, using Maple, shows that $S_{\mathbf{i}}^{\mathbf{j}}$ is linear on $C_{\mathbf{i}}$ for all pairs of reduced expressions $\mathbf{i},\mathbf{j}$. In particular, $S_{\mathbf{i}}^{\mathbf{l}}$ is linear on $C_{\mathbf{i}}$ for all $\mathbf{i}$. It turns out that $\mathbf{l}$ appears to be a reduced expression for which the calculation of $S_{\mathbf{i}}^{\mathbf{l}}$ is as simple as possible. Using Theorem 5.5 in the paper [15] by the second author, we have a description of this map, giving us $S_{\mathbf{i}}^{\mathbf{l}}(\mathbf{a}) = \mathbf{c}$ such that

$$\widetilde{F}_{i_1}^{a_1}\widetilde{F}_{i_2}^{a_2}\cdots\widetilde{F}_{i_k}^{a_k}\cdot 1 \equiv F_{\mathbf{l}}^{\mathbf{c}} \mod v\mathcal{L}'.$$

The coefficient of $F_{\mathbf{l}}^{\mathbf{c}}$ in the expression for b obtained by the above algorithm was then checked and seen to lie in $1 + v\mathbb{Z}[v]$. It follows from Lusztig [8, §§2.3, 3.2] that the coefficients of all other terms $F_{\mathbf{l}}^{\mathbf{d}}$ in the sum must lie in $v\mathbb{Z}[v]$, and thus that $b \equiv F_{\mathbf{l}}^{\mathbf{c}} \mod v\mathcal{L}$, and so $b \equiv F_{\mathbf{l}}^{\mathbf{c}} \mod v\mathcal{L}'$ (by [9, §2.3]). It follows that $b = F_{i_1}^{(a_1)}F_{i_2}^{(a_2)}\cdots F_{i_k}^{(a_k)} \equiv \widetilde{F}_{i_1}^{a_1}\widetilde{F}_{i_2}^{a_2}\cdots\widetilde{F}_{i_k}^{a_k}\cdot 1 \mod v\mathcal{L}'$ as required. □

We now give an example to demonstrate the above proof. We take $\mathbf{i} = (3,2,1,4,3,2,3,4,1,3)$, and $\mathbf{a} \in C_{\mathbf{i}}$. The corresponding monomial is

$$b = F_3^{(a_1)}F_2^{(a_2)}F_1^{(a_3)}F_4^{(a_4)}F_3^{(a_5)}F_2^{(a_6)}F_3^{(a_7)}F_4^{(a_8)}F_1^{(a_9)}F_3^{(a_{10})},$$

The ordering on the set of positive roots induced by $\mathbf{l} = (4,3,4,2,3,4,1,2,3,4)$ is

$$(\alpha_{45}, \alpha_{35}, \alpha_{34}, \alpha_{25}, \alpha_{24}, \alpha_{23}, \alpha_{15}, \alpha_{14}, \alpha_{13}, \alpha_{12}).$$

We begin the algorithm by writing b in the form

$$b = F_{34}^{(a_1)}F_{23}^{(a_2)}F_{12}^{(a_3)}F_{45}^{(a_4)}F_{34}^{(a_5)}F_{23}^{(a_6)}F_{34}^{(a_7)}F_{45}^{(a_8)}F_{12}^{(a_9)}F_{34}^{(a_{10})},$$

The first adjacent pair of root vectors appearing in the wrong order is $F_{12}^{(a_3)}F_{45}^{(a_4)}$. Applying relation (a) of Proposition 8.3 we obtain

$$b = F_{34}^{(a_1)}F_{23}^{(a_2)}F_{45}^{(a_4)}F_{12}^{(a_3)}F_{34}^{(a_5)}F_{23}^{(a_6)}F_{34}^{(a_7)}F_{45}^{(a_8)}F_{12}^{(a_9)}F_{34}^{(a_{10})}.$$

Next apply relation (a) to the adjacent pair $F_{23}^{(a_2)}F_{45}^{(a_4)}$ and obtain

$$b = F_{34}^{(a_1)}F_{45}^{(a_4)}F_{23}^{(a_2)}F_{12}^{(a_3)}F_{34}^{(a_5)}F_{23}^{(a_6)}F_{34}^{(a_7)}F_{45}^{(a_8)}F_{12}^{(a_9)}F_{34}^{(a_{10})}.$$

Now we apply relation (d) to the adjacent pair $F_{34}^{(a_1)}F_{45}^{(a_4)}$ and obtain

$$b=\sum v^{(a_1-p_1)(a_4-p_1)}F_{45}^{(a_4-p_1)}F_{35}^{(p_1)}F_{34}^{(a_1-p_1)}F_{23}^{(a_2)}F_{12}^{(a_3)}F_{34}^{(a_5)}F_{23}^{(a_6)}F_{34}^{(a_7)}F_{45}^{(a_8)}F_{12}^{(a_9)}F_{34}^{(a_{10})},$$

where the sum is over those p_1 satisfying $0\le p_1\le \min(a_1,a_4)$. Continuing the algorithm in this way we obtain, after 25 steps,

$$b=\sum_{(p_1,p_2,\dots p_{10})\in P}\lambda_{\mathbf{p},\mathbf{a}}F_{45}^{(f_1)}F_{35}^{(f_2)}F_{34}^{(f_3)}F_{25}^{(f_4)}F_{24}^{(f_5)}F_{23}^{(f_6)}F_{15}^{(f_7)}F_{14}^{(f_8)}F_{13}^{(f_9)}F_{12}^{(f_{10})},$$

where

$$\begin{aligned}\mathbf{f} &= (f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8,f_9,f_{10})=(a_4+a_8-p_1-p_6-p_7-p_8,p_1+p_8,\\ &\quad a_1+a_5+a_7+a_{10}-p_1-p_2-p_4-p_5-p_8-p_9-p_{10},p_7,p_2+p_5-p_7+p_{10},\\ &\quad a_2+a_6-p_2-p_3-p_5-p_{10},p_6,p_4-p_6+p_9,p_3-p_4-p_9,a_3+a_9-p_3).\end{aligned}$$

The lowest power of v in $\lambda_{\mathbf{p},\mathbf{a}}$ is $\mathbf{x}T\mathbf{x}^t$ where $\mathbf{x}=(a_1,a_2,\dots,a_{10},p_1,p_2,\dots,p_{10})$ and

$$T=\begin{pmatrix}
0&0&0&1&-1&0&-1&1&0&-1&-1&1&0&1&1&-1&-1&-1&1&1\\
0&0&0&0&1&-1&1&0&0&1&0&-1&1&-1&-1&0&0&0&-1&-1\\
0&0&0&0&0&1&0&0&-1&0&0&0&-1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&1&1&1&0&0\\
0&0&0&0&0&0&-1&1&0&-1&0&0&0&1&1&-1&-1&-1&1&1\\
0&0&0&0&0&0&1&0&0&1&0&0&0&-1&-1&0&0&0&-1&-1\\
0&0&0&0&0&0&0&1&0&-1&0&0&0&0&0&-1&-1&-1&1&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-1&1&0&1&-1&0&1&1&-1&0&-1&-1&1&1&0&-1&-1\\
0&0&0&0&-1&1&-1&0&0&-1&0&1&-1&1&0&0&0&1&1&0\\
0&0&0&0&0&-1&0&0&1&0&0&0&1&0&1&0&0&0&0&1\\
0&0&0&0&0&0&-1&0&0&-1&0&0&0&1&0&0&1&1&0&-1\\
0&0&0&0&0&0&-1&0&0&-1&0&0&0&1&1&0&0&1&1&0\\
0&0&0&0&0&0&0&-1&0&1&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&1&0&0&0&0&0&1&1&0&-1&0\\
0&0&0&0&0&0&0&-1&0&1&0&0&0&0&0&1&1&1&-1&-1\\
0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&1&1
\end{pmatrix}.$$

Here P is the set of $(p_1,p_2,\dots,p_{10})\in\mathbb{N}^{10}$ satisfying $p_1\le a_1,p_1\le a_4,p_2\le a_2,p_2\le a_5,p_3\le a_3,p_3\le a_6,p_4\le p_3,p_4\le a_7,p_5\le p_2-p_3+a_2+a_6,p_5\le -p_4+a_7,p_6\le p_4,p_6\le a_8,p_7\le p_2+p_5,p_7\le -p_6+a_8,p_8\le -p_1-p_2-p_4-p_5+a_1+a_5+a_7,p_8\le -p_6-p_7+a_8,p_9\le p_3-p_4,p_9\le a_{10},p_{10}\le -p_2-p_3-p_5+a_2+a_6,p_{10}\le -p_9+a_{10}$.

The function $S_{\mathbf{i}}^{\mathbf{l}}(\mathbf{a})=\mathbf{c}$ is given in this example by

$$\mathbf{c}=(-a_1+a_4,a_1,-a_2+a_5,-a_7+a_8,a_2+a_7-a_8,-a_3+a_6,a_7,a_{10},a_3-a_7-a_{10},a_9).$$

It can be checked that if $\mathbf{p}$ is taken as $(a_1,a_2,a_3,a_7,0,a_7,a_8-a_7,0,a_{10},0)$ then $\mathbf{p}$ satisfies the inequalities as given above and $\mathbf{f}=\mathbf{c}$. Furthermore, $\mathbf{x}T\mathbf{x}^t=0$, so that $\lambda_{\mathbf{p},\mathbf{a}}\in 1+v\mathbb{Z}[v]$. Since for any $\mathbf{p}$ satisfying the above inequalities we must have $\lambda_{\mathbf{a},\mathbf{p}}\in\mathbb{Z}[v]$ and the coefficient of the lowest power of v in $\lambda_{\mathbf{a},\mathbf{p}}$ is always 1, it follows that the coefficient of $F_{\mathbf{l}}^{\mathbf{c}}$ in the above sum, when terms in the same PBW basis element are collected together, is of the form $e+vh$, where $h\in\mathbb{Z}[v]$ and $e\in\mathbb{N},e\neq 0$. Since $b\in\mathbf{B}$, we know by [8, §§2.3, 3.2] that b has a unique expression as a $\mathbb{Z}[v]$-combination of elements $F_{\mathbf{l}}^{\mathbf{d}}$ as $\mathbf{d}$ varies, in which precisely one of the coefficients lies in $1 + v\mathbb{Z}[v]$. It follows that $e = 1$ and $b \equiv F_{\mathbf{l}}^{\mathbf{c}} \mod v\mathcal{L}$, whence $b \equiv F_{\mathbf{l}}^{\mathbf{c}} \mod v\mathcal{L}'$ by [9, §2.3]. But we know that

$$\widetilde{F}_{i_1}^{a_1}\widetilde{F}_{i_2}^{a_2}\cdots\widetilde{F}_{i_{10}}^{a_{10}} \equiv F_{\mathbf{l}}^{\mathbf{c}} \mod v\mathcal{L}',$$

since $S_{\mathbf{i}}^{\mathbf{l}}(\mathbf{a}) = \mathbf{c}$. It follows that

$$\widetilde{F}_{i_1}^{a_1}\widetilde{F}_{i_2}^{a_2}\cdots\widetilde{F}_{i_{10}}^{a_{10}} \cdot 1 \equiv F_{i_1}^{(a_1)}F_{i_2}^{(a_2)}\cdots F_{i_{10}}^{(a_{10})} \mod v\mathcal{L}'.\square$$

As a consequence of Theorem 8.4 we may deduce the relationships between the set $M_{\mathbf{i}}$ of tight monomials, the Lusztig cone $C_{\mathbf{i}}$, and the simplicial region of linearity $X_{\mathbf{i}}$ of the piecewise-linear function $R_{\mathbf{j}}^{\mathbf{j}'}$.

Theorem 8.5 *For each reduced expression* $\mathbf{i}$ *for* w_0 *let* $M_{\mathbf{i}} = \{F_{i_1}^{(a_1)}F_{i_2}^{(a_2)}\cdots F_{i_k}^{(a_k)} : \mathbf{a} \in C_{\mathbf{i}}\}$. *Suppose we have type* A_4. *Then* $M_{\mathbf{i}} \subseteq \mathbf{B}$. *Moreover, under Kashiwara's parametrization of* $\mathbf{B}$ *we have* $\psi_{\mathbf{i}}(M_{\mathbf{i}}) = C_{\mathbf{i}}$ *and under Lusztig's parametrization of* $\mathbf{B}$ *we have* $\phi_{\mathbf{j}}(M_{\mathbf{i}}) = X_{\mathbf{i}}$.

Proof: We recall that $M_{\mathbf{i}} \subseteq \mathbf{B}$ was shown in [14]. Theorem 8.4 shows that $\psi_{\mathbf{i}}(M_{\mathbf{i}}) = C_{\mathbf{i}}$. Proposition 6.1 shows that $S_{\mathbf{i}}^{\mathbf{j}}(C_{\mathbf{i}}) = X_{\mathbf{i}}$. Now $S_{\mathbf{i}}^{\mathbf{j}} = \phi_{\mathbf{j}}\psi_{\mathbf{i}}^{-1}$. Since $\psi_{\mathbf{i}}^{-1}(C_{\mathbf{i}}) = M_{\mathbf{i}}$ it follows that $\phi_{\mathbf{j}}(M_{\mathbf{i}}) = X_{\mathbf{i}}$. $\square$

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