Applying enriched categories to quasi-uniform spaces.

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Abstract: The represention of complete metric spaces of [Law73] by enrichments is extented to quasi-uniform spaces. Moreover quasi-uniformly continuous maps are described as enriched functors. The quasi-uniform space completion is also viewed as a Cauchy-completion. Super monoidal functors are introduced to obtain these results. A 2-category of enrichments over different bases is defined. In this general context the Cauchy-completion is still a universal construction.

keywords: Enriched categories, change of base, Cauchy-completion, uniform and quasi-uniform spaces.

1 Introduction

In [Law73] the Cauchy completion for enrichments over monoidal closed categories is defined. This completion process is shown to capture the metric space completion: generalised metric spaces correspond to enrichments and the complete metric spaces correspond to Cauchy-complete enrichments. In this paper, these results are extended to quasi-uniform spaces. Representations in terms of enrichments are given for quasi-uniformly continuous maps [1] and quasi-uniform space completion.

The notions of enrichments and Cauchy-completion are general enough and have been applied succesfully in various areas. Let us cite [Wal81] for sheaves, [Be-Ca-St-Wa83] for fibrations, [Rut96] for ultra-metric spaces, [Amb-Ver96] for ring representation... many others exist. Nevertheless it seems that the theory should be further developed. The original example of metric spaces is relevant.
(1) The enriched functors in [Law73] are less general than uniformly continuous maps, hence enriched functors may be too particular to describe morphisms.
(2) Given some quasi-uniform space, one could imagine that it occurs as some $\mathcal{V}$-enrichment for some $\mathcal{V}$ — and it does! — but this $\mathcal{V}$ should be built from the quasi-uniformity of that space. Therefore it seems unreasonable to represent the category of quasi-uniform spaces and uniformly continuous maps as the category of $\mathcal{V}$-enrichments for a fixed $\mathcal{V}$.

[1] For short we call "uniformly continuous maps" the quasi-uniformly continuous maps of [FL82]

Indeed quasi-uniform spaces (and also uniform ones) occur as particular enrichments over different bases. In our opinion the study of uniformly continuous maps and quasi-uniform space completion, may help to determine notions of morphisms between enrichments and Cauchy-completion, which are independent from base categories. Let us underline some results of this investigation. Super monoidal functors were introduced. These functors are interesting change of base categories. They serve to define a 2-category of enrichments over different bases. In this more general context the Cauchy-completion is still a universal construction. In this context also, the uniformly continuous maps occur as particular enriched functors and the quasi-uniform space completion is a Cauchy-completion.

The enriched category theory used is very simple: base categories are partial orders. "Non-pure" categorical concepts of smallness and skeletality have nice interpretations. The fact that Cauchy-completion preserves smallness is crucial. Skeletality for enrichments is the counterpart of separation for spaces. We also introduced the notion of reversibility for base categories and enrichments. There are open questions related to the axiomatisation of the base monoidal category. (For example what is a reasonable class $\mathcal{F}$ for theorem 4.9 ?)

We also had to look carefully at some topological aspects. According to the usual definition, a quasi-uniformity (X, U) consists in a filter U on $X \times X$ with extra properties. As suggested by the definition of enrichments, it seems natural to distinguish quasi-uniformities U from the set X of elements of the space. The idea is to define a lattice of properties U and a relation of satisfaction "$(x, y) \models u$" between pairs (x, y) in X and properties $u \in U$. This yields to abstract-quasi-uniformities. They occured as intermediate objects to relate quasi-uniformities to enrichments. Nevertheless to make clear the connection with enrichments on monoidal category, it was necessary to define the "dense" abstract quasi-uniformities.

The paper is organized as follows. Chapter 2 is a brief recall on uniform and quasi-uniform spaces. Chapter 3 is a brief recall of well-known facts from enriched category theory. Chapter 4 presents new categorical developments. Super monoidal functors are defined. They allow the definition of a 2-category *Enr* of enrichments over different bases. A general notion of Cauchy-completion holds in *Enr* (4.8 and 4.9). Chapter 5 introduces quasi-uniform triples. They correspond to an axiomatisation of the lattice of quasi-uniformities, they may be seen as quasi-uniformities "without points". We show how to build monoidal bi-closed categories from quasi-uniform triples. Chapter 6 introduces abstract quasi-uniformities and dense abstract quasi-uniformities. They consist in sets of points with a "general distance" valued in a quasi-uniform triple. Then abstract quasi-uniformities are related to enrichments. In Chapter 7, we prove the existence of completion for abstract quasi-uniformities. We relate this completion process to the categorical notion of Cauchy-completion. Chapter 8 relates (usual) quasi-uniformities to abstract ones. We show how to retrieve the usual completion of quasi-uniformities [FL82] from the more general completion of abstract quasi-uniformities. Chapter 9 treats reversibility. Abstract uniformities are introduced to generalise uniform spaces. We show that all the previous results may be specialised to the "reversible" case.

2 Uniformities and quasi-uniformities

Recall that a quasi-uniformity U on a set X, denoted (X, U), is a set of binary relations $U \subseteq \wp(X \times X)$ that satisfy:
- $\forall u \in U, \Delta \subseteq u$,
- U is filtered (non-empty, stable by finite intersection and arbitrary union),
- $\forall u \in U, \exists v \in U, v^2 \subseteq u$.

A quasi-uniformity (X, U) is a uniformity when $\forall u \in U, u^{-1} \in U$. Given quasi-uniformities (X, U)

and (Y, V), a map $f : X \to Y$ is uniformly continuous when $\forall v \in V, \exists u \in U, \forall x, y \in X, (x, y) \in u \Rightarrow (fx, fy) \in v$. Further on, $QUnif$ will denote the category of quasi-uniform spaces and uniformly continuous maps. $Unif$ will denote the full subcategory of uniformities.

Let (X, U) be a uniformity. A filter F on X is Cauchy (w.r.t U) when it satisfies: $\forall u \in U, \exists f \in F, f \times f \subseteq u$. The neighbourhood filter of some $x \in X$ is the filter consisting in the subsets $\{y/(x, y) \in u\}$, u ranging in U. Then (X, U) is *complete* when any minimal Cauchy filter occurs as the neighbourhood filter of some element of X. (X, U) is *separated* when the map sending elements of X to their neighbourhood filter is injective. Recall,

Theorem 2.1 *[Bou] Let X be a uniformity. There is a separated and complete uniformity $\overline{(X, U)}$, and a uniformly continuous map $i : (X, U) \to \overline{(X, U)}$ such that for any uniformly continuous map f from (X, U) to a separated and complete uniform space (Y, V), there exists one an only one uniformly continuous map $\bar{f} : \overline{(X, U)} \to (Y, V)$ such that $f = \bar{f} \circ i$.*

A generalization of this theorem to quasi-uniform spaces may be found in [FL82], (chapter 3).

Theorem 2.2 *For any quasi-uniform space (X, U) there is a separated and complete quasi-uniform space $\overline{(X, U)}$ with a uniformly continuous map $i_{(X,U)} : (X, U) \to \overline{(X, U)}$ such that for any uniformly continuous map $f : (X, U) \to (Y, V)$ with (Y, V) separated and complete there is a unique uniformly continuous $\bar{f} : \overline{(X, U)} \to (Y, V)$ such that $\bar{f} \circ i_{(X,U)} = f$.*

3 Enrichments over monoidal partial orders

All the results in this chapter are well-known and reproduced here for completness. We refer the reader to [Kel82] for a general presentation of enriched category theory.

We call a monoidal category that is also a partial order, a *monoidal partial order*. In this section, $\mathcal{V}$ will denote a monoidal partial order.

Definition 3.1 ($\mathcal{V}$-enrichment) *An* enrichment *A over $\mathcal{V}$ or a $\mathcal{V}$-*category *is a set $Obj(A)$ — the* objects *of A — with a mapping $A(-, -) : Obj(A) \times Obj(A) \longrightarrow Obj(\mathcal{V})$ satisfying:*
- for any object a of A, $I \leq A(a, a)$;
- for any objects a, b, c of A, $A(b, c) \otimes A(a, b) \leq A(a, c)$.
*Given $\mathcal{V}$-categories A and B, a $\mathcal{V}$-*functor *f from A to B is a map $f : Obj(A) \longrightarrow Obj(B)$ such that for any objects a, a' of A, $A(a, a') \leq B(fa, fa')$.*
*Given two $\mathcal{V}$-functors $f, g : A \to B$ there is a $\mathcal{V}$-*natural transformation *from f to g, — which is denoted $f \Rightarrow g$, when for any object a of A, $I \leq A(fa, ga)$.*

The usual definition of enrichment does not require $Obj(A)$ above to be a set. In this paper, we shall consider only "small" enrichments i.e. such that their sets of objects is small. The base categories will be also always small. This makes sense since the examples of enrichments that we will look at, are very "concrete", and the fact that Cauchy-completion (3.9) preserves smallness is essential.

Definition 3.2 *A $\mathcal{V}$-category A is called* skeletal *when for any objects a, b of A, if $I \leq A(a, b)$ and $I \leq A(b, a)$ then $a = b$.*

Fact 3.3 *$\mathcal{V}$-categories, $\mathcal{V}$-functors and, $\mathcal{V}$-natural transformations constitute a 2-category $\mathcal{V}-Cat$.*

Further on, $\mathcal{V}$ will be supposed cocomplete and bi-closed.

Definition 3.4 ($\mathcal{V}$-modules) *Given two $\mathcal{V}$-categories A and B, a* $\mathcal{V}$-module *φ from A to B — denoted $\varphi : A \multimap B$ — is a map $\varphi : Obj(B) \times Obj(A) \to Obj(\mathcal{V})$ such that:*
- for any objects a, a' of A, and b of B, $A(a,a') \otimes \varphi(b,a) \leq \varphi(b,a')$;
- for any objects a of A and, b, b' of B, $\varphi(b,a) \otimes B(b',b) \leq \varphi(b',a)$.
For any two $\mathcal{V}$-modules $\varphi : A \multimap B$ and $\psi : B \multimap C$, their composite $\psi \bullet \varphi : A \multimap C$ is defined by: $\forall a \in Obj(A), c \in Obj(C)$, $(\psi \bullet \varphi)(c,a) = \bigvee_{b \in Obj(B)} \varphi(b,a) \otimes \psi(c,b)$. The set $\mathcal{V} - Mod(A,B)$ of $\mathcal{V}$-modules from A to B is partially ordered by: $\varphi \leq \varphi' \Leftrightarrow \forall (b,a) \in Obj(B) \times Obj(A), \varphi(b,a) \leq \varphi'(b,a)$.

Fact 3.5 *$\mathcal{V}$-categories and $\mathcal{V}$-modules with partial orders defined above, constitute a locally posetal 2-category denoted $\mathcal{V} - Mod$.*

In $\mathcal{V} - Mod$, the identity in A is the module with underlying map $A(-,-) : Obj(A) \times Obj(A) \to Obj(\mathcal{V})$ sending any (a,a') to $A(a,a')$. For any $\mathcal{V}$-categories A and B, a $\mathcal{V}$-module $\varphi : A \multimap B$ has right adjoint $\psi : B \multimap A$ if and only if $A(-,-) \leq \psi \bullet \varphi$ and $\varphi \bullet \psi \leq B(-,-)$. For any left adjoint module φ, $\tilde{\varphi}$ will denote its (unique!) right adjoint. Any $\mathcal{V}$-functor $f : A \to B$ corresponds to a pair of adjoint modules $f_\diamond \dashv f^\diamond$, $f_\diamond : A \multimap B$, $f^\diamond : B \multimap A$, as follows: for any objects a of A, and b of B, $f_\diamond(b,a) = B(b,fa)$ and $f^\diamond(a,b) = B(fa,b)$. $\hat{1}$ denotes the enrichment on $\mathcal{V}$, with one object, say $*$, and $\hat{1}(*,*) = I$. If B is a $\mathcal{V}$-category, for any of its object b, $b : \hat{1} \to B$ denotes the $\mathcal{V}$-functor sending $*$ to b.

Further on $\mathcal{V} - AMod$ will denote the locally posetal 2-category with:
- objects $\mathcal{V}$-categories,
- arrows: *left adjoint* $\mathcal{V}$-modules;
- 2-cells: partial orders inherited from $\mathcal{V} - Mod$.

Fact 3.6 *There is a 2-functor $J_\mathcal{V} : \mathcal{V} - Cat \to \mathcal{V} - AMod$ as follows. It is the identity on objects, and is the map $(-)_\diamond$ on arrows sending any $\mathcal{V}$-functor f to the $\mathcal{V}$-module $f_\diamond$. For any $\mathcal{V}$-categories A and B:*
(i) $(-)_\diamond : \mathcal{V} - Cat(A,B) \to \mathcal{V} - Mod(A,B)$ is injective if and only if B is skeletal;
(ii) For any $\mathcal{V}$-functors $f,g : A \to B$, $f \Rightarrow g$ if and only if $f_\diamond \leq g_\diamond$.

Definition 3.7 (Cauchy-completeness) *A $\mathcal{V}$-category B is* Cauchy-complete *when one the following two equivalent assertions holds:*
(i) For any left adjoint $\mathcal{V}$-module $\varphi : A \multimap B$, $\varphi = f_\diamond$ for some $\mathcal{V}$-functor $f : A \to B$;
(ii) For any left adjoint $\mathcal{V}$-module $\varphi : \hat{1} \multimap B$, $\varphi = b_\diamond$ for some object b of B.

We define the following 2-categories.

- $\mathcal{V} - SkCcCat$:
  - objects: skeletal and Cauchy-complete $\mathcal{V}$-categories,
  - arrows and 2-cells: inherited from $\mathcal{V} - Cat$.

- $\mathcal{V} - SkCcAMod$:
  - objects: skeletal and Cauchy-complete $\mathcal{V}$-categories,
  - arrows and 2-cells: inherited from $\mathcal{V} - AMod$.

Fact 3.8 *The restriction of $J_\mathcal{V}$ on $\mathcal{V} - SkCcCat$ is an isomorphism onto $\mathcal{V} - SkCcAMod$.*

Definition 3.9 (Cauchy-completion) *Let A be a $\mathcal{V}$-category. The* Cauchy-completion *of A is the $\mathcal{V}$-category $\bar{A}$ defined as follows. Its objects are the left-adjoint $\mathcal{V}$-modules of the form $\varphi : \hat{1} \multimap A$. For convenience we will consider modules in $Obj(\bar{A})$ as well as their adjoints, as maps with domains A. For any $\varphi, \psi \in Obj(\bar{A})$, $\bar{A}(\varphi,\psi) = (\tilde{\varphi} \bullet \psi)(*,*) = \bigvee_{a \in Obj(A)} \psi(a) \otimes \tilde{\varphi}(a)$. The map $Obj(A) \to Obj(\bar{A})$ sending any a to the $\mathcal{V}$-module $a_\diamond$, defines a $\mathcal{V}$-functor $j_A : A \to \bar{A}$.*

Fact 3.10 *For any $\mathcal{V}$-category A, $\bar{A}$ is skeletal and Cauchy-complete.*

Lemma 3.11 *For any $\mathcal{V}$-category A, $(j_A)_\diamond : A \multimap \bar{A}$ and $(j_A)^\diamond : \bar{A} \multimap A$ are inverse modules.*

According to 3.11 and 3.10

Fact 3.12 *There is a 2-functor $S : \mathcal{V} - AMod \to \mathcal{V} - SkCcCat$ which is an equivalence of 2-categories. S is defined on objects by $S(A) = \bar{A}$ and on arrows by $S(\varphi) = f_\varphi$ where for any left adjoint $\mathcal{V}$-module $\varphi : A \multimap B$, $f_\varphi : \bar{A} \to \bar{B}$ is the unique $\mathcal{V}$-functor f satisfying $f_\diamond \bullet j_{A_\diamond} = j_{B_\diamond} \bullet \varphi$.*

Fact 3.13 *The inclusion 2-functor $\mathcal{V} - SkCcCat \to \mathcal{V} - Cat$ has a left 2-adjoint.*

3.13 implies that skeletal and Cauchy-complete $\mathcal{V}$-categories form a reflective subcategory of the underlying category of $\mathcal{V} - Cat$. Indeed, the left adjoint of the inclusion sends a $\mathcal{V}$-category A to $\bar{A}$ and the unit takes value $j_A : A \to \bar{A}$ in A.

According to 3.10 and 3.11, the following is coherent

Definition 3.14 (Morita-equivalence) *Two $\mathcal{V}$-categories A and B are* Morita-equivalent *when one the following equivalent assertions is satisfied:*
(i) They are isomorphic in $\mathcal{V} - Mod$;
(ii) Their Cauchy-completions are isomorphic in $\mathcal{V} - Cat$.

4 More on enrichments

4.1 About the change of base

The change of base for enrichments (over bicategories) has recently known some new developments [KLSS99]. Nevertheless we shall only use an old formulation of a classical result [Eil-Kel66]. Restricting ourselves to the case where bases are partial orders, we get:

Fact 4.1 *There is a 2-functor $(-)_@ : Mon \to 2 - Cat$ where Mon is the 2-category with:*
- objects: monoidal partial orders,
- arrows: monoidal functors,
- 2-cells: natural transformations [2]*,*
and $2 - Cat$ is the 2-category of 2-categories, 2-functors and 2-natural transformations.
It sends any category $\mathcal{V}$ to the 2-category $\mathcal{V} - Cat$. For any functor $F : \mathcal{V} \to \mathcal{W}$, $F_@ : \mathcal{V} - Cat \to: \mathcal{W} - Cat$, is as follows. For any $\mathcal{V}$-category A, $F_@A$ is the $\mathcal{W}$-category with the same objects as A and such that for any objects a, b of A, $F_@A(a,b) = FA(a,b)$. For any $\mathcal{V}$-functor $f : A \to B$, $F_@f : F_@A \to F_@B$ is the $\mathcal{W}$-functor with the same underlying map as f. For any monoidal transformation $F \leq G$, there is a natural transformation with value in A the $\mathcal{W}$-functor $F_@A \to G_@A$, with underlying map the identity of $Obj(A)$.

Further on we consider a monoidal functor between monoidal partial orders $F : \mathcal{V} \to \mathcal{W}$.

Fact 4.2 *There is a lax normal* [3] *functor $F_\sharp : \mathcal{V} - Mod \to \mathcal{W} - Mod$ that extends $F_@$, i.e. the diagram below commutes:*

[2] here, monoidal transformations are just natural transformations.
[3] A lax functor is *normal* when it preserves identities.

$$\begin{array}{ccc} \mathcal{V}-Cat & \xrightarrow{F_@} & \mathcal{W}-Cat \\ {\scriptstyle J_\mathcal{V}}\downarrow & & \downarrow{\scriptstyle J_\mathcal{W}} \\ \mathcal{V}-Mod & \xrightarrow[F_\sharp]{} & \mathcal{W}-Mod \end{array}$$

For any $\mathcal{V}$-module $\varphi:\ A \multimap B$, $F_\sharp(\varphi):\ F_@A \multimap F_@B$ is the map

$$\begin{cases} Obj(B)\times Obj(A)\to Obj(\mathcal{V}), \\ \forall a\in Obj(A), b\in Obj(B), F_\sharp(\varphi)(b,a)=F\varphi(b,a). \end{cases}$$

proof: Let $\varphi:\ A \multimap B$. For any objects a, a' of A and b of B,

$$\begin{aligned} F_@A(a,a')\otimes F_\sharp\varphi(b,a) &= FA(a,a')\otimes F\varphi(b,a) \\ &\le F(A(a,a')\otimes\varphi(b,a)) \\ &\le F(\varphi(b,a')). \end{aligned}$$

Similarly, for any object a of A and b, b' of B, $F_\sharp\varphi(b,a)\otimes F_@B(b',b)\le F_\sharp\varphi(b',a)$.

Now given $\varphi:\ A \multimap B$ and $\psi:\ B \multimap C$, for any objects a of A and c of C,

$$\begin{aligned} F_\sharp(\psi\bullet\varphi)(c,a) &= F(\psi\bullet\varphi)(c,a), \\ &= F(\textstyle\bigvee_{b\in Obj(B)}\varphi(b,a)\otimes\psi(c,b)), \\ &\ge \textstyle\bigvee_{b\in Obj(B)} F(\varphi(b,a)\otimes\psi(c,b)) \\ &\ge \textstyle\bigvee_{b\in Obj(B)} F\varphi(b,a)\otimes F\psi(c,b) \\ &= (F_\sharp\psi\bullet F_\sharp\varphi)(c,a). \end{aligned}$$

By definition $F_\sharp(f_\diamond)=(F_@f)_\diamond$ for any $\mathcal{V}$-functor f, thus for any $\mathcal{V}$-category A, $F_\sharp(1_{A\diamond})=(F_@A)_\diamond=(1_{F_@A})_\diamond$. ∎

Lemma 4.3 *Let $f:A\to B$ be a $\mathcal{V}-functor$. Then:*
(i) $F_\sharp(f_\diamond)\dashv F_\sharp(f^\diamond)$ in $\mathcal{W}-Mod$;
(ii) If $f_\diamond$ is an isomorphism in $\mathcal{V}-Mod$ then $F_\sharp(f^\diamond)\bullet F_\sharp(f_\diamond)=1$.

proof: (i): Results from the facts that for any $\mathcal{V}$-functor f, $F_\sharp(f_\diamond)=(F_@(f))_\diamond$ and also $F_\sharp(f^\diamond)=(F_@(f))^\diamond$.
(ii): Suppose $f_\diamond\bullet f^\diamond=1$ and $f^\diamond\bullet f_\diamond=1$. Composing the second inequality by $F_\sharp$ which is lax and normal, one gets $F_\sharp(f^\diamond)\bullet F_\sharp(f_\diamond)\le F_\sharp(f^\diamond\bullet f_\diamond)=F_\sharp(1)\le 1$. Thus $F_\sharp(f^\diamond)\bullet F_\sharp(f_\diamond)=1$ since $F_\sharp(f_\diamond)\dashv F_\sharp(f^\diamond)$. ∎

This motivates the definition

Definition 4.4 *A lax functor $G:\mathcal{V}\to\mathcal{W}$ is* super *when for any family of pairs of objects $(V_i,W_i)\in Obj(\mathcal{V})\times Obj(\mathcal{V})$, i ranging in I, if $I_\mathcal{V}\le\bigvee_{i\in I}V_i\otimes W_i$ then $I_\mathcal{W}\le\bigvee_{i\in I}G(V_i)\otimes G(W_i)$.*

Note that the composition of super functors and the identity functors are super.

Lemma 4.5 *If F is super then $F\bar{A}(-,-)\le F_\sharp(j_{A\diamond})\bullet F_\sharp(j_A{}^\diamond)$.*

proof: For any objects φ,ψ of $\bar{A}$,

$$\begin{aligned} F_\sharp(j_{A\diamond})\bullet F_\sharp(j_A{}^\diamond)(\varphi,\psi) &= \textstyle\bigvee_{a\in Obj(A)} Fj_A{}^\diamond(a,\psi)\otimes Fj_{A\diamond}(\varphi,a) \\ &= \textstyle\bigvee_{a\in Obj(A)} F\psi(a)\otimes F\tilde{\varphi}(a) \\ &\ge \textstyle\bigvee_{a\in Obj(A)}(F(\bar{A}(\varphi,\psi)\otimes\varphi(a))\otimes F\tilde{\varphi}(a)) \\ &= F\bar{A}(\varphi,\psi)\otimes\textstyle\bigvee_{a\in Obj(A)}(F\varphi(a)\otimes F\tilde{\varphi}(a)) \\ &\ge F\bar{A}(\varphi,\psi)\otimes I_\mathcal{W} \\ &\ge F\bar{A}(\varphi,\psi). \end{aligned}$$

■

According to this and 4.3,

Lemma 4.6 *If F is super then $F_\sharp(j_{A_\diamond})$ is an isomorphism, in this case $F_\sharp$ preserves the Morita-equivalence, i.e. if the $\mathcal{V}$-categories A and B are Morita equivalent then also are $F_@A$ and $F_@B$.*

One has also

Lemma 4.7 *If F is super then $F_\sharp : \mathcal{V} - Mod \to \mathcal{W} - Mod$ preserves adjoints.*

proof: Suppose that F is super. For any left adjoint $\varphi : A \longrightarrow\!\!\circ\!\!\!\rightarrow B$ in $\mathcal{V} - Mod$, and any objects a, a' of A, $I_\mathcal{V} \leq A(a,a) \leq \bigvee_{b \in Obj(B)}(\varphi(b,a) \otimes \tilde{\varphi}(a,b))$ thus
$I_\mathcal{W} \leq \bigvee_{b \in Obj(B)} F\varphi(b,a) \otimes F\tilde{\varphi}(a,b)$ and

$$\begin{aligned}
FA(a,a') &\leq FA(a,a') \otimes (\textstyle\bigvee_{b \in Obj(B)} F\varphi(b,a) \otimes F\tilde{\varphi}(a,b)) \\
&= \textstyle\bigvee_{b \in Obj(B)}((FA(a,a') \otimes F\varphi(b,a)) \otimes F\tilde{\varphi}(a,b)) \\
&\leq \textstyle\bigvee_{b \in Obj(B)}(F\varphi(b,a')) \otimes F\tilde{\varphi}(a,b)) \\
&= (F_\sharp\tilde{\varphi} \bullet F_\sharp\varphi)(a,a').
\end{aligned}$$

That $F_\sharp\varphi \bullet F_\sharp\tilde{\varphi} \leq 1$ results from the facts that $F_\sharp$ is lax and normal and $\varphi \bullet \tilde{\varphi} \leq 1$: $F_\sharp(\varphi) \bullet F_\sharp(\tilde{\varphi})$ $\leq F_\sharp(\varphi \bullet \tilde{\varphi}) \leq F_\sharp(1) = 1$. ■

4.2 The 2-categories Enr and SC-Enr

For convenience we write the "enrichment $(A, \mathcal{V})$" for the "$\mathcal{V}$-category A". Given a monoidal functor $F : \mathcal{V} \to \mathcal{W}$, enrichments $(A, \mathcal{V})$ and $(B, \mathcal{W})$, and a map $f : Obj(A) \to Obj(B)$, we say that F is *compatible* with the triple (f, A, B) when f defines a $\mathcal{W}$-functor $F_@A \to B$.

In this section $\mathcal{C}$ is the locally preordered 2-category with:
- objects: enrichments;
- arrows from $(A, \mathcal{V})$ to $(B, \mathcal{W})$: ordered pairs (f, F) where f is a map $Obj(A) \to Obj(B)$ and $F : \mathcal{V} \to \mathcal{W}$ is a monoidal functor compatible with (f, A, B);
- composition: the composite ($(B, \mathcal{W}) \xrightarrow{(g,G)} (C, \mathcal{X})$) $\circ$ ($(A, \mathcal{V}) \xrightarrow{(f,F)} (B, \mathcal{W})$) is the pair $(GF : \mathcal{V} \to \mathcal{X}, g \circ f)$;
- 2-cells : the 2-cell $(f, F) \Rightarrow (g, G) : A \to B$ exists when there is a (monoidal) natural transformation $F \leq G$ and for any object a of A, there exists the arrow $I \to B(fa, ga)$ in $\mathcal{W}$.

Enr, the locally preordered 2-category of enrichments has objects enrichments, arrows those (f, F) of $\mathcal{C}$ with F super and, local preorders inherited from $\mathcal{C}$.

It is straightforward to check that both $\mathcal{C}$ and *Enr* are well-defined 2-categories.

$SC - Enr$ will stand for the locally preordered 2-category with objects the skeletal and Cauchy-complete enrichments and with arrows and local preorders inherited from *Enr*.

Proposition 4.8 *The inclusion 2-functor $SC - Enr \hookrightarrow Enr$ has a 2-left-adjoint. Precisely, for any enrichment A, the arrow $A \xrightarrow{(j_A,1)} \bar{A}$ (that lies in Enr) is such that:*

- For any arrow $A \xrightarrow{(f,F)} B$ in Enr with B skeletal and Cauchy-complete, there is an unique arrow $(\bar{f}, \bar{F})$ that renders commutative the diagram in $\mathcal{C}$ below.

$$\begin{tikzcd}
A \arrow[r, "{(j_A,1)}"] \arrow[dr, "{(f,F)}"'] & \bar{A} \arrow[d, "{(\bar{f},\bar{F})}"] \\
& B
\end{tikzcd}$$

This arrow $(\bar{f},\bar{F})$ is in Enr. It is given by $\bar{F} = F$ and $\bar{f}_\diamond = f_\diamond \bullet F_\sharp(j_{A_\diamond})$.
- There is a 2-cell $(f,F) \leq (g,G) : A \to B$ if and only if there is one $(\bar{f},F) \leq (\bar{g},G) : \bar{A} \to B$.

proof: Let $(f,F) : A \to B$ be in Enr, $\bar{A}$ be the Cauchy-completion of A and, B be skeletal and Cauchy-complete. We are looking for some arrow $(\bar{f},\bar{F}) : \bar{A} \to B$ in Enr such that $(\bar{f},\bar{F}) \circ (j_A, 1_A) = (f,F)$. Necessarely $\bar{F} = F$. The composition on the right by the isomorphism of $\mathcal{W}-Mod$, $F_\sharp(j_A{}^\diamond)$ (see below),

$$\begin{tikzcd}
F_@(A) \arrow[r, "F_\sharp(j_{A_\diamond})", "\circ" marking] \arrow[dr, "f"'] & F_@(\bar{A}) \arrow[d, "\bar{f}"] \\
& B
\end{tikzcd}$$

yields an isomorphism of categories $\mathcal{W}-Mod(F_@\bar{A},B) \cong \mathcal{W}-Mod(F_@A,B)$. Thus by Cauchy-completness and skeletality of B, composition on the right by $F_@(j_A)$ is an isomorphism $\mathcal{W}-Cat(F_@\bar{A},B) \cong \mathcal{W}-Cat(F_@A,B)$. Therefore such an $\bar{f}$ exists and is unique, it satisfies $\bar{f}_\diamond = f_\diamond \bullet F_\sharp(j_{A_\diamond})$. Given another functor $(g,G) : A \to B$. We have to show that there is a natural transformation $(f,F) \Rightarrow (g,G)$ if and only if there is one $(\bar{f},F) \Rightarrow (\bar{g},G)$ where $\bar{g}$ is such that $\bar{g} \circ G_@(j_A) = g$. For any $\mathcal{V}$-category B, let $\theta_B : F_@B \to G_@B$ denote the $\mathcal{V}$-functor with underlying map the identity of $Obj(B)$. Then the situation in $\mathcal{W}-Mod$ is depicted below

$$\begin{tikzcd}[column sep=large, row sep=huge]
& F_@A \arrow[r, shift left, "F_\sharp(j_{A_\diamond})"] \arrow[r, phantom, "\cong"] \arrow[dl, "\theta_A"'] \arrow[drr, "f_\diamond" near start] \arrow[d, phantom, "(1)\ \Swarrow"] & F_@\bar{A} \arrow[l, shift left, "F_\sharp(j_A{}^\diamond)"] \arrow[dl, "\theta_{\bar{A}}" near end] \arrow[d, "\bar{f}_\diamond"] \\
G_@A \arrow[r, shift left, "G_\sharp(j_{A_\diamond})"] \arrow[r, phantom, "\cong"] \arrow[dr, "g_\diamond"'] & G_@\bar{A} \arrow[l, shift left, "G_\sharp(j_A{}^\diamond)"] \arrow[d, "\bar{g}_\diamond"] \arrow[r, phantom, "(2)\ \Swarrow"] & B \arrow[dl, "1"] \\
& B &
\end{tikzcd}$$

It is now straightforward that 2-cell (1) exists if and only 2-cell (2) exists. This is equivalent to the desired result since B is skeletal and Cauchy-complete.

∎

4.3 Enr° and derivatives

The category Enr° is as follows. It has the same objects as Enr. A morphism $(A,\mathcal{V}) \to (B,\mathcal{W})$, is a map $f : Obj(A) \to Obj(B)$ such that there is a super functor $F : \mathcal{V} \to \mathcal{W}$ compatible with

(f, A, B). There is an obvious forgetfull 2-functor $Enr \to Enr^\circ$ where Enr° is seen as a locally discrete 2-category. We define also the category $SC - Enr^\circ$ as the full subcategory of Enr° with objects skeletal and Cauchy-complete enrichments. A question is to know whether the inclusion functor $SC - Enr^\circ \hookrightarrow Enr^\circ$ admits some left adjoint. Indeed we could not answer it but the following related result appears relevant for our case of study.

Futher on $\mathcal{F}$ denotes a locally partially ordered 2-category with objects some monoidal partial orders, with arrows some super functors and, with 2-cells inherited from Mon. We will note $Enr(\mathcal{F})$ the locally preordered 2-category with objects enrichments over objects of $\mathcal{F}$, with arrows the (f, F) of Enr where F is an arrow of $\mathcal{F}$, and with local preorders inherited from Enr. $SC - Enr(\mathcal{F})$ will be the 2-category with objects skeletal and Cauchy-complete enrichments of $Enr(\mathcal{F})$ and, with arrows and 2-cells inherited from $Enr(\mathcal{F})$. Finally we extend the notation $(-)^\circ$ to define obviously the categories $Enr(\mathcal{F})^\circ$ and $SC - Enr(\mathcal{F})^\circ$. Then,

Theorem 4.9 *If $\mathcal{F}$ is locally filtered then the inclusion functor $SC - Enr(\mathcal{F})^\circ \hookrightarrow Enr(\mathcal{F})^\circ$ has a left adjoint.*

To see this, we will check that given any enrichment $(A, \mathcal{V})$ with $\mathcal{V}$ in $\mathcal{F}$, the morphism $(j_A, 1) : (A, \mathcal{V}) \to (\bar{A}, \mathcal{V})$ in $Enr(\mathcal{F})$, is such that for any morphism $(f, F) : (A, \mathcal{V}) \to (B, \mathcal{W})$ in $Enr(\mathcal{F})$ with $(B, \mathcal{W})$ skeletal and Cauchy-complete, there is a unique map $g : Obj(\bar{A}) \to Obj(B)$ such that:
- $g \circ j_A = f$;
- there exists a G in $\mathcal{F}$ compatible with $(g, \bar{A}, B)$.
We already know that such a map exists, namely $\bar{f}$ as defined in 4.8 works. Indeed, it is the unique such map. Given a map g (with some G) that satisfies the two conditions above, we shall prove $g = \bar{f}$.

By assumption there is an H in $\mathcal{F}$ with $H \leq F$, $H \leq G$. Note that this H is compatible with $(\bar{f}, \bar{A}, B)$ and $(g, \bar{A}, B)$

At this point, we need to introduce notations to avoid ambiguities. Mainly we shall distinguish enriched functors from their underlying maps. $j^F : F_@A \to F_@\bar{A}$, $j^G : G_@A \to G_@\bar{A}$, and $j^H : H_@A \to H_@\bar{A}$ will denote the $\mathcal{W}$-functors with underlying map $j_A : A \to \bar{A}$. So we define the $\mathcal{W}$-functors $f^F : F_@A \to B$, $g^G : G_@\bar{A} \to B$, $g^H : H_@\bar{A} \to B$, $\bar{f}^F : F_@\bar{A} \to B$ and, $\bar{f}^H : H_@\bar{A} \to B$ with respective underlying maps f, g, g, $\bar{f}$ and, $\bar{f}$.

Since F, G and H are super, the $\mathcal{W}$-modules $(j^F)_\diamond$, $(j^G)_\diamond$ and $(j^H)_\diamond$ are isomorphims with respective inverse $(j^F)^\diamond$, $(j^G)^\diamond$ and $(j^H)^\diamond$. Now since $H \leq F$ and $H \leq G$, the identity map $1_A : Obj(A) \to Obj(A)$ defines $\mathcal{W}$-functors $1_A^F : H_@A \to F_@A$ and $1_A^G : H_@A \to G_@A$. As well, the identity map $1_{\bar{A}} : Obj(\bar{A}) \to Obj(\bar{A})$ defines $\mathcal{W}$-functors $1_{\bar{A}}^F : H_@\bar{A} \to F_@\bar{A}$, $1_{\bar{A}}^G : H_@\bar{A} \to G_@\bar{A}$. The situation in $\mathcal{W} - Mod$ is depicted below.

$$
\begin{tikzcd}
F_{@}\bar{A} \arrow[d, "\cong"', "(j^F)^\diamond"] \arrow[dd, bend right=60, "1"'] \arrow[dr, phantom, "(III)"] & H_{@}\bar{A} \arrow[l, "(1^F_{\bar{A}})_\diamond"'] \arrow[r, "(1^G_{\bar{A}})_\diamond"] \arrow[d, "\cong"', "(j^H)^\diamond"] \arrow[dr, phantom, "(IV)"] & G_{@}\bar{A} \arrow[d, "\cong"', "(j^G)^\diamond"] \arrow[dd, bend left=60, "1"] \\
F_{@}A \arrow[d, "\cong"', "(j^F)_\diamond"] \arrow[dr, phantom, "(I)"] & H_{@}A \arrow[l, "(1^F_A)_\diamond"'] \arrow[r, "(1^G_A)_\diamond"] \arrow[d, "\cong"', "(j^H)_\diamond"] \arrow[dr, phantom, "(II)"] & G_{@}A \arrow[d, "\cong"', "(j^G)_\diamond"] \\
F_{@}\bar{A} \arrow[dr, "(\bar{f}^F)_\diamond"'] & H_{@}\bar{A} \arrow[l, "(1^F_{\bar{A}})_\diamond"'] \arrow[r, "(1^G_{\bar{A}})_\diamond"] & G_{@}\bar{A} \arrow[dl, "(g^G)_\diamond"] \\
 & B &
\end{tikzcd}
$$

Since $j^F \circ 1^F_A = 1^F_{\bar{A}} \circ j^H$, $(j^F)_\diamond \circ (1^F_A)_\diamond = (1^F_{\bar{A}})_\diamond \circ (j^H)_\diamond$ — this is the commutation of diagram (I). Multiplying on the left $(j^F)^\diamond$ and on the right by $(j^H)_\diamond$ the two sides of the previous equation, one gets $(j^F)^\diamond \circ (1^F_{\bar{A}})_\diamond = (1^F_A)_\diamond \circ (j^H)^\diamond$ which is the commutation of (III). Similarly (II) and (IV) commute. This shows that $(\bar{f}^F)_\diamond \circ (1^F_{\bar{A}})_\diamond = (g^G)_\diamond \circ (1^G_{\bar{A}})_\diamond$, thus $(\bar{f}^H)_\diamond = (g^H)_\diamond$ and $\bar{f} = g$ since B is skeletal and Cauchy-complete. ∎

5 Quasi-uniform triples

5.1 Definition

Definition 5.1 (Quasi-uniform triples) *A* quasi-uniform triple $(S, \cdot, \leq)$ *consists of a semi-group $(S, \cdot)$ with base S, composition operation "$\cdot$", with S partially ordered by $\leq$. It satisfies morever the following axioms:*
$(A1)$ *$\forall s, t \in S$, $(s \cdot t \geq s) \ \wedge \ (s \cdot t \geq t)$;*
$(A2)$ *$\forall s, t, u \in S, t \leq u \Rightarrow (s \cdot t \leq s \cdot u) \wedge (t \cdot s \leq u \cdot s)$;*
$(A3)$ *$\forall s \in S$, $\exists t \in S, t^2 \leq s$;*
$(A4)$ *S is filtered, i.e. $\forall s, t \in S, \exists u \in S, u \leq s \wedge u \leq t$.*

For the quasi-uniform triple $(S, \cdot, \leq)$, we define the following. The operation "$\cdot$" is extended to $\wp(S)$ by $V \cdot W = \{s \cdot t \mid s \in V, t \in W\}$. Further on, for some $V \subseteq S$, $[V]$ will denote the upper-closure of V with respect to $\leq$. $\mathcal{I}_S$ will stand for the set of upper-closed subsets of S. The binary operation $\otimes$ in $\mathcal{I}_S$ is given by $V \otimes W = [V \cdot W]$. We define the operation ρ by $\rho(V) = S \otimes V \otimes S$. Note that according to $(A3)$, $S \otimes S = S$. From this, one checks that ρ is indeed a lower closure operation (it is monotonous, decreasing with respect to the inclusion and idempotent). $\mathcal{O}_S$ will stand for the sets of fixpoints for ρ. It is stable for $\otimes$ since for any $V, W \in \mathcal{O}_S$,

$$
\begin{aligned}
V \otimes W &= S \otimes V \otimes S \otimes S \otimes W \otimes S \\
&= S \otimes S \otimes V \otimes S \otimes S \otimes W \otimes S \otimes S \\
&= S \otimes V \otimes W \otimes S.
\end{aligned}
$$

Therefore $\mathcal{O}_S$ provided with the restriction of $\otimes$ – still denoted $\otimes$, is a monoid with S as neutral element.

Observation 5.2 *Given a quasi-uniform triple $(S, \cdot, \leq)$, W and a family $(V_i)_{i\in I}$ both in $\mathcal{I}_S$, $W \otimes (\bigcup_{i\in I} V_i) = \bigcup_{i\in I}(W \otimes V_i)$ and $(\bigcup_{i\in I} V_i) \otimes W = \bigcup_{i\in I}(V_i \otimes W)$.*

Fact 5.3 *Given a quasi-uniform triple $(S, \cdot, \leq)$, a monoidal partial order $\mathcal{C}_S$ is defined by the following data:*
- objects: elements of $\mathcal{O}_S$;
- order: inclusion;
- identity: S;
- tensor product: $\otimes$.
$\mathcal{C}_S$ is moreover cocomplete and biclosed. For any family $(V_i)_{i\in I}$, its least upper bound is $\bigvee_{i\in I} V_i = \bigcup_{i\in I} V_i$, its greatest lower bound is $\bigwedge_{i\in I} V_i = \rho(\bigcap_{i\in I} V_i)$.

proof: Checking that $\mathcal{C}_S$ is a posetal monoidal category is straightforward. Let us see that it is cocomplete. For any family $(V_i)_{i\in I}$ of elements $\mathcal{O}_S$, $\bigcup_{i\in I} V_i$ lies in $\mathcal{O}_S$, since it is upper closed and, according to 5.2, $S \otimes (\bigcup_{i\in I} V_i) \otimes S = \bigcup_{i\in I} S \otimes V_i \otimes S = \bigcup_{i\in I} V_i$. Therefore $\bigcup_{i\in I} V_i$ is the least upper bound of the V_i's in $\mathcal{O}_S$. Since ρ is a right adjoint, it preserves greatest lower bounds. Thus $\mathcal{C}_S$ is cocomplete with greatest lower bounds as above. $\mathcal{C}_S$ with inclusion ordering has a least upper bound S and greatest lower bound $\emptyset$. $\mathcal{C}_S$ is biclosed: for any $V, V', W \in \mathcal{O}_S$,

$V' \otimes V \subseteq W$ if and only if
$V' \subseteq \{s \in S \mid \forall t \in V, s \cdot t \in W\}$ if and only if
$V' \subseteq \rho(\{s \in S \mid \forall t \in V, s \cdot t \in W\})$.

Analogously, $V \otimes V' \subseteq W$ if and only if $V' \subseteq \rho(\{s \in S \mid \forall t \in V, t \cdot s \in W\})$. ∎

5.2 Morphisms of quasi-uniform triples

Recall that given a binary relation $R \subseteq S \times T$. The maps

$$\begin{cases} \exists_R : \wp(S) \to \wp(T), \\ V \mapsto \{u' \mid \exists u \in V, R(u, u')\} \end{cases}$$

and

$$\begin{cases} \forall_R : \wp(T) \to \wp(S), \\ V' \mapsto \{u \mid \forall u' \in T, R(u, u') \Rightarrow u' \in V'\} \end{cases}$$

defines an adjoint pair $\exists_R \dashv \forall_R : \wp(S) \to \wp(T)$, the powersets $\wp(S)$ and $\wp(T)$ being partially ordered with inclusion. (For any $V \subseteq S$ and $V' \subseteq T$, $\exists_R(V) \subseteq V' \Leftrightarrow V \subseteq \forall_R(V')$).

Definition 5.4 *Given two quasi-uniform triples S and T, a morphism from S to T is a relation $R \subseteq S \times T$ such that:*
- (i) $\exists_R(S) = T$;
- (ii) $\forall s_1, s_2 \in S, \forall t \in T,\ R(s_2, t) \wedge (s_1 \leq s_2) \Rightarrow R(s_1, t)$;
- (iii) $\forall s \in S, \forall t_1, t_2 \in T,\ R(s, t_1) \wedge (t_1 \leq t_2) \Rightarrow R(s, t_2)$;
- (iv) $\forall s_1, s_2 \in S, \forall t_1, t_2 \in T,\ R(s_1, t_1) \wedge R(s_2, t_2) \Rightarrow R(s_1 s_2, t_1 t_2)$.

For any quasi-uniform triple S the partial order $\leq \subseteq S \times S$ is a morphism of quasi-uniform triples. It is straightforward to check that quasi-uniform triples and their morphisms form a locally posetal 2-category QUT, the local partial orders being the inclusion relations.

Fact 5.5 *QUT is locally finitely complete. Precisely, given morphisms of quasi-uniform triples $P, Q : S \to T$, the relation $P \cap Q$ defines a morphism $S \to T$.*

proof: For any $t \in T$, since P and Q satisfies (i), there are some $s_1, s_2 \in S$ such that $P(s_1, t)$ and $Q(s_2, t)$. Since S is filtered there is some $s \in S$ such that $s \leq s_1$ and $s \leq s_2$. Thus since P, Q satisfy (ii), $P(s,t)$ and $Q(s,t)$.
That $P \cap Q$ satisfies axioms (ii), (iii) (iv) is straightforward. ∎

Further on $R : S \to T$ is a morphism of quasi-uniform triples. Since R satisfies (ii) for any subsets $P \subseteq S$, $\exists_R(P)$ is upper closed, and dually since R satifies (iii), for any $Q \subseteq T$, $\forall_R(Q)$ is downward-closed.
Also, for any subsets P, Q of S,

$$\begin{aligned}\exists_R(P) \otimes \exists_R(Q) &= \uparrow \{v_1 \cdot v_2 \mid (\exists u_1 \in P, R(u_1, v_1)) \wedge (\exists u_2 \in Q, R(u_2, v_2))\} \\ &\subseteq \uparrow \{v \mid \exists u_1 \in P, u_2 \in Q, (u_1 \cdot u_2, v) \in R\} \text{ (according to } (vi)) \\ &= \exists_R(P \otimes Q).\end{aligned}$$

Definition 5.6 *The morphism R is said* dense *when for any $V \in \mathcal{I}_S$, $\exists_R(V)$ belongs to $\mathcal{O}_T$.*

Note that R is dense when one of the two equivalent assertions hold:
- for all $s \in S$, the upper set $\{t \in T \mid R(s,t)\}$ belongs to $\mathcal{O}_T$;
- $\forall s \in S, t \in T, R(s,t) \Rightarrow \exists t_1, t_2, t_3 \in T, t \geq t_1 t_2 t_3 \wedge R(s, t_2)$.

Fact 5.7 *There is larger dense morphism amongst subrelations of R, namely $\bar{R} \subseteq R$. It is defined for all $s \in S$ ans $t \in T$ by: $\bar{R}(s,t) \Leftrightarrow \exists t_1, t_2, t_3 \in T, t \geq t_1 t_2 t_3 \wedge R(s, t_2)$.*

proof: To check that $\bar{R}$ satifies axioms (i), (ii) and (iii), is straightforward. Let us see that it satisfies (iv). Suppose $s_1, s_2 \in S, t_1, t_2 \in T$, $\bar{R}(s_1, t_1)$ and $\bar{R}(s_2, t_2)$. Then there are $u_1, u_2, u_3, v_1, v_2, v_3 \in T$ such that $t_1 \geq u_1 u_2 u_3$, $R(s_1, u_2)$, $t_2 \geq v_1 v_2 v_3$ and $R(s_2, v_2)$. Then $t_1 t_2 \geq u_1 u_2 u_3 v_1 v_2 v_3 \geq u_1 u_2 v_2 u_3$ and $R(s_1 s_2, u_2 v_2)$. ∎

Note also that $\bar{\bar{R}} = \bar{R}$ and the composition of dense morphism is also dense.
Consider the partial order $\leq_S \subseteq S \times S$, we will write $\sqsubseteq_S$ for the relation $\bar{\leq}_S$. $\sqsubseteq_S$ is the relation given for all $s, t \in S$ by: $s \sqsubseteq_S t \Leftrightarrow \exists t_1, t_2, t_3 \in S, t \geq t_1 t_2 t_3 \wedge s \leq t_2$. Then $\bar{R}$ is just the composite $\sqsubseteq_T \circ R$. One has also

Fact 5.8 *If R is dense then $R \circ \sqsubseteq_S = R$.*

proof: Since $\sqsubseteq_S$ is a subrelation of $\leq_S$ then $(R \circ \sqsubseteq_S)$ is a subrelation of $(R \circ \leq_S) = R$. Conversely, suppose $R(s,t)$. Since R is dense, there are $t_1, t_2, t_3 \in T$ such that $t \geq t_1 t_2 t_3$ and $R(s, t_2)$. Now there are some $s_1, s_3 \in S$ such that $R(s_1, t_1)$ and $R(s_3, t_3)$. Thus $R(s_1 s s_3, t)$ showing $(R \circ \sqsubseteq_S)(s,t)$.
∎

5.3 Dense morphims and super functors

For any morphism of quasi-uniform triples $R : S \to T$, we define maps $F'(R) : \mathcal{I}_S \to \mathcal{I}_T$ and $F(R) : \mathcal{O}_S \to \mathcal{O}_T$ as follows. $F'(R)$ is the restriction of $\exists_R$ on $\mathcal{I}_S$ and and $F(R) = \rho \circ F'(R) \circ i$ where i denotes there the embedding $\mathcal{O}_S \to \mathcal{I}_S$ (see below)

$$\begin{tikzcd} \mathcal{I}_S \arrow[r, "F'(R)"] & \mathcal{I}_T \arrow[d, "\rho"] \\ \mathcal{O}_S \arrow[u, "i"] \arrow[r, "F(R)"'] & \mathcal{O}_T \end{tikzcd}$$

Then $\bar{R}$ is the unique dense morphism P such that $F(P) = F(R)$.

Facts 5.9 *Given morphisms of abstract quasi-uniform uples $R : S \to T$ and $P : T \to U$, one has:*
- $F'(R)$ preserves least upper bounds.
- $F(R)$ preserves least upper bounds.
- $F'(P \circ R) = F'(P) \circ F'(R)$
- $F(P \circ R) \supseteq F(P) \circ F(R)$.

proof: Since $\exists_R$ is a left adjoint and any union of upper-closed subsets is upper-closed, F' preserves least upper bound. Since i and ρ also preserves least upper bounds, F preserves least upper bounds. It is immediate that $F'(P \circ R) = F'(P) \circ F'(R)$. Thus

$$\begin{aligned} F(P) \circ F(R) &= \rho \circ F'(P) \circ i \circ \rho \circ F'(R) \circ i \\ &\subseteq \rho \circ F'(P) \circ F'(R) \circ i \\ &= \rho \circ F'(P \circ R) \circ i \\ &= F(P \circ R). \end{aligned}$$

∎

Fact 5.10 *For any morphism of quasi-uniform triples $R : S \to T$, the above map $F(R)$ defines a normal super monoidal functor $\mathcal{C}_S \to \mathcal{C}_T$*

proof: Since $F'(R)(S) = T$, $F(R)(S) = \rho \circ F'(R)(T) = \rho(T) = T$.
For any $P, Q \in \mathcal{O}_S$,

$$\begin{aligned} F(R)(P \otimes Q) &= \rho \circ F'(R) \circ i(P \otimes Q) \\ &\supseteq \rho \circ (F'(R) \circ i(P) \otimes F'(R) \circ i(Q)) \\ &\supseteq (\rho \circ F'(R) \circ i)(P) \otimes (\rho \circ F'(R) \circ i)(Q) \\ &= F(R)(P) \otimes F(R)(Q) \end{aligned}$$

Let us see that $F(R)$ is super. Let $(V_i, W_i)_{i \in I}$ be a family of pairs of objects in $\mathcal{C}_S$ such that $S \subseteq \bigcup_{i \in I} V_i \otimes W_i$. Given $s' \in T$ there is a $s'' \in T$ such that ${s''}^2 \le s'$. Since $T = F(R)(S)$ $= F(R)(\bigcup_{i \in I} V_i \otimes W_i) = \bigcup_{i \in I} F(R)(V_i \otimes W_i)$, for this s'' there is some $i \in I$, such that $s'' \in F(R)(V_i \otimes W_i)$. Thus $s'' \in F(R)(V_i) \cap F(R)(W_i)$ and $s' \in F(R)(V_i) \otimes F(R)(W_i)$. ∎

6 Abstract Quasi-uniformities

6.1 Definition

Definition 6.1 *An* abstract quasi-uniformities A *over a quasi-uniform triple S denoted also (A, S), is a set $Obj(A)$ together with a map: $A(-,-) : Obj(A) \times Obj(A) \to \mathcal{I}_S$ that satisfies:*
(i) $\forall x, y, z \in Obj(A)$, $A(x, y) \otimes A(y, z) \subseteq A(x, z)$,
(ii) $\forall x \in Obj(A)$, $S \subseteq A(x, x)$.

Let S, T, U be quasi-uniform triples, and A, B and, C be abstract quasi-uniformities respectively over S, T and U.

Definition 6.2 *Let f be a map $Obj(A) \to Obj(B)$. A relation $R \subseteq S \times T$ is said* compatible *with the triple (f, A, B) — or simply "with f", when there is no ambiguity — when $(*)$ for all $s, s' \in S$, for all $x, y \in Obj(A)$, $s \in A(x, y) \wedge R(s, s') \Rightarrow s' \in B(fx, fy)$.*

The set of compatible relations with some f is stable under arbitrary unions and intersections. The largest relation compatible with (f, A, B) is denoted $R_{f,A,B}$ (or simply R_f), it is defined for all $s \in S$ and $t \in T$ by $R_f(s,t)$ if and only if $\forall x, y \in Obj(A), s \in A(x,y) \Rightarrow t \in B(fx, fy)$ R_f satisfies the conditions (ii) and (iii) of 5.4. If the relation R is compatible with (f, A, B) and Q is compatible with (g, B, C) then $R \circ Q$ is compatible $(g \circ f, A, C)$.

A morphism of quasi-uniformities $(A, S) \to (B, T)$ consists in a pair (f, R) where f is a map $Obj(A) \to Obj(B)$ and R is a morphism $S \to T$ of quasi-uniform triples that is compatible with (f, A, B). Note that if $R : S \to T$ is a morphism then it is compatible with (f, A, B) if and only if for all objects x, y of A, $F'(R)A(x,y) \subseteq B(fx, fy)$.

Abstract quasi-uniformities and their morphisms form a locally preordered 2-category $AQUnif$. The preorder on the homsets $AQUnif((A,S),(B,T))$ is given by $(f,R) \leq (g,P)$ if and only if $R \subseteq P$ and for all object x of A, $T \subseteq B(fx, gx)$. The identity morphism in (A,S) is the pair $(1_A, \leq_S \subseteq S \times S)$.

The category $AQUnif^\circ$ is as follows. It has objects abstract quasi-uniformities and arrows $(A,S) \to (B,T)$, the maps $f : Obj(A) \to Obj(B)$ for which there is a morphism $R : S \to T$ such that the pair (f, R) becomes a morphism of abstract quasi uniformities $(A,S) \to (B,T)$. There is an evident "forgetfull" 2-functor $U : AQUnif \to AQUnif^\circ$ where $AQUnif^\circ$ is a considered as a locally discrete category.

Observation 6.3 *Given a morphism of abstract quasi-uniformities $(f,R) : (A,S) \to (B,T)$, since R satisfies (i) of 5.4 and is compatible with f, then*

$$\forall t \in T, \exists s \in S, \forall x, y \in Obj(A), s \in A(x,y) \Rightarrow t \in B(fx, fy)$$

6.2 Dense abstract quasi-uniformities

Let S be some quasi-uniform triple. An enrichment of the form $(A, \mathcal{C}_S)$ corresponds exactly to an abstract quasi-uniformity (A,S) where the map $A(-,-)$ takes its values in $\mathcal{O}_S$. Such an abstract quasi-uniformity will be called *dense*. A morphism between dense abstract quasi-uniformities $(f,R) : (A,S) \to (B,T)$ is said *dense* when R is dense. $DAQUnif$ will denote the locally preordered 2-category with:
- objects: dense abstract quasi-uniformities,
- arrows: dense morphisms,
- local partial orders inherited from $AQUnif$.

Note that in $DAQUnif$, the unit at (A,S) is the pair $(1_A, \sqsubseteq_S)$ where 1_A denotes the identity map on $Obj(A)$.

For any abstract quasi-uniformity A over S, the dense abstract quasi-uniformities $Int(A)$ over S, is as follows. It has the same objects as A and for any $x, y \in Obj(A)$, $Int(A)(x,y) = \rho(A(x,y))$. For any morphism of abstract quasi-uniformities $(f,R) : A \to B$, since for all $x, y \in Obj(A)$,

$$\begin{aligned} F'(\bar{R}) \circ \rho(A(x,y)) &= \rho \circ F'(R) \circ \rho(A(x,y)) \\ &\subseteq \rho \circ F'(R)(A(x,y)) \\ &\subseteq \rho(B(fx, fy)), \end{aligned}$$

we let $Int(f,R)$ denote the dense morphism $(f, \bar{R}) : Int(A) \to Int(B)$.

Fact 6.4 *The above data defines a normal lax functor $Int : AQUnif \to DAQUnif$.*

Let (A,S) be an abstract quasi-uniformity. Since for any objects x,y of A, $\rho(A(x,y)) \subseteq A(x,y)$, there is a morphism of abstract quasi-uniformities $Int(A) \to A$ defined by the pair $(1_A, \leq_S)$. Now $\sqsubseteq_S$ is compatible with $(1_A, A, Int(A))$. Thus the pair $(1_A, \sqsubseteq_S)$ defines a morphism $A \to Int(A)$. This shows that

Proposition 6.5 *There is an equivalence $Int^\circ : AQUnif^\circ \cong DAQUnif^\circ$.*

where $DAQUnif^\circ$ has an obvious meaning.

Given abstract quasi-uniformities (A,S), we write $E(A)$ for the enrichment over $\mathcal{C}_S$ that corresponds to $Int(A)$. For any arrow $(f,R) : (A,S) \to (B,T)$ in $AQUnif$, one gets the arrow $E(f,R) = (f, F(R)) : E(A) \to E(B)$. These assignments define a lax functor $E : AQUnif \to Enr$. It factorises as $E = E_D \circ Int$, where E_D is a lax functor $DAQUnif \to Enr$, that is such that the $E_D : DAQUnif(A,B) \to Enr(E(A), E(B))$ are full embeddings. From this, one deduces also two functors $E^\circ : AQUnif^\circ \to Enr^\circ$ and ${E_D}^\circ : DAQUnif^\circ \to Enr^\circ$, with $E^\circ = {E_D}^\circ \circ Int^\circ$. Since ${E_D}^\circ$ is faithfull then also is E°.

7 Completions

7.1 Cauchy filters / adjoint modules

Let (A,S) be an abstract quasi-uniformity.

Definition 7.1 (Cauchy filters) *A filter F on $Obj(A)$ is* Cauchy *when it satisfies*

$$(C)\ \forall u \in U, \exists f \in F, \forall x,y \in f, u \in A(x,y)$$

A filter basis is Cauchy when the filter that it generates is Cauchy.

In the definition above we could have replaced the condition (C) by:

$$(C') \quad \forall u \in U, \exists f \in F, \forall x,y \in f, u \in A(x,y) \otimes A(y,x).$$

Fact 7.2 *Let $(f,R) : (A,S) \to (B,T)$ be a morphism of abstract quasi-uniformities. If F is a Cauchy filter on (A,S) then $f(F)$, the direct image by of F by f is a Cauchy basis on (B,T).*

proof: Since the direct image of any filter is a filter basis, it remains to check that $f(F)$ is Cauchy. Let $t \in T$. According to 6.3, we may find some $s \in S$ such that $\forall x,y \in Obj(A), s \in A(x,y) \Rightarrow t \in B(fx,fy)$. Since F is Cauchy, there is $p \in F$ such that $\forall x,y \in Obj(A), x,y \in p \Rightarrow s \in A(x,y)$ Then $\forall x,y \in Obj(A), x,y \in p \Rightarrow t \in B(fx,fy)$. ∎

According to this and since for any abstract quasi-uniformities A, there are two morphisms $(1_A, \leq) : A \to Int(A)$ and $(1_A, \sqsubseteq) : A \to Int(A)$, Cauchy filters on A and on $Int(A)$ are the same and A is complete (resp. separated) if and only if $Int(A)$ is.

Further on we suppose that A is dense. We shall denote by A the corresponding $\mathcal{C}_S$-category. We shall relate adjoint pairs $\varphi \dashv \tilde{\varphi} :\ 1 \longrightarrow\!\!\!\!\circ\!\!\!\!\to A$ and Cauchy-filters on A (7.5). A Cauchy filter F is said *minimal* when it is minimal for the inclusion relation.

Fact 7.3 *To any adjoint pair of $\mathcal{C}_S$-modules $\varphi \dashv \tilde{\varphi}$ with $\varphi :\ \hat{1} \longrightarrow\!\!\!\!\circ\!\!\!\!\to A$ corresponds a Cauchy filter $\bar{\Gamma}(\varphi)$ on $Obj(A)$ with basis the family $\Gamma(\varphi)$ of sets $\Gamma(\varphi)(u) = \{x \in Obj(A) \mid u \in \varphi(x) \otimes \tilde{\varphi}(x)\}$, u ranging in S. $\bar{\Gamma}(\varphi)$ is a minimal Cauchy filter. Precisely any Cauchy filter compatible with $\Gamma(\varphi)$ is finer than $\Gamma(\varphi)$.*

proof: Consider an adjoint pair as above. By definition, the maps $\varphi : Obj(A) \to \mathcal{O}_S$, $\tilde{\varphi} : Obj(A) \to \mathcal{O}_S$ are such that

- $\forall x, y \in Obj(A), \varphi(x) \otimes A(y,x) \subseteq \varphi(y)$;
- $\forall x, y \in Obj(A), A(x,y) \otimes \tilde{\varphi}(x) \subseteq \tilde{\varphi}(y)$.

and

(1) $S \subseteq \bigcup_{x \in Obj(A)} \varphi(x) \otimes \tilde{\varphi}(x)$;
(2) $\forall x, y \in Obj(A), \tilde{\varphi}(y) \otimes \varphi(x) \subseteq A(x,y)$.

According to (1), for all u, $\Gamma(\varphi)(u)$ is non empty. Since S is filtered, for all $u, v \in S$, there is some $w \in S$ with $w \leq u$, $w \leq v$ and $\Gamma(\varphi)(w) \subseteq \Gamma(\varphi)(u) \cap \Gamma(\varphi)(v)$. Therefore the family $\Gamma(\varphi)$ is a filter basis. Let us see now that this basis is Cauchy. Let $u \in S$. Consider $v \in S$ such that $v^2 \leq u$ and let $x, y \in \Gamma(\varphi)(v)$. $v^2 \in \varphi(y) \otimes \tilde{\varphi}(y) \otimes \varphi(x) \otimes \tilde{\varphi}(x) \subseteq \tilde{\varphi}(y) \otimes \varphi(x)$. From (2) we have $u \in A(x,y)$.

Suppose now that F is a Cauchy filter compatible with $\Gamma(\varphi)$. We show that F is finer than $\Gamma(\varphi)$. Consider $u \in S$, we have to find some $f \in F$ such that $f \subseteq \Gamma(\varphi)(u)$. Let $v \in S$, such that $v^4 \leq u$. Choose some $f \in F$ such that $\forall x, y \in f, v \in A(x,y)$. Since F is compatible with $\Gamma(\varphi)$, $\Gamma(\varphi)(v) \cap f \neq \emptyset$. Let $\bar{x} \in \Gamma(\varphi)(v) \cap f$. Then for all $y \in f$, $u \geq v^4 \in \varphi(\bar{x}) \otimes A(y, \bar{x}) \otimes A(\bar{x}, y) \otimes \tilde{\varphi}(\bar{x}) \subseteq \varphi(y) \otimes \tilde{\varphi}(y)$. ∎

Fact 7.4 *If $F \in \wp(\wp(Obj(A)))$ satisfies:*

$$\forall u \in S, \exists f \in F, \forall x, y \in f, u \in A(x,y)$$

then the maps

$$\left\{ \begin{array}{l} \varphi : X \to \mathcal{O}_S, \\ x \mapsto \bigcup_{f \in F} \bigwedge_{y \in f} A(x,y) \end{array} \right. \text{ and,}$$

$$\left\{ \begin{array}{l} \tilde{\varphi} : X \to \mathcal{O}_S, \\ x \mapsto \bigcup_{f \in F} \bigwedge_{y \in f} A(y,x) \end{array} \right.$$

define respectively modules $\varphi : \hat{1} \longrightarrow\!\!\!\circ\!\!\!\rightarrow A$ and, $\tilde{\varphi} : A \longrightarrow\!\!\!\circ\!\!\!\rightarrow \hat{1}$. If F is moreover a filter basis then $\varphi \dashv \tilde{\varphi}$ and the Cauchy filter $\bar{\Gamma}(\varphi)$ (see 7.3) is less fine than F.

proof: φ is a module from $\hat{1}$ to A since for all objects x, y of A,

$$\begin{aligned} \varphi(x) \otimes A(y,x) &= (\bigcup_{f \in F} \bigwedge_{z \in f} A(x,z)) \otimes A(y,x) \\ &= \bigcup_{f \in F} ((\bigwedge_{z \in f} A(x,z)) \otimes A(y,x)) \\ &\subseteq \bigcup_{f \in F} \bigwedge_{z \in f} (A(x,z) \otimes A(y,x)) \\ &\subseteq \varphi(y). \end{aligned}$$

Analogously $\tilde{\varphi}$ is a module $A \longrightarrow\!\!\!\circ\!\!\!\rightarrow 1$.

Now suppose that F is moreover a filter basis. Let us see that $\varphi \dashv \tilde{\varphi}$. Let us check first $S \subseteq \bigcup_{x \in Obj(A)} \varphi(x) \otimes \tilde{\varphi}(x)$.

$$\begin{aligned} \bigcup_{x \in Obj(A)} \varphi(x) \otimes \tilde{\varphi}(x) &= \bigcup_{x \in Obj(A)} ((\bigcup_{f \in F} \bigwedge_{y \in f} A(x,y)) \otimes (\bigcup_{f \in F} \bigwedge_{y \in f} A(y,x))) \\ &= \bigcup_{x \in Obj(A)} \bigcup_{f_1, f_2 \in F} (\bigwedge_{y \in f_1} A(x,y) \otimes \bigwedge_{y \in f_2} A(y,x)) \\ &= \bigcup_{x \in Obj(A)} \bigcup_{f \in F} (\bigwedge_{y \in f} A(x,y) \otimes \bigwedge_{y \in f} A(y,x)) \text{ (F is filtered)} \end{aligned}$$

Let $u \in S$. There is some $v \in S$ such that $u \geq v^8$. By assumption there is some non-empty $f(v) \in F$ such that $x, y \in f(v) \Rightarrow v \in A(x,y)$. Then for some $x \in f(v)$, $v \in A(x,y) \cap A(y,x)$ for any $y \in f(v)$, $v^3 \in \bigwedge_{y \in f(v)} A(x,y) \cap \bigwedge_{y \in f(v)} A(y,x)$ and $u \in \bigwedge_{y \in f(v)} A(x,y) \otimes \bigwedge_{y \in f(v)} A(y,x)$.

Let us check that for any objects x, y of A, $\tilde{\varphi}(x) \otimes \varphi(y) \subseteq A(y,x)$. Let $x, y \in Obj(A)$,

$$\begin{aligned}\tilde{\varphi}(x)\otimes\varphi(y) &= (\bigcup_{f\in F}\bigwedge_{z\in f}A(z,x))\otimes(\bigcup_{f\in F}\bigwedge_{z\in f}A(y,z))\\ &= \bigcup_{f_1,f_2\in F}(\bigwedge_{z\in f_1}A(z,x)\otimes\bigwedge_{z\in f_2}A(y,z))\\ &= \bigcup_{f\in F}(\bigwedge_{z\in f}A(z,x)\otimes\bigwedge_{z\in f}A(y,z)).\end{aligned}$$

Let $f\in F$. Since f is non empty then for some $z_0\in f$, $\bigwedge_{z\in f}A(z,x)\subseteq A(z_0,x)$, $\bigwedge_{z\in f}A(y,z)\subseteq A(y,z_0)$ and, $\bigwedge_{z\in f}A(z,x)\otimes\bigwedge_{z\in f}A(y,z)\subseteq A(z_0,x)\otimes A(y,z_0)$. Thus $\tilde{\varphi}(x)\otimes\varphi(y)\subseteq A(y,x)$.

Now we prove that F is finer than $\Gamma(\varphi)$. Let $u\in S$. There is some $v\in S$ such that $v^2\leq u$. Since $\Gamma(\varphi)(v)\neq\emptyset$, we choose some $\bar{x}\in\Gamma(\varphi)(v)$ i.e. $v\in\varphi(\bar{x})\otimes\tilde{\varphi}(\bar{x})$, i.e. $v\geq v_1\cdot v_2$ for some $v_1\in\varphi(\bar{x})$ and $v_2\in\tilde{\varphi}(\bar{x})$. Thus there are some $f_1,f_2\in F$ such that

$$\begin{aligned}&v_1\in\bigwedge_{y\in f_1}A(\bar{x},y)\subseteq\bigcap_{y\in f_1}A(\bar{x},y)\text{ and,}\\ &v_2\in\bigwedge_{y\in f_2}A(y,\bar{x})\subseteq\bigcap_{y\in f_2}A(y,\bar{x}).\end{aligned}$$

Thus given $y\in f_1\cap f_2$, $v_1\in A(\bar{x},y)$ and $v_2\in A(y,\bar{x})$ and $u\geq v^2\geq v_1v_2v_1v_2\in\varphi(\bar{x})\otimes A(y,\bar{x})\otimes A(\bar{x},y)\otimes\tilde{\varphi}(\bar{x})\subseteq\varphi(y)\otimes\tilde{\varphi}(y)$. Thus $f_1\cap f_2\subseteq\Gamma(\varphi)(u)$. ∎

Fact 7.5 $\bar{\Gamma}$ *defines a bijection between adjoint pairs of modules of the form* $\varphi\dashv\tilde{\varphi}:\ 1 \multimap A$ *and minimal Cauchy-filters on* A.

This is a consequence of

Fact 7.6 *Given an adjoint pair* $\varphi\dashv\tilde{\varphi}:\ 1 \multimap A$ *then it is the pair corresponding to the Cauchy filter* $\bar{\Gamma}(\varphi)$ *as in 7.4, — i.e. for any object* x *of* A, $\varphi(x)=\bigcup_{u\in S}\bigwedge_{y\in\Gamma(\varphi)(u)}A(x,y)$

proof: It is enough to show that for any object x of A, $\varphi(x)\subseteq\bigcup_{u\in S}\bigwedge_{y\in\Gamma(\varphi)(u)}A(x,y)$ since one should get analogously $\tilde{\varphi}(x)\subseteq\bigcup_{u\in S}\bigwedge_{y\in\Gamma(\varphi)(u)}A(y,x)$ and we are considering two adjoint pairs. Let $x\in Obj(A)$ and $u\in\varphi(x)$. $u\geq v_1v_2u'v_3$ for some $v_1,v_2,v_3\in S$ and $u'\in\varphi(x)$. For any $y\in\Gamma(\varphi)(v_2)$, by definition $v_2\in\varphi(y)\otimes\tilde{\varphi}(y)$ thus $v_2\in\tilde{\varphi}(y)$ and $v_2u'\in\tilde{\varphi}(y)\otimes\varphi(x)\subseteq A(x,y)$. Therefore $u\geq v_1v_2u'v_3\in\bigwedge_{y\in\Gamma(\varphi)(v_2)}A(x,y)$. ∎

If F is Cauchy-filter on A, we shall write φ_F for the corresponding left adjoint module $1 \multimap A$ define in 7.4. 7.3 and 7.4 show that for a given Cauchy filter F there is one and only minimal Cauchy filter that is less fine than F. We will denote it $\bar{F}$. $\bar{F}=\bar{\Gamma}(\varphi_F)$.

7.2 Completion and separation / Cauchy-completion and skeletality

Let (A,S) be some abstract quasi-uniformities. Given some object x of A, let $B^1(x)$, $B^2(x)$, $B^3(x)$, $B^4(x)$ denote respectively the set of subsets,
- $\{y\in Obj(A)\mid u\in A(y,x)\cap A(x,y)\}$, u ranging in S;
- $\{y\in Obj(A)\mid u\in A(y,x)\otimes A(x,y)\}$, u ranging in S;
- $\{y\in Obj(A)\mid u\in\rho\circ A(y,x)\cap\rho\circ A(x,y)\}$ u ranging in S;
- $\{y\in Obj(A)\mid u\in\rho\circ A(y,x)\otimes\rho\circ A(x,y)\}$, u ranging in S.

Then $B_1(x)$, $B_2(x)$, $B_3(x)$, $B_4(x)$ are Cauchy bases. They are equivalent in the sense that they generate the same filter on $Obj(A)$. The filter that they generate is denoted $V(x)$. An element of $V(x)$ is called a *neigbourhood of* x (in (A,S)). If (A,S) is dense then $B_1(x)=B_3(x)$ and $B_2(x)=B_4(x)$. In this case, for any object x of A, $V(x)$ is just $\bar{\Gamma}(x_\diamond)$. According to 7.3 it is a minimal Cauchy filter on (A,S).

This motivates the definitions:

Definition 7.7 (A,S) *is said* complete *when one of the following equivalent assertion holds:*
- any Cauchy filter is finer than the filter of neighbourhoods for some x;
- any minimal Cauchy-filter occurs as some $V(x)$ for some object x of A.

Fact 7.8 *The following assertions are equivalent:*
- (A,S) is complete;
- $Int(A,S)$ is complete;
- $E(A,S)$ $(= E_D \circ Int(A))$ is Cauchy-complete.

We define (dually)

Definition 7.9 (A,S) *is said* separated *when one the following equivalent assertion holds:*
- (i) the map assigning any object x to the filter $V(x)$ is injective;
- (ii) For all objects x,y of $Obj(A)$, if $x \neq y$ then there is a $u \in S$ such that $u \notin A(x,y) \cap A(y,x)$.

The assertion (ii) is equivalent to the existence of a $u \in S$ such that $y \notin \Gamma(x)(u)$ or $x \notin \Gamma(y)(u)$. So clearly (ii) implies (i).
That (i) implies (ii) results from the following observations. Since any refinement of a Cauchy filter is Cauchy, two distinct minimal Cauchy filters could not have a common refinement. Thus given objects x and y of A, if $V(x) \neq V(y)$ then there are v and u such that $\Gamma(x)(u) \cap \Gamma(y)(v) = \emptyset$.

Fact 7.10 *The following assertions are equivalent:*
- (A,S) is separated;
- $Int(A,S)$ is separated;
- $E(A,S)$ is skeletal.

7.3 A completion for abstract quasi-uniformities

$SC - QAUnif$ is the locally preordered 2-category with:
- objects: separated and complete abstract quasi-uniformities;
- arrows and local preorders inherited from $QAUnif$.
$SC - DQAUnif$ is the locally preordered 2-category with:
- objects: separated and complete *dense* abstract quasi-uniformities;
- arrows and local preorders inherited from $DQAUnif$.
We extend the notation "$(-)^\circ$" to define the categories $SC - QAUnif^\circ$ and $SC - DQAUnif^\circ$.

Theorem 7.11 *The inclusion functor $SC - QAUnif^\circ \hookrightarrow QAUnif^\circ$ has a left adjoint.*

According to 6.5, this is equivalent to

Theorem 7.12 *The inclusion functor $SC - DQAUnif^\circ \hookrightarrow DQAUnif^\circ$ has a left adjoint.*

Since ${E_D}^\circ$ is faithfull, this holds due to 5.5 and 4.9.

Further on we shall write $\overline{(A,S)}$ for the completion of the abstract quasi-uniformity (A,S) (which is its image by the left adjoint in 7.11).

8 Quasi-uniformities

8.1 Quasi-uniformities/ abstract ones

An abstract uniformities (A,S) is said *concrete* when it satisfies:
$\forall s,t \in S$, $\forall x,y \in Obj(A)$, $t \circ s \in A(x,y) \Leftrightarrow \exists z \in Obj(A), s \in A(x,z) \wedge t \in A(z,y)$.

Given two abstract uniformities (A, S) and (B, T) with A concrete and a map $f : Obj(A) \to Obj(B)$, the relation $R_{f,A,B}$ (see section 6.1) satisfies conditions (iv) of 5.4, as shown below. For all $s, s' \in S$ and $t, t' \in T$, such that $R_f(s,t)$ and $R_f(s',t')$, one has:

$$\begin{aligned}\forall x, y \in Obj(A), s' \circ s \in A(x,y) &\Rightarrow \exists z \in Obj(A),\ s \in A(x,z) \wedge s' \in A(z,y)\\ &\Rightarrow t \in B(fx, fz) \wedge t' \in B(fz, fy)\\ &\Rightarrow t' \circ t \in B(fx, fy).\end{aligned}$$

Let (X, U) be a quasi-uniformity. One builts a quasi-uniform triple $U = (U, \circ, \subseteq)$, where $\circ$ denotes the composition of relations, $\subseteq$ is the inclusion ordering. For any pair (x, y) in X, we let $(x,y)^{\in} = \{u \in U \mid (x,y) \in U\}$. From (X, U), one can built $P(X, U)$, the abstract quasi-uniformity over U as follows. $P(X, U)$ has objects those of X, and for any $x, y \in X$, $P(X,U)(x,y) = (x,y)^{\in}$.

Let (X, U), (Y, T) be quasi-uniformities, $A = P(X, U)$, $B = P(Y, T)$ and let f be a map $X \to Y$. The relation $R_{f,A,B} \subseteq U \times T$ (section 6.1) is for all $u \in U$ and $v \in T$

$$(u, v) \in R_f \Leftrightarrow \forall x, y \in X, (x, y) \in u \Rightarrow (fx, fy) \in v.$$

Now f is uniformly continuous exactly when $R_f(U) = T$, that is condition (i) of 5.4. Thus if f is uniformly continuous, we let:

Definition 8.1 *$P(f)$ stands for the morphism $(f, R_f) : P(X,U) \to P(Y,T)$ in $AUnif$.*

Finally, if $QUnif$ is seen as a locally discrete 2-category denoted $QUnif^2$ then P defines a normal lax functor $QUnif^2 \to AQUnif$. We write P° for the corresponding functor $QUnif \to AQUnif^\circ$. P° is faithfull.

Let (A, S) be some abstract quasi-uniformity, one may built a quasi-uniformity denoted $Q(A)$ over $Obj(A)$ as follows. $Q(A)$ is the quasi-uniformity with basis $B(A) \subseteq \wp(Obj(A) \times Obj(A))$, that is the set of sets $\hat{u} = \{(x,y) \mid u \in A(x,y)\}$, u ranging in U. Observe that (A, S) is concrete exactly when for all $s, t \in S$, $\hat{s} \circ \hat{t} \subseteq \widehat{s \circ t}$. Let us check that $B(A)$ is a basis of a quasi-uniformity on $Obj(A)$. For any $s \in S$, $\Delta \subseteq \hat{s}$ since for any $s \in S$, $s \in A(x,x)$. The assignment $s \mapsto \hat{s}$ defines a monotonous map $S \to \wp(Obj(A) \times Obj(A))$, thus $B(A)$ is filtered since S is. Note that for any $s, t \in S$, $\hat{s} \circ \hat{t} \subseteq \widehat{s \circ t}$. So, for any $p \in B(A)$, since $p = \hat{s}$ for some $s \in S$ there is some $t \in U$ such that $t^2 \leq u$ and $(\hat{t})^2 \subseteq \hat{t^2} \subseteq p$.

Let $(f, R) : (A, S) \to (B, T)$ be a morphism of abstract quasi-uniformities. According to 6.3, f is a uniformely continuous map $Q(A) \to Q(B)$. We let $Q(f)$ denote this arrow in $QUnif$. Q defines a functor $AQUnif^\circ \to QUnif$.

Trivially the $Q \circ P^\circ \cong 1$.

Let (A, S) be an abstract quasi-uniformities. One may define a morphism $R_S : \hat{S} \to S$ by: $\forall s \in S, \forall p \in \hat{S}$, $R_S(p, s) \Leftrightarrow p \subseteq \hat{s}$. R_S is also compatible with $(1_A, P \circ Q(A), A)$. Suppose that the abstract quasi-uniformity (A, S) is moreover concrete. Then there is a morphism of quasi-uniform triples $R'_S : S \to \hat{S}$ defined for all $s \in S$ and $p \in P$ by $R'_S(s,p) \Leftrightarrow \hat{s} \subseteq p$. It is straightforward that R_S satisfies conditions (i), (ii), (iii), of 5.4. It satifies (iv) since for all $s, t \in S$, $p, q \in \hat{S}$ if $\hat{s} \subseteq p$ and $\hat{t} \subseteq q$ then $\widehat{s \circ t} = \hat{s} \circ \hat{t} \subseteq p \circ q$. This R'_S is compatible with $(1_A, A, P \circ Q(A))$.
This shows

Fact 8.2 *There is a coreflection $P^\circ \dashv Q : QUnif \to AQUnif^\circ$. $QUnif$ is equivalent to the full subcategory of $AQUnif^\circ$ of concrete abstract uniformities.*

8.2 Completions of quasi-uniformities

We show now how to infer 2.2 from 7.11.

It is straightforward to check that both P and Q preserve separation and completion. That is:
- For any quasi-uniformity (X,U), (X,U) is separated (resp. complete) if and only if $P(X,U)$ is.
- For any abstract quasi-uniformity (A,S), (A,S) is separated (resp. complete) if and only if $Q(A,S)$ is.

Let (X,U) be a quasi-abstract uniformity. Since the components of unit of the adjunction $P \dashv Q$ are isomorphisms, $\epsilon_{P(X,U)} : P \circ Q \circ P(X,U) \to P(X,U)$, the counit of the co-reflection 8.2 in $P(X,U)$ is an isomorphism. The situation is depicted by the commuting diagram in $QAUnif$ below.

$$\begin{tikzcd} P \circ Q \circ P(X,U) \arrow[r, "\epsilon_{P(X,U)}", "\cong"'] \arrow[d, "P\circ Q(i_{P(X,U)})"'] & P(X,U) \arrow[d, "i_{P(X,U)}"] \\ P \circ Q \circ \overline{P(X,U)} \arrow[r, "\epsilon_{\overline{P(X,U)}}"'] & \overline{P(X,U)} \end{tikzcd}$$

Since $P \circ Q \circ \overline{P(X,U)}$ is separated and complete, there is a unique morphism of quasi-uniformities r such that $r \circ i_{P(X,U)} = P \circ Q(i_{P(X,U)}) \circ \epsilon_{P(X,U)}{}^{-1}$. It satisfies $\epsilon_{\overline{P(X,U)}} \circ r = 1$. Since $\epsilon_{\overline{P(X,U)}}$ is a monomorphism, it is an isomorphism. From this, the fact that P is full and faithfull and, 7.11, one gets that for any uniformly continuous map $f : (X,U) \to (Y,V)$ where (Y,V) is separated and complete there is one and only one uniformly continuous map $\bar{f} : Q \circ \overline{P(X,U)} \to (Y,V)$ that makes the diagram below in $QUnif$ commute

$$\begin{tikzcd} (X,U) \arrow[r, "\cong"] \arrow[drr, "f"'] & Q \circ P(X,U) \arrow[r, "Q(i_{P(X,U)})"] & Q \circ \overline{P(X,U)} \arrow[d, "\bar{f}"] \\ & & (Y,V) \end{tikzcd}$$

9 Reversibility

9.1 Reversible enrichments

Definition 9.1 *A monoidal partial order $\mathcal{V}$ is* reversible *when there is an automorphism rev of the category $\mathcal{V}$ such that:*
- rev is involutive i.e. $rev^2 = 1$;
- for any objects a,b of $\mathcal{V}$, $rev(a \otimes b) = rev(b) \otimes rev(a)$;
- $rev(I) = I$.

Further on, $\mathcal{V}$ denotes a reversible closed monoidal partial order with automorphism rev.

Definition 9.2 *A $\mathcal{V}$-category A is* reversible *when the diagram of maps, below, commutes.*

$$\begin{tikzcd}
Obj(A) \times Obj(A) \arrow[d, "rev \times rev"'] \arrow[dr, "A(-,-)"] & \\
 & Obj(\mathcal{V}) \\
Obj(A) \times Obj(A) \arrow[ur, "A(-,-)"'] &
\end{tikzcd}$$

Observation 9.3 *Let A, B be $\mathcal{V}$-categories. If B is reversible then the preoder on $\mathcal{V} - Cat(A, B)$ is an equivalence, if B is moreover skeletal then this equivalence is the identity.*

A monoidal functor $F : \mathcal{V} \to \mathcal{W}$ between reversible categories *preserves reversibility* when the diagram in Cat, below, commutes:

$$\begin{tikzcd}
\mathcal{V} \arrow[r, "F"] \arrow[d, "rev"'] & \mathcal{W} \arrow[d, "rev"] \\
\mathcal{V} \arrow[r, "F"'] & \mathcal{W}
\end{tikzcd}$$

Note that if the monoidal functor $F : \mathcal{V} \to \mathcal{W}$ preserves reversibility and the enrichment $(A, \mathcal{V})$ is reversible then $F_{@}(A, \mathcal{V})$ is reversible. So we define $REnr$ as the locally preordered 2-category with:
- objects: reversible enrichments;
- arrows: the $(f, F) : (A, \mathcal{V}) \to (B, \mathcal{W})$ of Enr such that F preserves the reversibility.
- local pre-orders inherited from Enr.

9.2 Uniform 4-uples

A *uniform 4-uple* $(S, \cdot, \leq, rev)$ is a quasi-uniform triple $(S, \cdot, \leq)$ with a bijective map $rev : S \to S$ such that:
- $rev^2 = rev$,
- $\forall u, v \in S$, $rev(uv) = rev(v) \circ rev(u)$,
- $\forall u, v \in S$, $u \leq v \Rightarrow rev(u) \leq rev(v)$.

Observation 9.4 *If $(S, \cdot, \leq, rev)$ is a quasi-uniform 4-uple, then $\mathcal{C}_S$ is reversible with the direct image of rev as automorphism.*

Given uniform 4-uples S and T, we say that a relation $R \subseteq S \times T$ *preserves reversibility* when $\forall s \in S, \forall t \in T, R(s, t) \Leftrightarrow R(rev(s), rev(t))$. The composition of reversibility-preserving relations preserves reversibility. For any uniform 4-uple S, $\leq_S$ and $\sqsubseteq_S$ preserve reversibility. We define the morphisms of uniform 4-uples as the morphisms of uniform triples that preserve reversibility. For such a morphism $R : S \to T$, $\bar{R}$ also preserves reversibility, and $F(R) : \mathcal{C}_S \to \mathcal{C}_T$ preserves reversibility.

9.3 Abstract uniformities

An *abstract uniformity* (A, S) is a quasi-abstract uniformity A on a uniform 4-uple S, such that the diagram of maps below commutes

$$\begin{tikzcd}
Obj(A) \times Obj(A) \arrow[d, "rev \times rev"'] \arrow[dr, "A(-,-)"] & \\
& \mathcal{I}_S \\
Obj(A) \times Obj(A) \arrow[ur, "A(-,-)"'] &
\end{tikzcd}$$

The definitions and the various observations above allow us to define of the following 2-categories. $AUnif$ is the local preordered 2-category of abstract uniformities. It consists in the following data:
- objects: abstract uniformities,
- arrows $A \to B$: the $(f, R) : (A, S) \to (B, T)$ in $QAUnif$ with R preserving reversibility,
- local pre-orders inherited from $QAUnif$.

$DAUnif$ is the locally preordered 2-category with:
- objects: dense abstract uniformities,
- arrows $A \to B$: the $(f, R) : A \to B$ of $QAUnif$ with R dense and preserving reversibility,
- local pre-orders inherited from $QAUnif$.

There are inclusion 2-functors $AUnif \hookrightarrow QAUnif$ and $DAUnif \hookrightarrow DQAUnif$. We define the 2-functor Int_R and the lax functors E_R, E_{RD}, as the unique lax-functors making the diagrams below commute.

$$\begin{tikzcd}
AQUnif \arrow[r, "Int"] \arrow[rr, bend left, "E"] & DAQUnif \arrow[r, tail, "E_D"] & Enr \\
AUnif \arrow[u, tail] \arrow[r, "Int_R"'] \arrow[rr, bend right, "E_R"'] & DAUnif \arrow[u, tail] \arrow[r, tail, "E_{RD}"'] & REnr \arrow[u, tail]
\end{tikzcd}$$

Again we will obviously extend the notation $(-)^\circ$. We get an equivalence of categories $Int_R{}^\circ : AUnif^\circ \cong DAUnif^\circ$. The functors $E_R{}^\circ : AUnif^\circ \to REnr^\circ$ and $E_{RD}{}^\circ : DAUnif^\circ \to REnr^\circ$ satisfy $E_R{}^\circ = E_{RD}{}^\circ \circ Int_R{}^\circ$. Since $E_{RD}{}^\circ$ is faithfull then also is $E_R{}^\circ$.

Recall that the Cauchy-completion does not generally preserve reversibility [Be-Wal82]. But according to 7.5, one sees that for any uniform 4-uple S, the Cauchy-completion of any reversible enrichment over $\mathcal{C}_S$ is also reversible. Thus from 7.11, one deduces

Theorem 9.5 *The inclusion functor $SC - AUnif^\circ \hookrightarrow AUnif^\circ$ has a left adjoint.*

Precisely the restriction to the category $AUnif^\circ$ of $Comp$, the left adjoint to $SC - AQUnif^\circ \hookrightarrow SC - AQUnif^\circ$, defines a functor $Comp_R : AUnif^\circ \to SC - AUnif^\circ$ that is left adjoint to the inclusion $SC - AUnif^\circ \hookrightarrow AUnif^\circ$. The situation is depicted below by the commuting diagrams in Cat.

$$
\begin{tikzcd}
AQUnif^{\circ} \arrow[r, "Comp"] & SC - AQUnif^{\circ} \arrow[r, tail] & AQUnif^{\circ} \\
AUnif^{\circ} \arrow[u, tail] \arrow[r, "Comp_R"'] & AUnif^{\circ} \arrow[u, tail] \arrow[r, tail] & AQUnif^{\circ} \arrow[u, tail]
\end{tikzcd}
$$

9.4 Abstract uniformities/ uniform spaces

Any uniformity (X, U) obviously defines a uniform 4-uple $(U, \subseteq, \circ, (-)^{-1})$ where $(U, \subseteq, \circ)$ is the quasi-uniform triple defined in 8.1 and $(-)^{-1}$ is the map sending $u \in U$ to its inverse u^{-1}. Now the abstract-quasi uniformity $P(X, U)$ is easily seen to be an abstract uniformity, and one may define a full and faithfull $P_R : Unif \to AUnif^{\circ}$ by considering the restriction of P on $Unif$. Obviously P_R has a right adjoint Q_R deduced from Q by restriction on $AUnif^{\circ}$. Analogous arguments to those used in 8.2 prove 2.1.

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