MA1251 Chaos and Fractals: Problem Set No. 2

Due: Tuesday, 13 May 2003, 11:30am

1. (Linear stability analysis for maps) In the lectures, you have seen the analysis of the dynamics of a one-dimensional flow in the vicinity of a fixed point. In particular, if $dx/dt = f(x)$ is the flow with a fixed point x^*, then the solution in the vicinity of the fixed point is given by

$$x(t) = x^* + \delta(0)\mathrm{e}^{\lambda t} ,$$

where $\lambda = f'(x^*)$ is the function derivative evaluated at the fixed point.

Similar analysis can be also performed for maps. To this end, consider a map $x_{n+1} = f(x_n)$ with a fixed point given by $x^* = f(x^*)$ and an orbit starting in the vicinity of the fixed point at $x_0 = x^* + \delta_0$, where δ_0 is small.

(a) Show that, if $x^* + \delta_1$ is the next point of the orbit, then, in the linear approximation,

$$\delta_1 = f'(x^*)\delta_0 .$$

(b) Show also that for subsequent map iterates, where $x^* + \delta_n = f(x^* + \delta_{n-1})$,

$$\delta_n = [f'(x^*)]^n \, \delta_0 . \tag{1}$$

(c) A fixed point is stable if the nearby orbits converge to it. Use Eq. (1) to deduce that a fixed point is stable if

$$|f'(x^*)| < 1 .$$

2. (Stability of fixed points of the Logistic map) Consider the logistic map $x_{n+1} = rx_n(1 - x_n)$.

(a) Show that the logistic map has two fixed points: $x^* = 0$ and $x^* = 1 - 1/r$. *Hint:* Fixed points x^* of a map $f(x)$ can be found by solving the equation $x^* = f(x^*)$.

(b) In what range of values of parameter r is the fixed point $x^* = 0$ stable?

(c) In what range of values of parameter r is the fixed point $x^* = 1 - 1/r$ stable?