MA1251 Chaos and Fractals: Problem Set No. 1

Due: Tuesday, 6 May 2003, 11:30am

1. (Tumour growth model) The growth of cancerous tumours can be modelled by the Gompertz law

$$\dot{N} = -aN \log(bN)\,,$$

where $N(t)$ is proportional to the number of cells in the tumour, and $a, b > 0$ are parameters. Note that $\dot{N} = \frac{dN}{dt}$ denotes the derivative of $N(t)$ with respect to t. The predictions of this simple model agree surprisingly well with data on tumour growth, as long as N is not too small.

   (a) Sketch the function of the right-hand side and the direction of the flow for various initial values of N.

   (b) Find all fixed points and classify their stability.

   (c) Interpret a and b biologically.

2. (The Allee effect) For certain species of organisms, the effective growth rate $\dot{N}/N$ is highest at intermediate N. This is the so-called *Allee effect*. For example, imagine that it is too hard to find mates when N is very small, and there is too much competition for food and other resources when N is large.

   (a) Show that the model

$$\dot{N}/N = r - a(N - b)^2$$

   provides an example of Allee effect, if r, a, and b satisfy certain constraints, to be determined.

   (b) Find all the fixed points of the model and classify their stability. *Hint:* Depending on the values of parameters r, a, and b, there are either two or three fixed points.

   (c) Sketch the directions of the flow for various initial conditions and compare to those for the logistic equation discussed in the lectures. What are the qualitative differences, if any?

3. (Linear stability analysis) Find the analytic solutions of the Gompertz model in the vicinity of the fixed points, by deriving and solving the linearised system. *Hint:* This is similar to the logistic equation example discussed in the lectures.